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ON INVOLUTIVE MONOIDAL CATEGORIES

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ABSTRACT. In this paper, we consider a non-posetal analogue of the notion of *involutive quantale* [MP92]; specifically, a (planar) monoidal category equipped with a covariant involution that reverses the order of tensoring. We study the coherence issues that inevitably result when passing from posets to categories; we also link our subject with other notions already in the literature, such as *balanced monoidal categories* [JS91] and *dagger pivotal categories* [Sel09].

1. Introduction

The *algebraic theory of involutive monoids* is obtained by adding a unary operation $\overline{(\ )}$ and the axioms

$$\overline{p} \cdot \overline{q} = \overline{q \cdot p} \quad , \quad \overline{\overline{r}} = r$$

to that of monoids. Clearly, this theory can be modelled in any symmetric monoidal category: familiar examples of involutive monoids in $(\mathbf{Ab}, \otimes, \mathbb{Z})$ include $\mathbb{C}$ and $\mathbb{H}$; involutive monoids in $(\mathbf{Sup}, \otimes, 2)$ are precisely the *involutive quantales* referred to in the abstract. Since $(\mathbf{Cat}, \times, 1)$ is a symmetric monoidal category, we obtain a notion of (small and) *strict involutive monoidal category*. In this paper, we shall consider a (large and) non-strict variant of the latter; it is defined and justified in section 2. A coherence theorem for involutive monoidal categories is stated and proven in section 3.

Every *symmetric monoidal category* [ML63] can be made into an involutive monoidal category for which both $\overline{(\ )}$ and the chosen isomorphism $\overline{\overline{r}} \xrightarrow{\sim} r$ are identities. But it is not true, as one might guess, that an involutive monoidal category with such a trivial involution necessarily arises in this way. Indeed, one finds that in the dawn of monoidal category theory, Bénabou defined a *commutation* [Bén64] on a monoidal category $\mathfrak{K} = (\mathcal{K}, \otimes, k, \alpha, \lambda, \rho)$ to be a natural transformation $\otimes^{\mathsf{rev}} \xrightarrow{\chi} \otimes$ satisfying $\chi^{\mathsf{rev}} = \chi^{-1}$, and such that $(\mathrm{Id}_{\mathcal{K}}, \chi, \mathrm{id}_k)$ forms a monoidal functor $\mathfrak{K}^{\mathsf{rev}} \longrightarrow \mathfrak{K}$. (Here $p \otimes^{\mathsf{rev}} q := q \otimes p$, $\chi^{\mathsf{rev}}_{p,q} := \chi_{q,p}$, and $\mathfrak{K}^{\mathsf{rev}} = (\mathcal{K}, \otimes^{\mathsf{rev}}, k, \alpha^{-1}, \rho, \lambda)$.) An involutive monoidal category with a trivial involution (in the sense above) amounts to no more than a monoidal category equipped with a commutation, and an example of a commutation which is not a symmetry was given by Kasangian and Rossi [KR81]. (What they show, of course, is that their

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example is not a *braiding* [JS93], but that term did not then exist.) This apparently irredeemable situation is thoroughly explored in section 4.

In section 5, we consider *involutive ambimonoidal categories* (where we use *ambimonoidal* to mean *linearly distributive* in the sense of Cockett and Seely [CS97b]) and, in particular, *involutive star-autonomous categories*. In section 6, we show that *dagger pivotal categories* [Sel09] are a (well-motivated) special case of the latter.

1.1. Remark. Two things were brought to the author's attention after this paper was submitted for publication: that some of its material has been prefigured in [BM09], and that a handful of other authors have simultaneously and independently become interested in related topics [Jac10, Sel10]. Further details of how our results overlap with those of *opp. cit.* are contained in a remark at the end of each section.

1.2. Remark. The astute reader may wonder why we do not consider *involutive bicategories* (generalising *involutive quantaloids* [Pas99, Gyl99]), and treat involutive monoidal categories as the one-object case thereof. The answer is simple expediency: given the topicality of involutive monoidal categories, the extra time required to rewrite our entire exposition (which, unfortunately, was begun in the less general case) did not seem worth it. We hope that the reader interested in extra generality will have little difficulty in extrapolating the correct definitions and theorems from those contained herein.

2. Involutive monoidal categories

2.1. Definition. *An* involution *for a monoidal category* $\mathfrak{K} = (\mathcal{K}, \otimes, k, \alpha, \lambda, \rho)$ *consists of a functor* $\mathcal{K} \xrightarrow{\overline{(\,)}} \mathcal{K}$ *and natural isomorphisms*

$$\overline{p} \otimes \overline{q} \xrightarrow{\chi_{p,q}} \overline{q \otimes p} \qquad\qquad \overline{\overline{r}} \xrightarrow{\varepsilon_r} r$$

such that the diagrams

$$\begin{array}{ccc}
(\overline{p} \otimes \overline{q}) \otimes \overline{r} & \xrightarrow{\alpha_{\overline{p},\overline{q},\overline{r}}} & \overline{p} \otimes (\overline{q} \otimes \overline{r}) \\
\downarrow{\scriptstyle \chi_{p,q} \otimes \mathrm{id}_{\overline{r}}} & & \downarrow{\scriptstyle \mathrm{id}_{\overline{p}} \otimes \chi_{q,r}} \\
\overline{q \otimes p} \otimes \overline{r} & (F_3) & \overline{p} \otimes \overline{r \otimes q} \\
\downarrow{\scriptstyle \chi_{q\otimes p,r}} & & \downarrow{\scriptstyle \chi_{p,r\otimes q}} \\
\overline{r \otimes (q \otimes p)} & \xleftarrow[\overline{\alpha_{r,q,p}}]{} & \overline{(r \otimes q) \otimes p}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overline{\overline{p}} \otimes \overline{\overline{q}} & \xrightarrow{\chi_{\overline{p},\overline{q}}} & \overline{\overline{q} \otimes \overline{p}} \\
\downarrow{\scriptstyle \varepsilon_p \otimes \varepsilon_q} & (N_2) & \downarrow{\scriptstyle \overline{\chi_{q,p}}} \\
p \otimes q & \xleftarrow[\varepsilon_{p\otimes q}]{} & \overline{\overline{p \otimes q}}
\end{array}$$

hold, and also the equation $\overline{\overline{\overline{r}}} \xrightarrow{\overline{\varepsilon_r} = \varepsilon_{\overline{r}}} \overline{r}$, *which is denoted* (A).

An involutive monoidal category *is a monoidal category equipped with an involution.*

2.2. Examples. A simple example of a non-symmetric involutive monoidal category is $(\mathbf{Cat}, +_{lex}, 0, (\ )^{\mathsf{op}})$, where $+_{lex}$ denotes the *ordinal sum* of categories [Law69]. Of course, $(\mathbf{Cat}, \times, 1, (\ )^{\mathsf{op}})$ is also an involutive monoidal category; hence it is possible for a symmetric monoidal category to also admit an entirely non-trivial involutive structure.

Restricting $+_{lex}$ to the full subcategory of $\mathbf{Cat}$ determined by finite linearly ordered sets, $\mathbf{Flo}$, we see that it is possible for a monoidal category to have an involution which is trivial in the sense that $p^{\mathsf{op}} \cong p$ for every object p, but which is not naturally trivial in the sense that there is no natural isomorphism $p^{\mathsf{op}} \xrightarrow{\sim} p$. (Of course, this is just another way of looking at the celebrated fact that $(\mathbf{Flo}, +_{lex}, 0)$ is not a symmetric monoidal category despite having $p +_{lex} q \cong q +_{lex} p$ for every pair of objects p and q.)

A class of examples somewhat nearer the author's heart (and which go some way to explaining our preferred notation) are monoidal categories of the form $({}_k\mathbf{Mod}_k, \otimes_k, k)$, where k is an involutive ring, such as $\mathbb{C}$ or $\mathbb{H}$. Then we have an involution defined as follows: given a (k,k)-module p (with left and right scalar multiplications denoted by $\triangleright$ and $\triangleleft$), $\overline{p}$ has the same underlying abelian group as p, but with scalar multiplications $\overline{\triangleright}$ and $\overline{\triangleleft}$ defined by

$$\begin{array}{rcl} \alpha \mathbin{\overline{\triangleright}} \psi & = & \psi \triangleleft \overline{\alpha} \\ \psi \mathbin{\overline{\triangleleft}} \alpha & = & \overline{\alpha} \triangleright \psi \end{array}$$

for all $\alpha \in k$ and $\psi \in p$.

In the case $k = \mathbb{C}$, we can restrict to the case of bimodules satisfying $\alpha \triangleright \psi = \psi \triangleleft \alpha$—*i.e.*, to the case of (mere) vector spaces over $\mathbb{C}$. Note that this example is similar both to $(\mathbf{Cat}, \times, 1, (\ )^{\mathsf{op}})$ in that $\otimes$ happens to be symmetric, and to $(\mathbf{Flo}, +_{lex}, 0, (\ )^{\mathsf{op}})$ in that we have unnatural isomorphisms $\overline{v} \xrightarrow{\sim} v$. Both $(\mathbf{Ban}, \hat{\otimes}, \mathbb{C})$ and $(\mathbf{Ban}, \check{\otimes}, \mathbb{C})$, where $\hat{\otimes}$ and $\check{\otimes}$ respectively denote the *projective* and *injective* tensor product of Banach spaces, inherit an involutive structure from $(\mathbf{Vec}, \otimes, \mathbb{C}, \overline{(\ )})$; more elaborate examples arising in functional analysis will be discussed in a future paper.

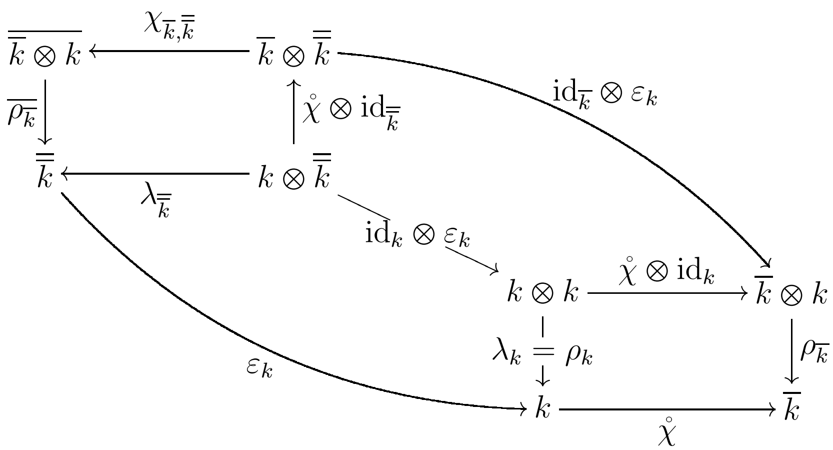
In all of the examples above, ε happens to be the identity natural transformation; yet we feel that the full generality of Definition 2.1 is justified by Theorem 2.4 below.

2.3. Lemma. *For every involutive monoidal category, there exists a unique arrow* $k \xrightarrow{\mathring{\chi}} \overline{k}$ *such that the diagrams*

$$\begin{array}{ccc} k \otimes \overline{q} & \xrightarrow{\ \mathring{\chi} \otimes \mathrm{id}_{\overline{q}}\ } & \overline{k} \otimes \overline{q} \\ {\scriptstyle \lambda_{\overline{q}}}\downarrow & (F_1^{\ell}) & \downarrow{\scriptstyle \chi_{k,q}} \\ \overline{q} & \xleftarrow[\ \overline{\rho_q}\ ]{} & \overline{q \otimes k} \end{array} \qquad \begin{array}{ccc} \overline{q} \otimes k & \xrightarrow{\ \mathrm{id}_{\overline{q}} \otimes \mathring{\chi}\ } & \overline{q} \otimes \overline{k} \\ {\scriptstyle \rho_{\overline{q}}}\downarrow & (F_1^{r}) & \downarrow{\scriptstyle \chi_{q,k}} \\ \overline{q} & \xleftarrow[\ \overline{\lambda_q}\ ]{} & \overline{k \otimes q} \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\ \mathring{\chi}\ } & \overline{k} \\ \Vert & (N_0) & \downarrow{\scriptstyle \overline{\mathring{\chi}}} \\ k & \xleftarrow[\ \varepsilon_k\ ]{} & \overline{\overline{k}} \end{array}$$

hold. Moreover, this arrow is invertible.

PROOF. Uniqueness is easy: by considering (F_1^ℓ) in the case $q = \overline{k}$, and adjoining some naturality squares, one obtains the following diagram.



Since $\overline{\rho_{\overline{k}}}$, $\chi_{\overline{k},\overline{\overline{k}}}$ and $\lambda_{\overline{\overline{k}}}$ are all invertible, the outer perimeter of this diagram commutes; hence, $\mathring{\chi}$ must factor as follows.

$$k \xrightarrow{\varepsilon_k^{-1}} \overline{\overline{k}} \xrightarrow{\overline{\rho_{\overline{k}}}^{-1}} \overline{\overline{k} \otimes k} \xrightarrow{\chi_{k,\overline{k}}^{-1}} \overline{k} \otimes \overline{\overline{k}} \xrightarrow{\mathrm{id}_{\overline{k}} \otimes \varepsilon_k} \overline{k} \otimes k \xrightarrow{\rho_{\overline{k}}} \overline{k}$$

That this arrow satisfies the listed diagrams is merely a tedious exercise. ■

We note in passing that the natural isomorphisms

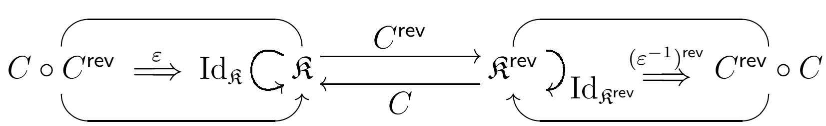
$$\overline{k} \otimes q \xrightarrow{\mathrm{id}_{\overline{k}} \otimes \varepsilon_q^{-1}} \overline{k} \otimes \overline{\overline{q}} \xrightarrow{\chi_{k,\overline{q}}} \overline{\overline{q} \otimes k} \xrightarrow{\overline{\rho_{\overline{q}}}} \overline{\overline{q}} \xrightarrow{\varepsilon_q} q$$

$$q \otimes \overline{k} \xrightarrow{\varepsilon_q^{-1} \otimes \mathrm{id}_{\overline{k}}} \overline{\overline{q}} \otimes \overline{k} \xrightarrow{\chi_{\overline{q},k}} \overline{k \otimes \overline{q}} \xrightarrow{\overline{\lambda_{\overline{q}}}} \overline{\overline{q}} \xrightarrow{\varepsilon_q} q$$

(denoted λ'_q, ρ'_q respectively) satisfy the coherence axioms necessary for $(\mathcal{K}; \otimes, \overline{k}; \alpha, \lambda', \rho')$ to be a monoidal category; moreover, the arrow $\mathring{\chi}$ constructed in the proof above is, predictably, nothing more than $k \xrightarrow{(\lambda'_k)^{-1}} \overline{k} \otimes k \xrightarrow{\rho_{\overline{k}}} \overline{k}$.

Let **Lax** denote the 2-category of monoidal categories, (lax) monoidal functors, and monoidal natural transformations; let $(\ )^{\mathsf{rev}}$ denote the (strict) involution $\mathbf{Lax} \longrightarrow \mathbf{Lax}$ which takes a monoidal category $\mathfrak{K}$ to $\mathfrak{K}^{\mathsf{rev}}$, a monoidal functor $(M, \mu, \mathring{\mu})$ to $(M, \mu^{\mathsf{rev}}, \mathring{\mu})$, and a monoidal natural transformation θ to itself.

2.4. THEOREM. *An involutive monoidal category is nothing more than an equivalence, internal to* **Lax**, *which happens to be of the form below.*



Proof. That $C = (\overline{(\ )}, \chi, \mathring{\chi})$ constitutes a monoidal functor $\mathfrak{K}^{\mathsf{rev}} \longrightarrow \mathfrak{K}$ is simply a restatement of the axiom (F_3) together with the diagrams (F_1^ℓ), (F_1^r). That ε constitutes a monoidal natural transformation $C \circ C^{\mathsf{rev}} \longrightarrow \mathrm{Id}_{\mathfrak{K}}$ is simply a restatement of the axiom (N_2) together with the diagram (N_0). That $(\varepsilon^{-1})^{\mathsf{rev}}$ and ε form the unit and the counit of an adjunction is simply a restatement of axiom (A).

Finally, note that it is superfluous to take the invertibility of χ as an axiom: it follows from naturality and the axioms (N_2) and (A) that

$$\overline{q \otimes p} \xrightarrow{\overline{\varepsilon_q^{-1} \otimes \varepsilon_p^{-1}}} \overline{\overline{\overline{q}} \otimes \overline{\overline{p}}} \xrightarrow{\overline{\chi_{\overline{q},\overline{p}}}} \overline{\overline{\overline{p} \otimes \overline{q}}} \xrightarrow{\varepsilon_{\overline{p}\otimes\overline{q}}} \overline{p} \otimes \overline{q}$$

is the inverse of $\chi_{p,q}$. ∎

2.5. Remark. The concept of *involutive monoidal category* described above is equivalent to that of *strong bar category* contained in [BM09]. To see this, it suffices to note that, once the following substitution of symbols has been made,

Bar	I.M.
Φ	α
l	ρ^{-1}
r	λ^{-1}
Υ	χ^{-1}
bb	ε^{-1}
$\star$	$\mathring{\chi}$

the five axioms of a *bar category* [BM09, Definition 2.1] are identical to the diagrams (F_1^ℓ), (F_1^r), (F_3), (N_0) and (A), and that the additional axiom of a *strong bar category* [BM09, Definition 2.3] equals the diagram (N_2).

Jacobs' concept of an involutive monoidal category [Jac10, Definition 4.1] is radically different from ours because he is primarily interested in the symmetric case; we shall have more to say about this in Remark 4.6.

3. Coherence for involutive monoidal categories

We recall that every monoidal category, $\mathfrak{K}$, is equivalent to a strict monoidal category, $\mathfrak{K}_{str}$; we prove the analogous theorem for involutive monoidal categories.

Both strictification theorems can be seen as corollaries of the fact that suitably constructed *free* (involutive) monoidal categories are strict.[1] For instance, in the non-involutive case, $\mathfrak{K}_{str}$ (as described, for example, in [JS91, p. 59]), can be abstractly understood as the (bijective-on-objects,full-and-faithful)-factorisation of the canonical map

[1] In other words, free (involutive) monoidal categories can be constructed to be strict.

$\mathcal{F}_m\mathcal{U}_m\mathfrak{K} \xrightarrow{[\![\ ]\!]} \mathfrak{K}$; *i.e.*, the counit of the pseudo-adjunction

$$\mathbf{Strong} \underset{\mathcal{F}_m}{\overset{\mathcal{U}_m}{\rightleftarrows}} \mathbf{Cat}$$

where **Strong** denotes the locally full sub-2-category of **Lax** determined by the strong monoidal functors. If we denote said factorisation as follows,

$$\begin{array}{ccc} \mathcal{F}_m\mathcal{U}_m\mathfrak{K} & \xrightarrow{\quad [\![\ ]\!] \quad} & \mathfrak{K} \\ & {\scriptstyle [\![\ ]\!]_{\mathrm{BO}}} \searrow \quad \nearrow {\scriptstyle [\![\ ]\!]_{\mathrm{FF}}} & \\ & \mathfrak{K}_{str} & \end{array}$$

then the strictness of $\mathfrak{K}_{str}$ follows from that of $\mathcal{F}_m\mathcal{U}_m\mathfrak{K}$ and $[\![\ ]\!]_{\mathrm{BO}}$, and its equivalence to $\mathfrak{K}$ from the fact that $[\![\ ]\!]$, and therefore also $[\![\ ]\!]_{\mathrm{FF}}$, is surjective on objects.

3.1. LEMMA. *Given an arbitrary category $\mathbb{A}$, $\mathcal{F}_m(\mathbb{A}+\mathbb{A})$ admits a strict involution given by the formula*

$$\overline{\mathrm{in}_{b_1}p_1 \cdots \mathrm{in}_{b_n}p_n} = \mathrm{in}_{1-b_n}p_n \cdots \mathrm{in}_{1-b_1}p_1$$

where $b_1, \ldots, b_n \in \{0,1\}$ and $p_1, \ldots, p_n \in \mathsf{ob}\,\mathbb{A}$.

We write $\mathcal{F}_{im}\mathbb{A}$ for the strict involutive monoidal category consisting of $\mathcal{F}_m(\mathbb{A}+\mathbb{A})$ together with the strict involution described in Lemma 3.1. It seems obvious that $\mathcal{F}_{im}$ is the object part of a 2-functor, and this proves to be the case as soon as we make the obvious definitions of involutive monoidal functor and involutive monoidal natural transformation.

3.2. DEFINITION. *Let $\mathfrak{J} = (\mathcal{J}, \otimes, j, \overline{(\ )})$ and $\mathfrak{K} = (\mathcal{K}, \otimes, k, \overline{(\ )})$ be involutive monoidal categories.*

1. *An* involutive monoidal functor $\mathfrak{J} \longrightarrow \mathfrak{K}$ *consists of a monoidal functor*

$$(\mathcal{J}, \otimes, j) \xrightarrow{(M, \mu, \mathring{\mu})} (\mathcal{K}, \otimes, k)$$

together with a natural transformation $\overline{Mp} \xrightarrow{\sigma_p} M\overline{p}$ such that the diagrams

$$\begin{array}{ccc} \overline{Mp} \otimes \overline{Mq} & \xrightarrow{\chi_{Mp,Mq}} & \overline{Mq \otimes Mp} \\ {\scriptstyle \sigma_p \otimes \sigma_q} \downarrow & & \downarrow {\scriptstyle \overline{\mu_{q,p}}} \\ M\overline{p} \otimes M\overline{q} & & \overline{M(q \otimes p)} \\ {\scriptstyle \mu_{\overline{p},\overline{q}}} \downarrow & & \downarrow {\scriptstyle \sigma_{q \otimes p}} \\ M(\overline{p} \otimes \overline{q}) & \xrightarrow{M\chi_{p,q}} & M\overline{q \otimes p} \end{array} \qquad \begin{array}{ccc} \overline{\overline{Mr}} & \xrightarrow{\overline{\sigma_r}} & \overline{M\overline{r}} \\ {\scriptstyle \varepsilon_{Mr}} \downarrow & & \downarrow {\scriptstyle \sigma_{\overline{r}}} \\ Mr & \xleftarrow{M\varepsilon_r} & M\overline{\overline{r}} \end{array}$$

hold.

2. *An* involutive monoidal natural transformation $(M, \mu, \mathring{\mu}, \sigma) \xrightarrow{\theta} (N, \nu, \mathring{\nu}, \varsigma)$ *is a monoidal natural transformation* $(M, \mu, \mathring{\mu}) \xrightarrow{\theta} (N, \nu, \mathring{\nu})$ *which respects involution, in the sense that the diagram*

$$\begin{array}{ccc} \overline{Mr} & \xrightarrow{\sigma_r} & M\overline{r} \\ {\scriptstyle \overline{\theta_r}}\big\downarrow & & \big\downarrow{\scriptstyle \theta_{\overline{r}}} \\ \overline{Nr} & \xrightarrow[\varsigma_r]{} & N\overline{r} \end{array}$$

holds.

An involutive monoidal functor is called strong *if its underlying monoidal functor is strong. We write* **StrongInvol** *for the 2-category of involutive monoidal categories, strong involutive monoidal functors and involutive monoidal natural transformations.*

3.3. Examples. If $\mathfrak{J}$ is the terminal involutive monoidal category, then (not necessarily strong) involutive monoidal functors $\mathfrak{J} \longrightarrow \mathfrak{K}$ can be called simply involutive monoids in $\mathfrak{K}$.[2] If a symmetric monoidal category is regarded as a trivially involutive monoidal category, then the two notions of involutive monoid (that here, and that in the first paragraph of section 1) coincide. In particular, an involutive monoid in $(\mathbf{Sup}, \oslash, 2, \mathrm{Id}_{\mathbf{Sup}})$ is an involutive quantale; but an involutive monoid in $(\mathbf{Vec}, \otimes, \mathbb{C}, \overline{(\ )})$ is an involutive $\mathbb{C}$-algebra. An involutive monoid in $(\mathbf{Ban}, \boxtimes, \mathbb{C}, \overline{(\ )})$ is what is nowadays often called a B^*-algebra[3]; this is similar to a C^*-algebra, except that it need not satisfy the C^*-identity.

The functor $\mathbf{Ban} \xrightarrow{P_c} \mathbf{Sup}$ assigning to each Banach space a its lattice of closed subspaces, $P_c(a)$, underlies an involutive monoidal functor

$$(\mathbf{Ban}, \boxtimes, \mathbb{C}, \overline{(\ )}) \xrightarrow{(P_c, \mu, \mathring{\mu}, \sigma)} (\mathbf{Sup}, \oslash, 2, \mathrm{Id}_{\mathbf{Sup}})$$

and hence gives rise to a functor from the category of B^*-algebras to that of involutive quantales. (In the case of a C^*-algebra, A, the resultant involutive quantale is called the *spectrum* of A and denoted $\mathrm{Max}(A)$ in [MP02, KPRR03].)

3.4. Lemma. *Given an arbitrary functor* $\mathbb{A} \xrightarrow{f} \mathbb{B}$*, the (strict) monoidal functor* $\mathcal{F}_{im} f := \mathcal{F}_m(f + f)$ *(explicitly given by* $\mathrm{in}_{b_1} p_1 \cdots \mathrm{in}_{b_n} p_n \mapsto \mathrm{in}_{b_1} f p_1 \cdots \mathrm{in}_{b_n} f p_n$*) satisfies*

$$\overline{(\mathcal{F}_{im} f) w} = (\mathcal{F}_{im} f) \overline{w}$$

[2] More generally—and this is the one place we really regret not having embraced the extra generality discussed in Remark 1.2—there exists a notion of *involutive morphism* of involutive bicategories and, in particular, of *involutive polyad* in an involutive bicategory; just as a polyad in a one-object bicategory corresponds to an enriched category [Bén67], so an involutive polyad in a one-object involutive bicategory should be called an *enriched dagger category*. From this point of view, and involutive monoid might be equally well called a *dagger monoid*.

[3] Formerly, B^**-algebra* was used as a synonym for (abstract) C^*-algebra.

for every $w \in \mathsf{ob}\,\mathcal{F}_{im}\mathbb{A}$*; thus it is a (strict) involutive monoidal functor* $\mathcal{F}_{im}\mathbb{A} \longrightarrow \mathcal{F}_{im}\mathbb{B}$.

Similarly, given an arbitrary natural transformation $\mathbb{A} \overset{f \overset{\lambda}{\Rightarrow} g}{\longrightarrow} \mathbb{B}$*, the monoidal natural transformation* $\mathcal{F}_{im}\lambda := \mathcal{F}_m(\lambda + \lambda)$ *satisfies* $\overline{\lambda_w} = \lambda_{\overline{w}}$ *for every* $w \in \mathsf{ob}\,\mathcal{F}_{im}\mathbb{A}$—i.e., *it is an involutive monoidal natural transformation.*

3.5. Theorem. *There exists a 2-natural transformation,* $\mathrm{Id}_{\mathbf{Cat}} \overset{\mathrm{in}_0}{\longrightarrow} \mathcal{U}_{im}\mathcal{F}_{im}$*, and a pseudo-natural transformation,* $\mathcal{F}_{im}\mathcal{U}_{im} \overset{[\![\ ]\!]}{\longrightarrow} \mathrm{Id}_{\mathbf{StrongInvol}}$*, manifesting* $\mathcal{F}_{im}$ *as left pseudo-adjoint to the forgetful functor* $\mathbf{StrongInvol} \overset{\mathcal{U}_{im}}{\longrightarrow} \mathbf{Cat}$

Proof. Let $\mathfrak{K} = (\mathcal{K}, \otimes, k, \overline{(\ )})$ be an arbitrary involutive monoidal category. The functor $\mathcal{U}_{im}\mathfrak{K} + \mathcal{U}_{im}\mathfrak{K} \overset{[\mathrm{Id}, \overline{(\ )}]}{\longrightarrow} \mathcal{U}_{im}\mathfrak{K} = \mathcal{U}_m\mathcal{U}_i\mathfrak{K}$ determines a monoidal functor $\mathcal{F}_m(\mathcal{U}_{im}\mathfrak{K} + \mathcal{U}_{im}\mathfrak{K}) \longrightarrow \mathcal{U}_i\mathfrak{K}$ explicitly given by the following inductive definition.

$$\begin{aligned} [\![\emptyset]\!] &:= k \\ [\![\mathrm{in}_0 p]\!] &:= p \\ [\![\mathrm{in}_1 p]\!] &:= \overline{p} \\ [\![w \cdot \mathrm{in}_b p]\!] &:= [\![w]\!] \otimes [\![\mathrm{in}_b p]\!] \end{aligned}$$

Let σ be the natural transformation inductively defined as follows.

$$\begin{array}{rcl} \overline{[\![\emptyset]\!]} = \overline{k} & \overset{\mathring{\chi}^{-1}}{\longrightarrow} & k = [\![\emptyset]\!] = [\![\overline{\emptyset}]\!] \\ \overline{[\![\mathrm{in}_0 p]\!]} = \overline{p} & \overset{\mathrm{id}_{\overline{p}}}{\longrightarrow} & \overline{p} = [\![\mathrm{in}_1 p]\!] = [\![\overline{\mathrm{in}_0 p}]\!] \\ \overline{[\![\mathrm{in}_1 p]\!]} = \overline{\overline{p}} & \overset{\varepsilon_p}{\longrightarrow} & p = [\![\mathrm{in}_0 p]\!] = [\![\overline{\mathrm{in}_1 p}]\!] \\ \overline{[\![w \cdot \mathrm{in}_b p]\!]} = \overline{[\![w]\!] \otimes [\![\mathrm{in}_b p]\!]} & \overset{\chi^{-1}_{[\![\mathrm{in}_b p]\!], [\![w]\!]}}{\longrightarrow} & \overline{[\![\mathrm{in}_b p]\!]} \otimes \overline{[\![w]\!]} \\ & \overset{\sigma_{\mathrm{in}_b p} \otimes \sigma_w}{\longrightarrow} & [\![\overline{\mathrm{in}_b p}]\!] \otimes [\![\overline{w}]\!] \\ & \overset{\mu_{\overline{\mathrm{in}_b p}, \overline{w}}}{\longrightarrow} & [\![\overline{\mathrm{in}_b p} \cdot \overline{w}]\!] = [\![\overline{w \cdot \mathrm{in}_b p}]\!] \end{array}$$

It is then a routine inductive argument to show that $[\![\ ]\!]$ and σ form an involutive monoidal functor $\mathcal{F}_{im}\mathcal{U}_{im}\mathfrak{K} \longrightarrow \mathfrak{K}$.

The unit $\mathbb{A} \longrightarrow \mathcal{U}_{im}\mathcal{F}_{im}\mathbb{A}$ is simply given by $p \mapsto \mathrm{in}_0 p$. The remaining details: the triangle identities (which both hold on the nose), the naturality of the unit and the pseudo-naturality of the counit, are left as exercises to the reader. ■

3.6. Corollary. *Every involutive monoidal category* $\mathfrak{K}$ *is equivalent to a strict involutive monoidal category* $\mathfrak{K}_{str}$.

4. Twisted involutive monoidal categories

In this section we address the perceived disconnect, alluded to in the introduction, between the theory of symmetric monoidal categories and involutive monoidal categories with trivial involution. We begin with a class of examples, apparently due to Lack [Sel08], of the phenomenon first observed by Kasangian and Rossi.

4.1. EXAMPLE. Let a be some fixed set, and let $\mathbf{Set} \times \mathbf{Set} \xrightarrow{\oplus_a} \mathbf{Set}$ be defined by

$$p \oplus_a q := p + (p \times a \times q) + q.$$

Then $(\mathbf{Set}, \oplus_a, 0)$ is a monoidal category; moreover, the obvious bijections $p \oplus_a q \xrightarrow{\chi_{p,q}} q \oplus_a p$ form a commutation (in the sense of Bénabou) for $(\mathbf{Set}, \oplus_a, 0)$—but they form a braiding (and therefore a symmetry) if and only if $\#a < 2$.

In order to obtain a positive result (Corollary 4.5), we consider coherent natural isomorphisms $\overline{p} \longrightarrow p$, rather than (possibly unnatural, possibly incoherent) identities. Along the way, we obtain a quite unexpected connection to the theory of balanced braided monoidal categories (Theorem 4.4).

We recall that, given a braiding β for a monoidal category $\mathfrak{K} = (\mathcal{K}, \otimes, k)$, both $(\mathrm{Id}_\mathcal{K}, \beta, \mathrm{id}_k)$ and $(\mathrm{Id}_\mathcal{K}, \beta^{-1}, \mathrm{id}_k)$ define strong monoidal functors $\mathfrak{K}^{\mathsf{rev}} \longrightarrow \mathfrak{K}$. A *balance* for β can be defined as a monoidal natural isomorphism $(\mathrm{Id}_\mathcal{K}, \beta^{-1}, \mathrm{id}_k) \longrightarrow (\mathrm{Id}_\mathcal{K}, \beta, \mathrm{id}_k)$; more concretely, it is a natural transformation $\mathrm{Id}_\mathcal{K} \xrightarrow{\xi} \mathrm{Id}_\mathcal{K}$ such that the diagram

$$\begin{array}{ccc} p \otimes q & \xrightarrow{\beta^{-1}_{q,p}} & q \otimes p \\ {\scriptstyle \xi_p \otimes \xi_q}\downarrow & & \downarrow{\scriptstyle \xi_{q\otimes p}} \\ p \otimes q & \xrightarrow[\beta_{p,q}]{} & q \otimes p \end{array}$$

holds. (It is superfluous to assert $\xi_k = \mathrm{id}_k$ as many authors do. In fact, any semigroupal natural isomorphism between strong monoidal functors is automatically monoidal [EM11b, Lemma A.1].)

A *balanced monoidal category* is a braided monoidal category further equipped with a balance. The graphical calculus for balanced monoidal categories, developed in [JS91], models objects as ribbons in three-dimensional space, components of $\beta^{\pm 1}$ as crossings of ribbons, and components of $\xi^{\pm 1}$ as $\pm 2\pi$-*twists*; components of $\alpha^{\pm 1}$, $\lambda^{\pm 1}$ and $\rho^{\pm 1}$ are rendered invisible—*i.e.*, treated as identities—a step which is justified by the coherence theorem for monoidal categories.

4.2. LEMMA. *Let $(\mathcal{K}, \otimes, k, \overline{(\ )})$ be an involutive monoidal category. For a natural isomorphism $\overline{(\ )} \xrightarrow{\varsigma} \mathrm{Id}_\mathcal{K}$, the following two diagrams are equivalent.*

$$\begin{array}{ccccc}
(\overline{\overline{p}} \otimes \overline{\overline{q}}) \otimes \overline{\overline{r}} & \xrightarrow{(\chi\,;\,\varsigma)\otimes \mathrm{id}} & (\overline{q}\otimes\overline{p})\otimes\overline{\overline{r}} & \xrightarrow{\alpha} & \overline{q}\otimes(\overline{p}\otimes\overline{\overline{r}}) \\
{\scriptstyle (\varsigma\otimes\varsigma)\otimes\varsigma}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}\otimes(\chi\,;\,\varsigma)} \\
(\overline{p}\otimes\overline{q})\otimes\overline{r} & & (T^\ell) & & \overline{q}\otimes(\overline{r}\otimes p) \\
{\scriptstyle (\chi\otimes\mathrm{id})\,;\,\chi\,;\,\varsigma}\downarrow & & & & \downarrow{\scriptstyle \alpha^{-1}} \\
r\otimes(q\otimes p) & \xleftarrow[\alpha]{} & (r\otimes q)\otimes p & \xleftarrow[(\chi\,;\,\varsigma)\otimes\mathrm{id}]{} & (\overline{q}\otimes\overline{r})\otimes p
\end{array}$$

$$\begin{array}{ccccc}
\overline{\overline{p}} \otimes (\overline{\overline{q}} \otimes \overline{\overline{r}}) & \xrightarrow{\mathrm{id}\otimes(\chi\,;\,\varsigma)} & \overline{\overline{p}}\otimes(\overline{r}\otimes\overline{q}) & \xrightarrow{\alpha^{-1}} & (\overline{\overline{p}}\otimes\overline{r})\otimes\overline{q} \\
{\scriptstyle \varsigma\otimes(\varsigma\otimes\varsigma)}\downarrow & & & & \downarrow{\scriptstyle (\chi\,;\,\varsigma)\otimes\mathrm{id}} \\
\overline{p}\otimes(\overline{q}\otimes\overline{r}) & & (T^r) & & (\overline{r}\otimes\overline{p})\otimes\overline{q} \\
{\scriptstyle (\mathrm{id}\otimes\chi)\,;\,\chi\,;\,\varsigma}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\
(r\otimes q)\otimes p & \xleftarrow[\alpha^{-1}]{} & r\otimes(q\otimes p) & \xleftarrow[\mathrm{id}\otimes(\chi\,;\,\varsigma)]{} & r\otimes(\overline{p}\otimes\overline{q})
\end{array}$$

PROOF. Firstly, note that, from the invertibility of ς and the naturality square below,

$$\begin{array}{ccc}
\overline{\overline{\overline{r}}} & \xrightarrow{\varsigma_{\overline{r}}} & \overline{r} \\
{\scriptstyle \overline{\varsigma_r}}\downarrow & & \downarrow{\scriptstyle \varsigma_r} \\
\overline{r} & \xrightarrow[\varsigma_r]{} & r
\end{array}$$

we immediately obtain $\overline{\varsigma_r} = \varsigma_{\overline{r}}$.

Now, because of the coherence theorem for involutive monoidal categories (Corollary 3.6), it suffices to treat only the strict case, where (T^ℓ) and (T^r) reduce to

$$\begin{array}{ccccccccc}
\overline{q}\otimes\overline{p}\otimes\overline{\overline{r}} & \xleftarrow{\varsigma\otimes\mathrm{id}} & \overline{\overline{q}\otimes\overline{p}}\otimes\overline{\overline{r}} & = & \overline{\overline{p}}\otimes\overline{\overline{q}}\otimes\overline{\overline{r}} & = & \overline{\overline{p}}\otimes\overline{\overline{r}\otimes\overline{q}} & \xrightarrow{\mathrm{id}\otimes\varsigma} & \overline{\overline{p}}\otimes\overline{r}\otimes\overline{q} \\
\| & & & & {\scriptstyle \varsigma\otimes\varsigma\otimes\varsigma}\downarrow & & & & \| \\
\| & & & & \overline{p}\otimes\overline{q}\otimes\overline{r} & & & & \| \\
\overline{q}\otimes\overline{\overline{r}\otimes p} & & (T^\ell_{str}) & & \| & & (T^r_{str}) & & \overline{r\otimes\overline{p}}\otimes\overline{q} \\
{\scriptstyle \mathrm{id}\otimes\varsigma}\downarrow & & & & \overline{r\otimes q\otimes p} & & & & \downarrow{\scriptstyle \varsigma\otimes\mathrm{id}} \\
\downarrow & & & & {\scriptstyle \varsigma}\downarrow & & & & \downarrow \\
\overline{q}\otimes\overline{r}\otimes p & = & \overline{r\otimes q}\otimes p & \xrightarrow[\varsigma\otimes\mathrm{id}]{} & r\otimes q\otimes p & \xleftarrow[\mathrm{id}\otimes\varsigma]{} & r\otimes\overline{q\otimes p} & = & r\otimes\overline{p}\otimes\overline{q}
\end{array}$$

respectively.

Conjugating either (T^ℓ_{str}) or (T^r_{str}) and repeatedly applying $\overline{\psi\otimes\omega} = \overline{\omega}\otimes\overline{\psi}$ (which is the naturality of χ) and $\overline{\varsigma_r} = \varsigma_{\overline{r}}$ (discussed above), results in a relabelled version of the other diagram. ∎

4.3. Definitions. *A* twist *for an involutive monoidal category* $(\mathcal{K}, \otimes, k, \overline{(\ )})$ *is a natural isomorphism satisfying either/both of diagrams* (T^ℓ) *and* (T^r). *A twist is called* involutive *if it further satisfies* $\varepsilon_r = \overline{\varsigma_r}\ ;\ \varsigma_r$ *for every object* r. *An* (involutively) twisted involutive monoidal category *is an involutive monoidal category equipped with an (involutive) twist.*

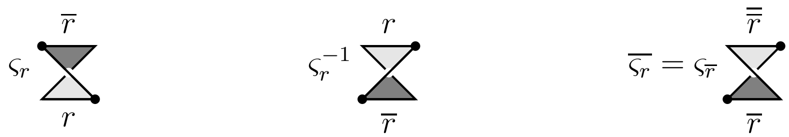
4.4. Theorem. *Given a twisted involutive monoidal category* $(\mathcal{K}, \otimes, k, \overline{(\ )})$, *the composites*

$$p \otimes q \xrightarrow{\ \varsigma_p^{-1} \otimes \varsigma_q^{-1}\ } \overline{p} \otimes \overline{q} \xrightarrow{\ \chi_{p,q}\ } \overline{q \otimes p} \xrightarrow{\ \varsigma_{q \otimes p}\ } q \otimes p$$

$$r \xrightarrow{\ \varepsilon_r^{-1}\ } \overline{\overline{r}} \xrightarrow{\ \varsigma_{\overline{r}}\ } \overline{r} \xrightarrow{\ \varsigma_r\ } r$$

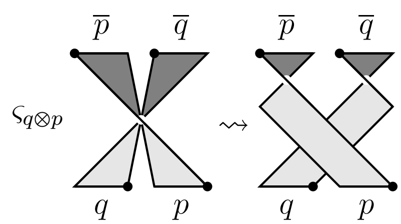
define, respectively, a braiding β *and a balance* ξ.

The proof of Theorem 4.4 is mostly uninteresting, but that does not entail that the result is devoid of meaning! Our intuition relies on extending the graphical calculus of balanced monoidal categories so that the components of $\varsigma^{\pm 1}$ are modelled by $\pm\pi$-twists.



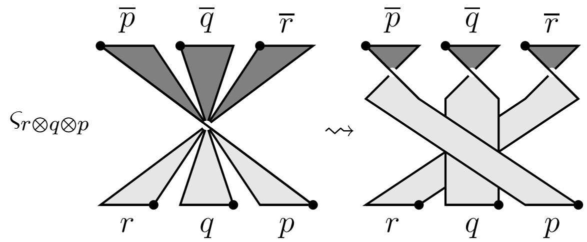
We use a bullet, rather than the more traditional arrowhead, to indicate the direction of a vector, and shading to distinguish one side of a ribbon from the other; $\overline{(\ )}$ is modelled by rotation about an *axis of time* (which, in our case, points down the page), which reverses the orientation of a vector and exposes the other side of a ribbon; components of $\chi^{\pm 1}$ and $\varepsilon^{\pm 1}$ can, in accordance with the coherence theorem for involutive monoidal categories (Corollary 3.6), be made invisible.

In terms of this intuition, the meaning of the axioms contained in Definition 4.3 and of the arrows defined in Theorem 4.4 become clear: a π-twist applied to a parallel pair of ribbons results in a π-twist on each ribbon and a crossing of one over the other.

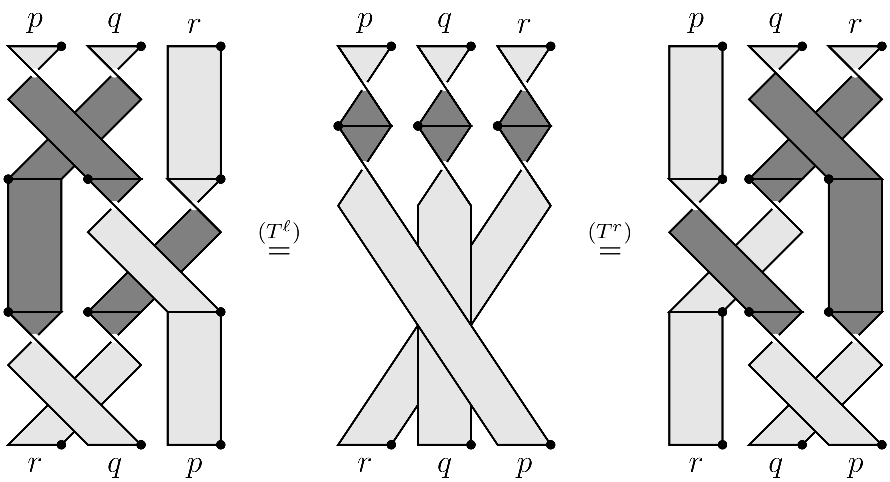


Hence, attaching a $(-\pi)$-twist to each ribbon really ought to result in a braiding; similarly, composing two π-twists really ought to result in a 2π-twist.

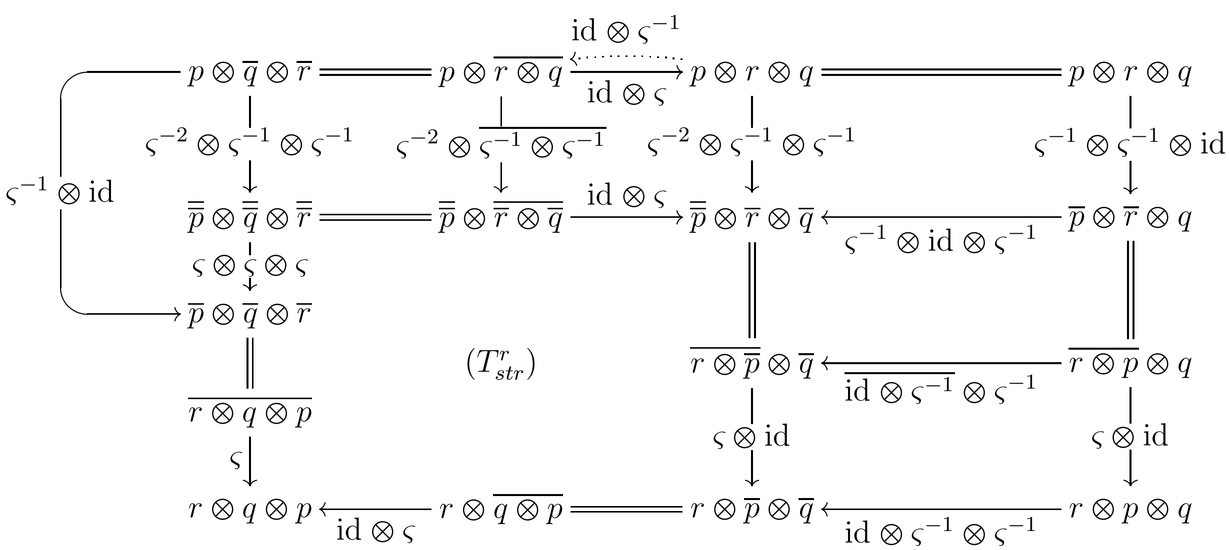
Moreover, a π-twist applied to threesome of parallel ribbons can be similarly decomposed.



Thus the axioms of a twisted involutive monoidal category assert the following truths, whose connection to the axioms for a braiding are self-evident.



Proof of Theorem 4.4 Again, we consider only the strict case. Attaching various naturality squares to (T^r_{str}), we obtain the following diagram



—the perimeter of which is one of the two braiding axioms. The other braiding axiom can be derived from (T^ℓ_{str}) in exactly the same manner.

That ξ is a balance for β can be derived abstractly: having established that β is indeed a braiding, we know that $(\mathrm{Id}_{\mathcal{K}}, \beta, \mathrm{id}_k)$ and $(\overline{(\ )}, \chi, \mathring{\chi})$ both define (strong) monoidal functors $(\mathcal{K}, \otimes, k) \longrightarrow (\mathcal{K}, \otimes^{\mathsf{rev}}, k)$; ς defines a monoidal natural transformation $(\overline{(\ )}, \chi, \mathring{\chi}) \longrightarrow (\mathrm{Id}_{\mathcal{K}}, \beta, \mathrm{id}_k)$—in fact, this is implicit in the definition of β! Now, it follows from general nonsense that ς must have a *mate*,[4] which is a monoidal natural transformation $(\mathrm{Id}_{\mathcal{K}}, \beta^{-1}, \mathrm{id}_k) \longrightarrow (\overline{(\ )}, \chi, \mathring{\chi})$; its components are, in fact, given by $\varepsilon^{-1}\ ;\ \overline{\varsigma}$. Composing the latter with ς therefore results in monoidal natural transformation $(\mathrm{Id}_{\mathcal{K}}, \beta^{-1}, \mathrm{id}_k) \longrightarrow (\mathrm{Id}_{\mathcal{K}}, \beta, \mathrm{id}_k)$, which is exactly what a balance is. ∎

4.5. Corollary. *Given an involutively twisted involutive monoidal category* $(\mathcal{K}, \otimes, k, \overline{(\ )})$, *the braiding* β *defined in Theorem 4.4 is in fact a symmetry.*

4.6. Remark. The notion of *star bar category* which appears in [BM09, Definition 4.5] is similar to that of involutively twisted involutive monoidal category, except that it lacks our axioms (T^ℓ) and (T^r), and of course also (N_2). On the other hand, axioms (T^ℓ) and (T^r) bear a striking resemblance to axiom (A6) of [Sel10], except that the latter is phrased in terms of a contravariant functor, $(\ )^*$, rather than a covariant functor, $\overline{(\ )}$; we shall have more to say about this in Remark 6.8.

If a monoidal category is equipped with both an involution and a symmetry, then it is natural to ask for these to cohere with one another (see diagram (B_+) below); doing so results in a notion (which we call *symmetric involutive monoidal category*) which is equivalent to that of *involutive symmetric monoidal category* contained in [Jac10, Definition 4.1].

But if a monoidal category is equipped with both an involution and a braiding, there are two possible ways of asking these structures to cohere.

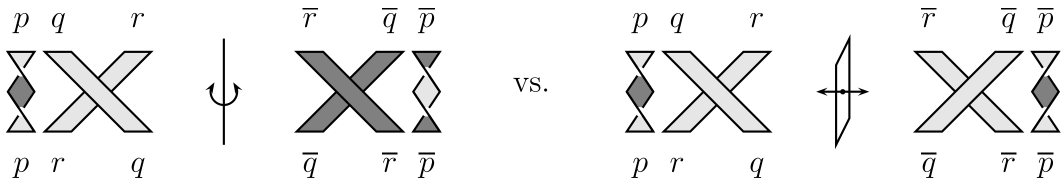
$$
\begin{array}{ccc}
\overline{q \otimes p} & \xrightarrow{\ \chi^{-1}_{p,q}\ } & \overline{p} \otimes \overline{q} \\
{\scriptstyle \overline{\beta_{q,p}}}\downarrow & (B_+) & \downarrow{\scriptstyle \beta_{\overline{p},\overline{q}}} \\
\overline{p \otimes q} & \xleftarrow[\ \chi_{q,p}\ ]{} & \overline{q} \otimes \overline{p}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overline{q \otimes p} & \xrightarrow{\ \chi^{-1}_{p,q}\ } & \overline{p} \otimes \overline{q} \\
{\scriptstyle \overline{\beta_{q,p}}}\downarrow & (B_-) & \downarrow{\scriptstyle \beta^{-1}_{\overline{p},\overline{q}}} \\
\overline{p \otimes q} & \xleftarrow[\ \chi_{q,p}\ ]{} & \overline{q} \otimes \overline{p}
\end{array}
$$

Both cases are considered in [BM09, Definition 4.1], where a braiding for an involutive monoidal category is called: *real*, if it satisfies axiom (B_+); *antireal*, if it satisfies axiom (B_-). It is natural to similarly describe a balance as: *real*, if it satisfies $\overline{\xi_r} = \xi_{\overline{r}}$; *antireal*, if it satisfies $\overline{\xi_r} = \xi^{-1}_{\overline{r}}$.

A strongly related observation is that there are also two natural graphical calculi available to deal with braided involutive monoidal categories: the one we have been using so far, where $\overline{(\ )}$ is modelled by a rotation; and an alternative one in which $\overline{(\ )}$ is modelled by reflection through the plane perpendicular to the axis along which objects are aligned.

[4] Mates can only exist where there are adjunctions: the equivalence of Theorem 2.4 is a special case of an adjunction $(\overline{(\ )}, \chi, \mathring{\chi}) \dashv (\overline{(\ )}, \chi^{\mathsf{rev}}, \mathring{\chi})$; similarly, also $(\mathrm{Id}_{\mathcal{K}}, \beta^{-1}, \mathrm{id}_k) \dashv (\mathrm{Id}_{\mathcal{K}}, \beta^{\mathsf{rev}}, \mathrm{id}_k)$. Technically, therefore, it is ς^{rev} whose mate is being used above, but since this has the same components as ς, we did not bother to make this distinction above.

The illustration below indicates that the former graphical calculus is compatible with the notion of real braidings and balances, and the latter with antireal braidings and balances.



It is no coincidence that our arguments for twisted involutive monoidal categories were developed in one graphical calculus and not the other: as noted in the proof of Lemma 4.2, naturality forces $\overline{\varsigma_r} = \varsigma_{\overline{r}}$, and this is incompatible with the latter graphical calculus—unless, of course, we restrict our attention to involutive twists. It is both an easy exercise and an unsurprising result that the braiding and balance constructed in Theorem 4.4 are both "real". (In particular, an involutively twisted involutive monoidal category is a symmetric involutive monoidal category.)

But the reader is to be warned that subsequent investigations (to be published separately) have led the author to conclude that antireal braidings and balances are of much greater importance than their real counterparts. Therefore, the graphical calculus used prior to this remark also seems to us, now, of less importance than its rival.

5. Involutive star-autonomous categories

We recall that a *star-autonomous category* [Bar95] can be equivalently defined, either as a closed monoidal category with a dualising object, or as an *linearly distributive category* with (chosen) left- and right- duals for every object [CS97b]. [But, as noted in the Introduction, we prefer the term *ambimonoidal* over the phrase *linearly distributive*; they are synonyms.] Correspondingly, one expects that an *involutive star-autonomous category* can be equivalently defined, either as a closed involutive monoidal category with an *involutive dualising object*, or as an *involutive ambimonoidal category* with (chosen) duals.

We begin with the latter approach: just as an involutive monoidal category is a certain kind of equivalence internal to the bicategory **Lax**, so an involutive ambimonoidal category should be the same kind of equivalence internal to the bicategory **Frob**, which consists of ambimonoidal categories, *Frobenius functors* [Egg10] and morphisms thereof. The practical import of this intuition is spelt out below.

5.1. Definition. *An* involutive ambimonoidal category *is an ambimonoidal category* $(\mathcal{K}, \boxtimes, e, \boxtimes, d)$ *together with a functor* $\overline{(\ )}$ *and natural isomorphisms*

$$\overline{p} \boxtimes \overline{q} \xrightarrow{\hat{\chi}_{p,q}} \overline{q \boxtimes p} \qquad \overline{p \boxtimes q} \xrightarrow{\check{\chi}_{p,q}} \overline{q} \boxtimes \overline{p}$$

$$\overline{\overline{r}} \xrightarrow{\varepsilon_r} r$$

such that both $(\mathcal{K}, \owedge, e, \hat{\chi}, \varepsilon)$ *and* $(\mathcal{K}, \ovee, d, \check{\chi}^{-1}, \varepsilon)$ *are involutive monoidal categories, and such that the diagrams*

$$\begin{array}{ccccc}
\overline{p \owedge (q \ovee r)} & \xleftarrow{\hat{\chi}_{q \ovee r, p}} & \overline{q \ovee r} \owedge \overline{p} & \xrightarrow{\check{\chi}_{q,r} \owedge \mathrm{id}_{\overline{p}}} & (\overline{r} \ovee \overline{q}) \owedge \overline{p} \\
{\scriptstyle \overline{\kappa_{p,q,r}}}\downarrow & & (D_1) & & \downarrow{\scriptstyle \kappa_{\overline{r},\overline{q},\overline{p}}} \\
\overline{(p \owedge q) \ovee r} & \xrightarrow[\check{\chi}_{p \owedge q, r}]{} & \overline{r} \ovee \overline{p \owedge q} & \xleftarrow[\mathrm{id}_{\overline{r}} \ovee \hat{\chi}_{\overline{q},\overline{p}}]{} & \overline{r} \ovee (\overline{q} \owedge \overline{p})
\end{array}$$

$$\begin{array}{ccccc}
\overline{p} \owedge (\overline{q} \ovee \overline{r}) & \xleftarrow{\mathrm{id}_{\overline{p}} \owedge \check{\chi}_{q,r}} & \overline{p} \owedge \overline{r \ovee q} & \xrightarrow{\hat{\chi}_{p, r \ovee q}} & \overline{(r \ovee q) \owedge p} \\
{\scriptstyle \kappa_{\overline{p},\overline{q},\overline{r}}}\downarrow & & (D_2) & & \downarrow{\scriptstyle \overline{\kappa_{r,q,p}}} \\
(\overline{p} \owedge \overline{q}) \ovee \overline{r} & \xrightarrow[\hat{\chi}_{\overline{q},\overline{p}} \ovee \mathrm{id}_{\overline{r}}]{} & \overline{q \owedge p} \ovee \overline{r} & \xleftarrow[\check{\chi}_{r, q \owedge p}]{} & \overline{r \ovee (q \owedge p)}
\end{array}$$

hold.

By Lemma 2.3, this entails the existence of canonical isomorphisms $e \longrightarrow \overline{e}$ and $\overline{d} \longrightarrow d$; these will be denoted $\mathring{\hat{\chi}}$ and $\mathring{\check{\chi}}$, respectively.

5.2. Lemma. *If* $p \owedge q \xrightarrow{\gamma} d$, $e \xrightarrow{\tau} q \ovee p$ *constitute a (linear) adjunction in an involutive ambimonoidal category, then so do*

$$\overline{q} \owedge \overline{p} \xrightarrow{\hat{\chi}_{p,q}} \overline{p \owedge q} \xrightarrow{\overline{\gamma}} \overline{d} \xrightarrow{\mathring{\check{\chi}}} d,$$

$$e \xrightarrow{\mathring{\hat{\chi}}} \overline{e} \xrightarrow{\overline{\tau}} \overline{q \ovee p} \xrightarrow{\check{\chi}_{q,p}} \overline{p} \ovee \overline{q}.$$

Hence, if every object p *has a chosen right dual,* $p^{\perp}$*, then one may choose* ${}^{\perp}p := \overline{\overline{p}^{\perp}}$ *(together with arrows as above) as a left dual for every* p.

Proof. This is, in fact, a special case of a theorem about Frobenius functors; see [Egg10]. ■

Conversely, if every object q has a chosen left dual, ${}^{\perp}q$, then one may choose $q^{\perp} := \overline{{}^{\perp}\overline{q}}$ (together with arrows as above) as a right dual for every q. In general, we find it more practical to assume that every object r has independently chosen left and right duals; then the lemma simply entails the existence of canonical isomorphisms $\overline{r^{\perp}} \longrightarrow {}^{\perp}\overline{r}$ and $\overline{r}^{\perp} \longrightarrow \overline{{}^{\perp}r}$, which we denote $\hat{\sigma}_r$ and $\check{\sigma}_r$, respectively.

This approach is particularly useful when considering *cyclicity*, as we shall do below and in section 6. We refer the reader to [EM11b] for a comprehensive treatment of cyclic star-autonomous categories; a *cycle* is there defined to be a natural isomorphism $r^{\perp} \xrightarrow{\varphi_r} {}^{\perp}r$ satisfying some coherence axioms, and a *cyclic star-autonomous category* to be a star-autonomous category equipped with a cycle. This definition proves to be equivalent to earlier one of [BLR02], which is phrased in terms of a different set of data.

5.3. Definitions. *An* involutive star-autonomous category *is an involutive ambimonoidal category together with chosen left- and right- duals for every object r.*

A cyclic involutive star-autonomous category *is an involutive star-autonomous category together with a cycle $r^{\perp} \xrightarrow{\varphi_r} {}^{\perp}r$ which satisfies the diagram below.*

$$\begin{array}{ccc} \overline{p}^{\perp} & \xrightarrow{\quad\varphi_{\overline{p}}\quad} & {}^{\perp}\overline{p} \\ {\scriptstyle \check{\sigma}_p}\big\downarrow & (C) & \big\uparrow{\scriptstyle \hat{\sigma}_p} \\ \overline{{}^{\perp}p} & \xrightarrow[\quad\overline{\varphi_p^{-1}}\quad]{} & \overline{p^{\perp}} \end{array}$$

A fuller study of the interaction between cyclicity and involutivity will be undertaken in [EM11a]. We now proceed to explore the alternative, and in some ways more conventional, approach to defining (involutive) star-autonomous categories.

5.4. Lemma. *An involutive monoidal category $(\mathcal{K}, \otimes, e, \overline{(\ )})$ is left closed if and only if it is right closed.*

Proof. Given a left closed structure $\multimap$, then $r \multimapinv q := \overline{\overline{q} \multimap \overline{r}}$ is a right closed structure. Dually, given a right closed structure $\multimapinv$, then $p \multimap r := \overline{\overline{r} \multimapinv \overline{p}}$ is a left closed structure. ■

But, as before, we find it more convenient to define a *closed involutive monoidal category* to be an involutive monoidal category with chosen left and right closed structures; by uniqueness of adjoints one obtains $p \multimap r \xrightarrow{\sim} \overline{\overline{r} \multimapinv \overline{p}}$ and $r \multimapinv q \xrightarrow{\sim} \overline{\overline{q} \multimap \overline{r}}$; equivalently, $\overline{p \multimap r} \xrightarrow{\sim} \overline{r} \multimapinv \overline{p}$ and $\overline{r \multimapinv q} \xrightarrow{\sim} \overline{q} \multimap \overline{r}$.

5.5. Definitions. *Given an (arbitrary) involutive monoidal category, an* involutive object *is a pair (d, ς) where d is an object and ς is an arrow $\overline{d} \longrightarrow d$ satisfying $\varepsilon_d = \overline{\varsigma} \,;\, \varsigma$.*

Given a closed involutive monoidal category, an involutive dualising object *is an involutive object (d, ς) such that d is dualising*—i.e.*, such that the natural transformation $p \longrightarrow d \multimapinv (p \multimap d)$ is invertible.*

5.6. Theorem. *An involutive star-autonomous category is the same thing as a closed involutive monoidal category together with an involutive dualising object.*

Proof. Given an involutive star-autonomous category (in the sense of Definition 5.5 above), one induces (left and right) closed structures by setting $p \multimap r := p^{\perp} \parr r$ and $r \multimapinv q := r \parr {}^{\perp}q$ as per usual; moreover, one chooses $\varsigma = \mathring{\check{\chi}}$.

Conversely, given a closed involutive monoidal category and an involutive dualising object (d, ς), one sets $p^{\perp} := p \multimap d$, ${}^{\perp}q := d \multimapinv q$ and $p \parr q$ to be any one of the following objects

$${}^{\perp}(q^{\perp} \otimes p^{\perp}) \cong p \multimapinv (q^{\perp}) \cong ({}^{\perp}p) \multimap q \cong ({}^{\perp}q \otimes {}^{\perp}p)^{\perp}$$

(as per usual). We then set $\check{\chi}_{p,q}$ to equal

$$\overline{p \parr q} \xrightarrow{\sim} \overline{({}^{\perp}p) \multimap q} \xrightarrow{\quad\sim\quad} \overline{q} \multimapinv \overline{{}^{\perp}p} \xrightarrow{\mathrm{id}_{\overline{q}} \multimapinv \check{\sigma}_p} \overline{q} \multimapinv \overline{p}^{\perp} \xrightarrow{\sim} \overline{q} \parr \overline{p}$$

where $\breve{\sigma}_p$ abbreviates $\overline{p}^{\perp} := \overline{p} \multimap d \xrightarrow{\mathrm{id}_{\overline{p}} \multimap (\varsigma^{-1})} \overline{p} \multimap \overline{\overline{d}} \xrightarrow{\sim} \overline{d \multimapinv p} =: \overline{{}^{\perp}p}$. It remains to show that $\breve{\chi}$ satisfies the necessary coherence conditions; but this is routine. ∎

5.7. Remark. What we call involutive objects in Definition 5.5, are called $\star$-*objects* in [BM09, Definition 2.10] and *self-conjugates* in [Jac10, Definition 3.1]; [BM09, Proposition 6.2] can be considered as a special case of our Lemma 5.2.

6. Dagger pivotal categories

We recall that a *dagger structure* on a category $\mathcal{K}$ is a contravariant (strict) involution $(\ )^{\dagger} : \mathcal{K}^{\mathsf{op}} \longrightarrow \mathcal{K}$ which is also identity on objects—that is, we should have $p^{\dagger} = p$ for every object p, and $\omega^{\dagger\dagger} = \omega$ for every arrow ω. A *pivotal category* can be defined as a monoidal category $(\mathcal{K}, \otimes, k)$ such that every object p is equipped with a right (or, equivalently, left) adjoint, $\widetilde{p}$, together with monoidal natural isomorphism $r \xrightarrow{\zeta_r} \widetilde{\widetilde{r}}$.

A *dagger pivotal category* [Sel09] is a pivotal category together with a compatible dagger structure. Here, compatibility means that: $(\ )^{\dagger}$ should commute with $\otimes$ and $\widetilde{(\ )}$—that is, $(\omega \otimes \psi)^{\dagger} = \omega^{\dagger} \otimes \psi^{\dagger}$ and $\widetilde{\omega}^{\dagger} = \widetilde{\omega^{\dagger}}$ should hold for all arrows ψ and ω; and all of the structural isomorphisms—including the de Morgan laws $\widetilde{p} \otimes \widetilde{q} \xrightarrow{\delta_{p,q}} \widetilde{q \otimes p}$ as well as the more obvious associativity and unit laws—should be unitary. (An arrow ω in a dagger category is called *unitary* if it is invertible with $\omega^{\dagger} = \omega^{-1}$.)

6.1. Lemma. *Every dagger pivotal category* $(\mathcal{K}, \otimes, k, \widetilde{(\ )}, (\ )^{\dagger})$ *has an underlying cyclic involutive star-autonomous structure.*

Proof. It is well-known that every pivotal category gives rise to a cyclic star-autonomous category, obtained by setting $\boxslash := \otimes =: \boxbslash$, $(\ )^{\perp} := \widetilde{(\ )} =: {}^{\perp}(\ )$, and so on; all that needs to be done is to describe a suitable involutive structure.

We set $\overline{(\ )} := \widetilde{(\ )}^{\dagger}$. Since $\overline{p} = \widetilde{p}$ as objects, we can choose $\chi = \delta$ and $\varepsilon = \zeta^{-1}$.

$$\begin{array}{ccc} \overline{p} \otimes \overline{q} & \xrightarrow{\chi_{p,q}} & \overline{q \otimes p} \\ \| & & \| \\ \widetilde{p} \otimes \widetilde{q} & \xrightarrow{\delta_{p,q}} & \widetilde{q \otimes p} \end{array} \qquad\qquad \begin{array}{ccc} \overline{\overline{r}} & \xrightarrow{\varepsilon_r} & r \\ \| & & \| \\ \widetilde{\widetilde{r}} & \xrightarrow{\zeta_r^{-1}} & r \end{array}$$

Then axioms (F_3) and (N_2) are equivalent to

$$\begin{array}{ccc} (\widetilde{p}\otimes\widetilde{q})\otimes\widetilde{r} & \xrightarrow{\alpha_{\widetilde{p},\widetilde{q},\widetilde{r}}} & \widetilde{p}\otimes(\widetilde{q}\otimes\widetilde{r}) \\ {\scriptstyle \delta_{p,q}\otimes \mathrm{id}_{\widetilde{r}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\widetilde{p}}\otimes\delta_{q,r}} \\ \widetilde{q\otimes p}\otimes\widetilde{r} & & \widetilde{p}\otimes\widetilde{r\otimes q} \\ {\scriptstyle \delta_{q\otimes p,r}}\downarrow & & \downarrow{\scriptstyle \delta_{p,r\otimes q}} \\ \widetilde{p\otimes(q\otimes r)} & \xleftarrow[\widetilde{\alpha^{\dagger}_{r,q,p}}]{} & \widetilde{(r\otimes q)\otimes p} \end{array} \qquad \begin{array}{ccc} \widetilde{\widetilde{p}}\otimes\widetilde{\widetilde{q}} & \xrightarrow{\delta_{\widetilde{p},\widetilde{q}}} & \widetilde{\widetilde{q}\otimes\widetilde{p}} \\ {\scriptstyle \zeta_p\otimes\zeta_q}\uparrow & & \downarrow{\scriptstyle \widetilde{\delta^{\dagger}_{q,p}}} \\ p\otimes q & \xrightarrow{\zeta_{p\otimes q}} & \widetilde{\widetilde{p\otimes q}} \end{array}$$

respectively, and axiom (A) amounts to

$$\widetilde{(\zeta_r^{-1})^{\dagger}} = \zeta_{\widetilde{r}}^{-1}.$$

All of these certainly hold in a dagger pivotal category.

Finally, (C) holds because both $\hat{\sigma}_p$ and $\check{\sigma}_p$ turn out to be the identity on $\widetilde{\widetilde{p}}$. ∎

Now suppose we have an involutive star-autonomous category with a fortuitous isomorphism $p \xrightarrow{\sim} \overline{p}^{\perp}$ for every object p; then one can derive isomorphisms

$$e \xrightarrow{\sim} \overline{e}^{\perp} \xrightarrow{\sim} e^{\perp} \xrightarrow{\sim} d$$

$$p \boxtimes q \xrightarrow{\sim} (\overline{p \boxtimes q})^{\perp} \xrightarrow{\sim} (\overline{q} \boxtimes \overline{p})^{\perp} \xrightarrow{\sim} \overline{p}^{\perp} \uplus \overline{q}^{\perp} \xrightarrow{\sim} p \uplus q$$

$$p^{\perp} \xrightarrow{\sim} \overline{\overline{p}}^{\perp} \xrightarrow{\sim} \overline{p} \xrightarrow{\sim} \overline{\overline{p}^{\perp}} \xrightarrow{\sim} \overline{\overline{{}^{\perp}p}} \xrightarrow{\sim} {}^{\perp}p$$

and one also obtains a contravariant functor

$$(p \xrightarrow{\omega} q) \qquad \mapsto \qquad (q \xrightarrow{\sim} \overline{q}^{\perp} \xrightarrow{\omega^{\dagger} := \overline{\omega}^{\perp}} \overline{p}^{\perp} \xrightarrow{\sim} p)$$

which is the identity on objects. Thus we arrive at something that looks like a dagger pivotal category; to ensure that it is a dagger pivotal category, one must add some axioms. Note that any arrow $p \longrightarrow \overline{p}^{\perp}$ can be deCurryed into a *sesquilinear form* $\overline{p} \boxtimes p \longrightarrow d$.

6.2. Definitions. *Let (d, ς) be an involutive object in some involutive monoidal category $\mathcal{K} = (\mathcal{K}, \boxtimes, e, \overline{(\ )})$; then a (d-valued) sesquilinear form $\overline{p} \boxtimes p \xrightarrow{\gamma} d$ is called (ς-)*Hermitian *if the diagram*

$$\begin{array}{ccccc} \overline{p}\boxtimes\overline{\overline{p}} & \xrightarrow{\chi_{p,\overline{p}}} & \overline{\overline{p}\boxtimes p} & \xrightarrow{\overline{\gamma_p}} & \overline{d} \\ {\scriptstyle \mathrm{id}_{\overline{p}}\boxtimes\varepsilon_p}\downarrow & & & & \downarrow{\scriptstyle \varsigma} \\ \overline{p}\boxtimes p & & \xrightarrow{\gamma_p} & & d \end{array}$$

holds. A Hermitian system *for* $(\mathcal{K}, d, \varsigma)$ *is an* $\mathsf{ob}\,\mathcal{K}$*-indexed family of Hermitian forms* $\overline{p} \otimes p \xrightarrow{\gamma_p} d$*; in other words, it is a not-necessarily-natural transformation* $\overline{(\ )} \otimes (\ ) \longrightarrow d$*.*

In the case where $\mathcal{K}$ *is closed, we call a Hermitian form* $\overline{p} \otimes p \xrightarrow{\gamma} d$ exact *if its Curry,* $p \xrightarrow{\upsilon} (\overline{p} \multimap d)$*, is invertible; a Hermitian system is called exact if each of its components has this property.*

Of course, the main case of interest is when $\mathcal{K}$ is closed and (d, ς) dualising. In this case, an exact Hermitian system also induces isomorphisms $\overline{p} \xrightarrow{\pi_p} {}^{\perp}p$; it will be convenient to abbreviate the composite

$$\overline{q} \xrightarrow{\ \pi_q\ } {}^{\perp}q \xrightarrow{\ {}^{\perp}\omega\ } {}^{\perp}p \xrightarrow{\ \pi_p^{-1}\ } \overline{p}$$

to $\widetilde{\omega}$, and to set $\widetilde{p} = \overline{p}$.

6.3. Lemma. *Given an exact Hermitian system on an involutive star-autonomous category* $\mathcal{K}$*,*

$$\begin{array}{ccc} p & & \\ \big\downarrow{\scriptstyle \omega} & \mapsto & \\ q & & \end{array} \qquad \begin{array}{ccc} p & \xleftarrow{\ \upsilon_p^{-1}\ } & \overline{p}^{\perp} \\ {\scriptstyle \omega^{\dagger}}\big\uparrow & := & \big\uparrow{\scriptstyle \overline{\omega}^{\perp}} \\ q & \xrightarrow[\ \upsilon_q\ ]{} & \overline{q}^{\perp} \end{array}$$

defines a dagger structure for $\mathcal{K}$*.*

Proof. The functoriality of $(\ )^{\dagger}$ is trivial, despite the unnaturality of υ_p. All that remains to be shown is the involutivity property: $\omega^{\dagger\dagger} = \omega$ for all ω.

By standard categorical yoga, Hermitianness is equivalent to the square in the diagram below (the triangle is a tautology for any star-autonomous category)

$$\begin{array}{ccccc} p & \xrightarrow{\ \eta_p\ } & ({}^{\perp}p)^{\perp} & \xrightarrow{\ \varepsilon_{{}^{\perp}p}{}^{\perp}\ } & \overline{\overline{{}^{\perp}p}}^{\perp} \\ & \searrow{\scriptstyle \upsilon_p} & \big\downarrow{\scriptstyle \pi_p{}^{\perp}} & & \big\downarrow{\scriptstyle \overline{\breve{\sigma}_p}^{\perp}} \\ & & \overline{p}^{\perp} & \xleftarrow[\ \overline{\upsilon_p}^{\perp}\ ]{} & \overline{\overline{p}^{\perp}}^{\perp} \end{array}$$

—hence,

$$\begin{array}{ccccccccccc} p & \xrightarrow{\ \upsilon_p\ } & \overline{p}^{\perp} & \xrightarrow{\ \overline{\upsilon_p^{-1}}^{\perp}\ } & \overline{\overline{p}^{\perp}}^{\perp} & \xleftarrow{\ \overline{\breve{\sigma}_p}^{\perp}\ } & \overline{\overline{{}^{\perp}p}}^{\perp} & \xleftarrow{\ \varepsilon_{{}^{\perp}p}{}^{\perp}\ } & ({}^{\perp}p)^{\perp} & \xleftarrow{\ \eta_p\ } & p \\ \big\downarrow{\scriptstyle \omega^{\dagger\dagger}} & := & \big\downarrow{\scriptstyle \overline{\omega^{\dagger}}^{\perp}} & := & \big\downarrow{\scriptstyle \overline{\overline{\omega}^{\perp}}^{\perp}} & & \big\downarrow{\scriptstyle \overline{\overline{{}^{\perp}\omega}}^{\perp}} & & \big\downarrow{\scriptstyle ({}^{\perp}\omega)^{\perp}} & & \big\downarrow{\scriptstyle \omega} \\ q & \xleftarrow{\ \upsilon_q^{-1}\ } & \overline{q}^{\perp} & \xleftarrow{\ \overline{\upsilon_q}^{\perp}\ } & \overline{\overline{q}^{\perp}}^{\perp} & \xleftarrow{\ \overline{\breve{\sigma}_q}^{\perp}\ } & \overline{\overline{{}^{\perp}q}}^{\perp} & \xleftarrow{\ \varepsilon_{{}^{\perp}q}{}^{\perp}\ } & ({}^{\perp}q)^{\perp} & \xleftarrow{\ \eta_q\ } & q \end{array}$$

as desired. ■

Note that the Hermitianness axiom is slightly stronger than needed: it would suffice to know that the composite arrows

$$p \xrightarrow{\ \upsilon_p\ } \overline{p}^{\perp} \xrightarrow{\ \overline{\upsilon_p^{-1}}^{\perp}\ } \overline{\overline{p}^{\perp}}^{\perp} \xrightarrow{\ \overline{\check{\sigma}_p^{-1}}^{\perp}\ } \overline{\overline{{}^{\perp}p}}^{\perp} \xrightarrow{\ (\varepsilon_{{}^{\perp}p}^{-1})^{\perp}\ } ({}^{\perp}p)^{\perp} \xrightarrow{\ \eta_p^{-1}\ } p$$

form the components of a natural endomorphism of the identity functor. Indeed, the lemma could be sharpened to an "if and only if", were we willing to take this into account; but it is unclear whether the extra generality is of any interest.

Note also that the diagram

$$\begin{array}{ccc} \overline{p} \owedge p & \xrightarrow{\ \gamma_p\ } & d \\ {\scriptstyle \overline{\omega} \owedge \omega}\downarrow & & \| \\ \overline{q} \owedge q & \xrightarrow{\ \gamma_q\ } & d \end{array}$$

is equivalent to

$$\begin{array}{ccc} p & \xrightarrow{\ \upsilon_p\ } & \overline{p}^{\perp} \\ {\scriptstyle \omega}\downarrow & & \uparrow{\scriptstyle \overline{\omega}^{\perp}} \\ q & \xrightarrow{\ \upsilon_q\ } & \overline{q}^{\perp} \end{array}$$

and hence to $\omega\ ;\ \omega^{\dagger} = \mathrm{id}_p$. But the latter does not hold for every arrow of an arbitrary dagger pivotal category—*e.g.*, the category of finite-dimensional Hilbert spaces and arbitrary linear maps, $\mathbf{Hilb}_{fd}$ [Sel09]; this is why we do not assume γ to be a natural transformation.

6.4. Lemma. *Given a exact Hermitian system on an involutive star-autonomous category $\mathcal{K}$, the equation $(\psi \owedge \omega)^{\dagger} = \psi^{\dagger} \owedge \omega^{\dagger}$ holds if and only if the composites*

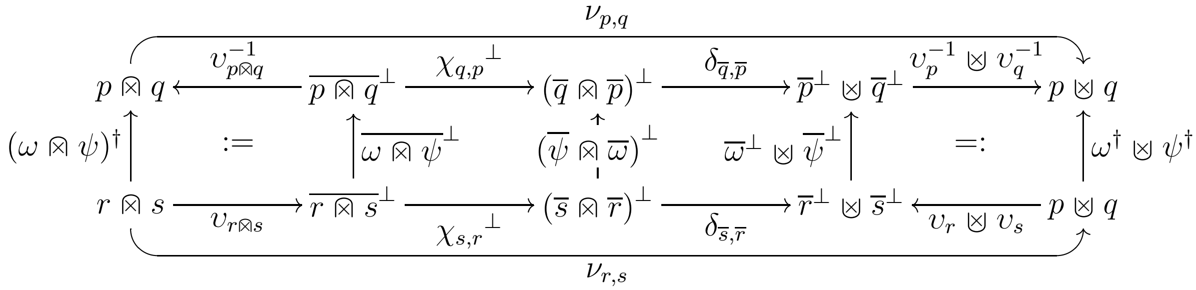
$$p \owedge q \xrightarrow{\ \upsilon_{p \owedge q}\ } (\overline{p \owedge q})^{\perp} \xrightarrow{\ \chi_{q,p}{}^{\perp}\ } (\overline{q} \owedge \overline{p})^{\perp} \xrightarrow{\ \delta_{\overline{q},\overline{p}}\ } \overline{p}^{\perp} \ovee \overline{q}^{\perp} \xrightarrow{\ \upsilon_p^{-1} \ovee \upsilon_q^{-1}\ } p \ovee q$$

(henceforth denoted $\nu_{p,q}$) form the components of a natural transformation $\owedge \longrightarrow \ovee$, and the equation $\widetilde{\omega}^{\dagger} = \widetilde{\omega^{\dagger}}$ holds if and only if the composites

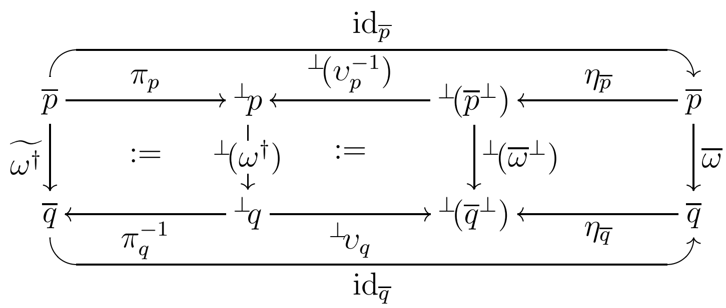
$$r^{\perp} \xrightarrow{\ \varepsilon_r{}^{\perp}\ } \overline{\overline{r}}^{\perp} \xrightarrow{\ \upsilon_{\overline{r}}^{-1}\ } \overline{r} \xrightarrow{\ \pi_r\ } {}^{\perp}r$$

(henceforth denoted ϑ_r) form the components of a natural transformation $(\ )^{\perp} \longrightarrow {}^{\perp}(\ )$.

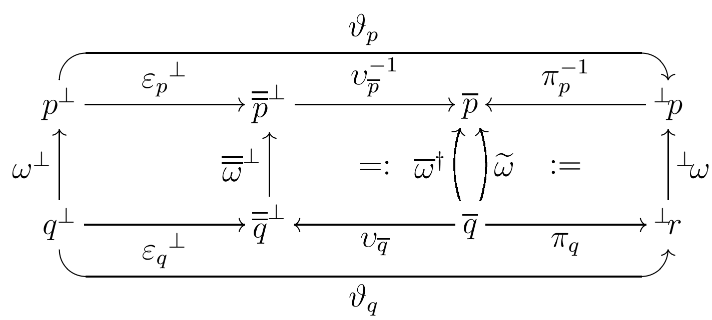
Proof. The naturality of χ and δ, together with the definition of $(\ )^\dagger$ entail that



—hence the assertion $(\omega \owedge \psi)^\dagger = \omega^\dagger \owedge \psi^\dagger$ is equivalent to the naturality of ν relative to all pairs of maps of the form $\omega^\dagger, \psi^\dagger$. But, by the preceding lemma, every pair of arrows is of this form. Moreover, it is immediate that $\widetilde{\omega^\dagger} = \overline{\omega}$:



—hence, the equation $\widetilde{\omega}^\dagger = \widetilde{\omega^\dagger}$ is equivalent to $\widetilde{\omega}^\dagger = \overline{\omega}$, and therefore also to $\widetilde{\omega} = \overline{\omega}^\dagger$. The diagram



shows that the naturality of ϑ is equivalent to the equation $\overline{\omega}^\dagger = \widetilde{\omega}$, as desired. ■

6.5. Lemma. *Given an exact Hermitian system on an involutive star-autonomous category $\mathcal{K}$ satisfying the equivalent conditions of the preceding lemma, the associativity isomorphism $(p \owedge q) \owedge r \xrightarrow{\alpha_{p,q,r}} p \owedge (q \owedge r)$ is unitary if and only if $\owedge \xrightarrow{\nu} \ovee$ satisfies the* *coherence axiom*

$$\begin{array}{ccccc}
(p \boxtimes q) \boxtimes r & \xrightarrow{\nu_{p \boxtimes q, r}} & (p \boxtimes q) \circledvee r & \xrightarrow{\nu_{p,q} \circledvee \mathrm{id}_r} & (p \circledvee q) \circledvee r \\
{\scriptstyle \alpha_{p,q,r}} \downarrow & & & & \downarrow {\scriptstyle \alpha_{p,q,r}} \\
p \boxtimes (q \boxtimes r) & \xrightarrow[\nu_{p, q \boxtimes r}]{} & p \circledvee (q \boxtimes r) & \xrightarrow[\mathrm{id}_p \circledvee \nu_{q,r}]{} & p \circledvee (q \circledvee r).
\end{array}$$

Similarly, the unit isomorphisms $e \boxtimes q \xrightarrow{\lambda_q} q$ *and* $q \boxtimes e \xrightarrow{\rho_q} q$ *are unitary if and only if* $\boxtimes \xrightarrow{\nu} \circledvee$ *and the arrow*

$$e \xrightarrow{\upsilon_e} \overline{e}^{\perp} \xrightarrow{\hat{\mathring{\chi}}^{\perp}} e^{\perp} \xrightarrow{\mathring{\delta}} d$$

(henceforth denoted $e \xrightarrow{\mathring{\nu}} d$*) satisfy the coherence axioms*

$$\begin{array}{ccc}
e \boxtimes q & \xrightarrow{\mathring{\nu} \boxtimes \mathrm{id}_q} & d \boxtimes q \\
{\scriptstyle \lambda_q} \downarrow & & \downarrow {\scriptstyle \nu_{d,q}} \\
q & \xleftarrow[\lambda_q]{} & d \circledvee q
\end{array}
\qquad\qquad
\begin{array}{ccc}
q \boxtimes e & \xrightarrow{\mathrm{id}_q \boxtimes \mathring{\nu}} & q \boxtimes d \\
{\scriptstyle \rho_q} \downarrow & & \downarrow {\scriptstyle \nu_{q,d}} \\
q & \xleftarrow[\rho_q]{} & q \circledvee d
\end{array}$$

respectively.

Furthermore, assuming the above holds, $\overline{p} \boxtimes \overline{q} \xrightarrow{\chi_{p,q}} \overline{q \boxtimes p}$ *is unitary if and only if* $(\ )^{\perp} \xrightarrow{\vartheta} {}^{\perp}(\ )$ *satisfies the coherence axiom*

$$\begin{array}{ccc}
(p \boxtimes q)^{\perp} & \xrightarrow{\vartheta_{p \boxtimes q}} & {}^{\perp}(p \boxtimes q) \\
{\scriptstyle \delta_{p,q}} \downarrow & & \downarrow {\scriptstyle \delta_{p,q}} \\
q^{\perp} \circledvee p^{\perp} & \xrightarrow{\vartheta_p \circledvee \vartheta_q} & {}^{\perp}q \circledvee {}^{\perp}p.
\end{array}$$

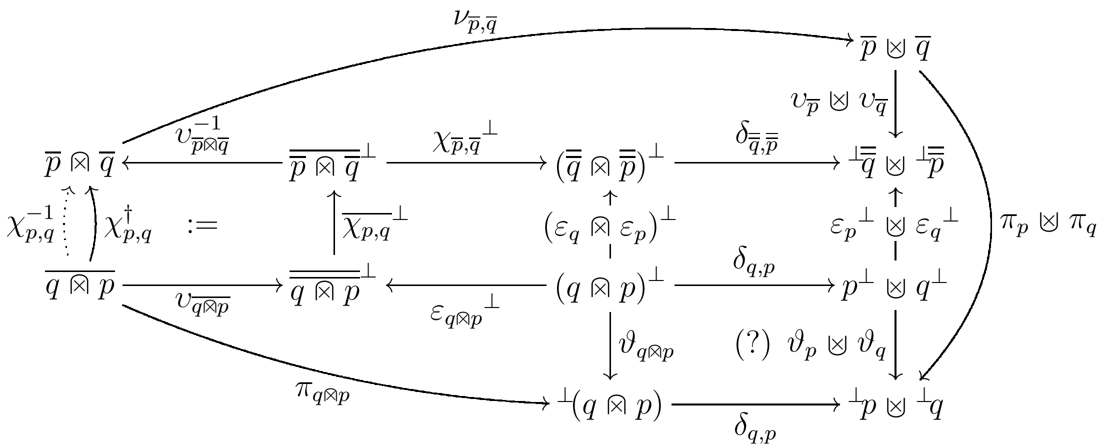
Proof. It is a simple exercise to show that

$$\begin{array}{ccccc}
(p \boxtimes q) \boxtimes r & \xrightarrow{\nu_{p \boxtimes q, r}} & (p \boxtimes q) \circledvee r & \xrightarrow{\nu_{p,q} \circledvee \mathrm{id}_r} & (p \circledvee q) \circledvee r \\
\uparrow {\scriptstyle \alpha^{\dagger}_{p,q,r}} & & & & {\scriptstyle \alpha^{-1}_{p,q,r}} \uparrow \\
p \boxtimes (q \boxtimes r) & \xrightarrow[\nu_{p, q \boxtimes r}]{} & p \circledvee (q \boxtimes r) & \xrightarrow[\mathrm{id}_p \circledvee \nu_{q,r}]{} & p \circledvee (q \circledvee r)
\end{array}$$

commutes; hence, the assertion $\alpha^{\dagger}_{p,q,r} = \alpha^{-1}_{p,q,r}$ is equivalent to the first coherence axiom.

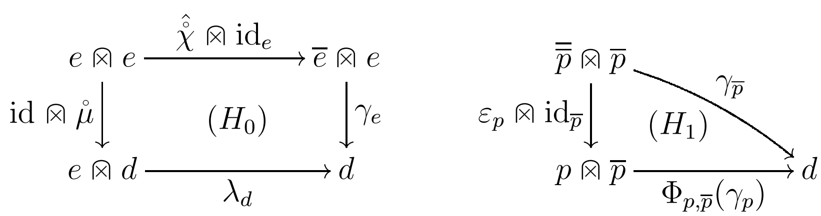
The case of the two unit laws, λ and ρ, is similar.

The last statement is a bit trickier: since the diagram below consists exclusively of invertible arrows, the commutativity of (?) is equivalent to that of the perimeter of the solid diagram (*i.e.*, excluding the dotted arrow); but the the definition of ν is equivalent to the perimeter of the whole diagram (*i.e.*, including the dotted arrow).

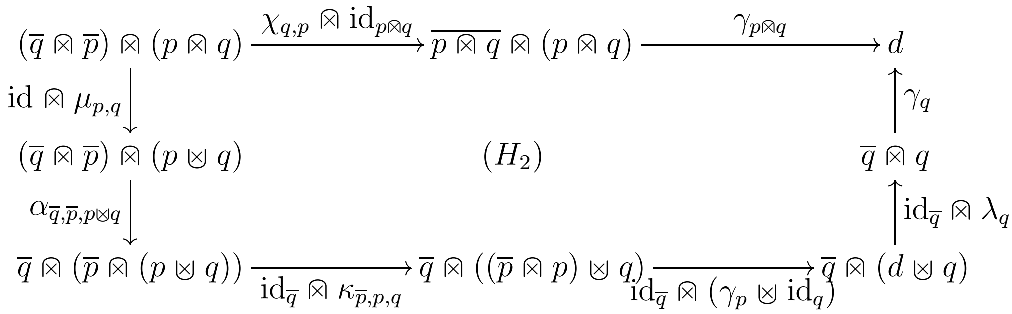


Again using that all of the depicted arrows are invertible, we can conclude that the commutativity of (?) is equivalent to the equality of $\chi^{\dagger}_{p,q}$ and $\chi^{-1}_{p,q}$. ■

6.6. Definition. *Let* $e \xrightarrow{\mathring{\mu}} d$ *and* $p \boxtimes q \xrightarrow{\mu_{p,q}} p \boxtimes q$ *be* mix rules *[CS97a] for some involutive ambimonoidal category* $\mathcal{K}$, $r^{\perp} \xrightarrow{\varphi_r} {}^{\perp}r$ *a cycle for it, and* Φ *the corresponding natural transformation* $\hom_{\mathcal{K}}(p \boxtimes t, d) \longrightarrow \hom_{\mathcal{K}}(t \boxtimes p, d)$*; then, a Hermitian system is called (*μ*-,*φ*-)*hereditary *if the diagrams*



and



hold for all p, q.

We remark that (H_0) and (H_2) generalise the very natural intuitions that $\langle a, b\rangle = \overline{a} \cdot b$ should hold for scalars, and that

$$\langle u \otimes x, v \otimes y\rangle = \langle x, \langle u, v\rangle \cdot y\rangle \quad (= \langle u, v\rangle \cdot \langle x, y\rangle, \text{ in the presence of symmetry})$$

should hold for pure tensors; (H_1) appears to say that the structure of $\overline{p}$ is essentially the same as that of p, which is also reasonable.

6.7. THEOREM. *For an exact Hermitian system on an involutive star-autonomous category $\mathcal{K}$, the following are equivalent:*

1. $(\mathcal{K}, \otimes, e, \widetilde{(\ )}, (\ )^\dagger)$ *is a dagger pivotal category;*
2. $\mathcal{K}$ *admits mix rules* $e \xrightarrow{\mathring{\mu}} d$ *and* $p \otimes q \xrightarrow{\mu_{p,q}} p \mathbin{\rotatebox[origin=c]{180}{$\otimes$}} q$ *and a cycle* $r^\perp \xrightarrow{\varphi_r} {}^\perp r$ *for which the Hermitian system is hereditary;*
3. $\mathcal{K}$ *admits invertible mix rules* $e \xrightarrow{\mathring{\mu}} d$ *and* $p \otimes q \xrightarrow{\mu_{p,q}} p \mathbin{\rotatebox[origin=c]{180}{$\otimes$}} q$ *for which the Hermitian system is hereditary.*

PROOF. By the preceding lemmata, $(\mathcal{K}, \otimes, e, (\ )^\dagger)$ is a dagger monoidal category if and only if $\nu_{p,q}, \mathring{\nu}$ form mix rules. But, by standard categorical yoga, axioms (H_2) and (H_0) are equivalent to $\mu_{p,q} = \nu_{p,q}$ for all p and q, and $\mathring{\mu} = \mathring{\nu}$ respectively. (In particular, these two axioms force $\mu_{p,q}$ and $\mathring{\mu}$ to be invertible, even if they were not assumed to be so at the beginning.)

Moreover, (H_2) entails that $\widetilde{p} \otimes \widetilde{q} \xrightarrow{\chi_{p,q}} \widetilde{q \otimes p}$ is the de Morgan law associated to $\widetilde{(\ )}$. Hence the preceding lemmata also show that $(\mathcal{K}, \otimes, e, \widetilde{(\ )}, (\ )^\dagger)$ is a dagger pivotal category if and only if, in addition, ϑ_r forms a cycle. In strict analogy with the other heredity axioms, (H_1) turns out to be equivalent to $\varphi_r = \vartheta_r$ for all r. ∎

6.8. REMARK. Combining the results of this section with those of section 4, we see that it is possible to obtain pivotal categories for which every object p (being isomorphic to its conjugate, $\overline{p}$) is isomorphic to its own dual, ${}^\perp p$; a comprehensive study of pivotal categories with this property is undertaken in [Sel10], and it seems that the results contained therein are closely related to ours. But these connections remain to be explored in detail.

References

[Bar95] Michael Barr. Nonsymmetric ∗-autonomous categories. *Theoret. Comput. Sci.*, 139(1-2):115–130, 1995. doi:10.1016/0304-3975(94)00089-2.

[Bén64] Jean Bénabou. Algèbre élémentaire dans les catégories avec multiplication. *C. R. Acad. Sci. Paris*, 258:771–774, 1964.

[Bén67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[BLR02] Richard F. Blute, François Lamarche, and Paul Ruet. Entropic Hopf algebras and models of non-commutative logic. *Theory Appl. Categ.*, 10:No. 17, 424–460 (electronic), 2002.

[BM09] E. J. Beggs and S. Majid. Bar categories and star operations. *Algebr. Represent. Theory*, 12(2-5):103–152, 2009. doi:10.1007/s10468-009-9141-x.

[CS97a] J. R. B. Cockett and R. A. G. Seely. Proof theory for full intuitionistic linear logic, bilinear logic, and MIX categories. *Theory Appl. Categ.*, 3:No. 5, 85–131 (electronic), 1997.

[CS97b] J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. *J. Pure Appl. Algebra*, 114(2):133–173, 1997.

[Egg10] J. M. Egger. The Frobenius relations meet linear distributivity. *Theory Appl. Categ.*, 24:No. 2, 25–38, 2010. http://www.tac.mta.ca/tac/volumes/24/2/24-02abs.html.

[EM11a] J.M. Egger and M.B. McCurdy. Cyclic linearly distributive categories, 2011. To appear.

[EM11b] J.M. Egger and M.B. McCurdy. On cyclic star-autonomous categories, 2011. Accepted for publication in Journal of Pure and Applied Algebra.

[Gyl99] R. P. Gylys. Involutive and relational quantaloids. *Liet. Mat. Rink.*, 39(4):475–492, 1999. doi:10.1007/BF02465588.

[Jac10] Bart Jacobs. Involutive categories and monoids, with a GNS-correspondence. 7th workshop on Quantum Physics and Logic (QPL 2010), 2010. http://www.cs.ru.nl/B.Jacobs/PAPERS/invcatmon-qpl10.pdf.

[JS91] André Joyal and Ross Street. The geometry of tensor calculus. I. *Adv. Math.*, 88(1):55–112, 1991. doi:10.1016/0001-8708(91)90003-P.

[JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993. doi:10.1006/aima.1993.1055.

[KPRR03] David Kruml, Joan Wick Pelletier, Pedro Resende, and Jivrḱ Rosický. On quantales and spectra of C^*-algebras. *Appl. Categ. Structures*, 11(6):543–560, 2003. doi:10.1023/A:1026106305210.

[KR81] Stefano Kasangian and Fabio Rossi. Some remarks on symmetry for a monoidal category. *Bull. Austral. Math. Soc.*, 23(2):209–214, 1981. doi:10.1017/S0004972700007061.

[Law69] F. William Lawvere. Ordinal sums and equational doctrines. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 141–155. Springer, Berlin, 1969.

[ML63] Saunders Mac Lane. Natural associativity and commutativity. *Rice Univ. Studies*, 49(4):28–46, 1963.

[MP92] Christopher J. Mulvey and Joan Wick Pelletier. A quantisation of the calculus of relations. In *Category theory 1991 (Montreal, PQ, 1991)*, volume 13 of *CMS Conf. Proc.*, pages 345–360. Amer. Math. Soc., Providence, RI, 1992.

[MP02] Christopher J. Mulvey and Joan Wick Pelletier. On the quantisation of spaces. *J. Pure Appl. Algebra*, 175(1-3):289–325, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.
doi:10.1016/S0022-4049(02)00139-1.

[Pas99] J. Paseka. Hilbert Q-modules and nuclear ideals in the category of $\bigvee$-semilattices with a duality. In *CTCS '99: Conference on Category Theory and Computer Science (Edinburgh)*, volume 29 of *Electron. Notes Theor. Comput. Sci.*, pages Paper No. 29019, 19 pp. (electronic). Elsevier, Amsterdam, 1999.

[Sel08] Peter Selinger. Re: Set as a monoidal category. Posting on categories mailing list, 14 August 2008.
http://permalink.gmane.org/gmane.science.mathematics.categories/4477.

[Sel09] Peter Selinger. A survey of graphical languages for monoidal categories, 2009.
http://arXiv.org/abs/0908.3347.

[Sel10] Peter Selinger. Autonomous categories in which $A \cong A^*$ (extended abstract). 7th workshop on Quantum Physics and Logic (QPL 2010), 2010.
http://www.mscs.dal.ca/~selinger/papers/halftwist.pdf.

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