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SPAN, COSPAN, AND OTHER DOUBLE CATEGORIES

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Abstract. Given a double category $\mathbb{D}$ such that $\mathbb{D}_0$ has pushouts, we characterize oplax/lax adjunctions $\mathbb{D} \rightleftarrows \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ for which the right adjoint is normal and restricts to the identity on $\mathbb{D}_0$, where $\mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ is the double category on $\mathbb{D}_0$ whose vertical morphisms are cospans. We show that such a pair exists if and only if $\mathbb{D}$ has companions, conjoints, and 1-cotabulators. The right adjoints are induced by the companions and conjoints, and the left adjoints by the 1-cotabulators. The notion of a 1-cotabulator is a common generalization of the symmetric algebra of a module and Artin-Wraith glueing of toposes, locales, and topological spaces.

1. Introduction

Double categories, first introduced by Ehresmann [2], provide a setting in which one can simultaneously consider two kinds of morphisms (called *horizontal* and *vertical* morphisms). Examples abound in many areas of mathematics. There are double categories whose objects are sets, rings, categories, posets, topological spaces, locales, toposes, quantales, and more.

There are general examples, as well. If $\mathcal{D}$ is a category with pullbacks, then there is a double category $\mathbb{S}\mathrm{pan}(\mathcal{D})$ whose objects and horizontal morphisms are those of $\mathcal{D}$, and vertical morphisms $X_0 \nrightarrow X_1$ are spans, i.e., morphisms $X_0 \leftarrow X \rightarrow X_1$ of $\mathcal{D}$, with vertical compositions via pullback. If $\mathcal{D}$ is a category with pushouts, then $\mathbb{C}\mathrm{ospan}(\mathcal{D})$ is defined dually, in the sense that vertical morphisms $X_0 \nrightarrow X_1$ are cospans $X_0 \rightarrow X \leftarrow X_1$ with vertical compositions via pushout. Moreover, if $\mathcal{D}$ has both, then pushout of spans and pullback of cospans induce an oplax/lax adjunction (in the sense of Grandis-Paré [5, 12])

$$\mathbb{S}\mathrm{pan}(\mathcal{D}) \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}\mathrm{ospan}(\mathcal{D})$$

which restricts to the identity on the horizontal category $\mathcal{D}$.

Now, suppose $\mathcal{D}$ is a category with pushouts, and we replace $\mathbb{S}\mathrm{pan}(\mathcal{D})$ by a double category $\mathbb{D}$ whose horizontal category is also $\mathcal{D}$. Then several questions arise. Under what conditions on $\mathbb{D}$ is there an oplax/lax adjunction $\mathbb{D} \rightleftarrows \mathbb{C}\mathrm{ospan}(\mathcal{D})$ which restricts to the identity on $\mathcal{D}$? In particular, if $\mathbb{D}$ has cotabulators, then there is an induced oplax functor

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$F\colon \mathbb{D} \longrightarrow \mathbb{C}\mathrm{ospan}(\mathcal{D})$, and so one can ask when this functor F has a right adjoint. Similarly, if $\mathbb{D}$ has companions and conjoints, there is a normal lax functor $G\colon \mathbb{C}\mathrm{ospan}(\mathcal{D}) \longrightarrow \mathbb{D}$ which takes a cospan $X_0 \xrightarrow{c_0} X \xleftarrow{c_1} X_1$ to the composite $X_0 \xrightarrow{c_{0*}} X \xrightarrow{c_1^*} X_1$, and so one can ask when the induced functor G has a left adjoint.

We will see that these questions are related. In particular, there is an oplax/lax adjunction

$$\mathbb{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}\mathrm{ospan}(\mathcal{D})$$

such that G is normal and restricts to the identity on $\mathcal{D}$ precisely when $\mathbb{D}$ has 1-cotabulators, conjoints, and companions (where for 1-cotabulators we drop the tetrahedron condition in the definition of cotabulator). Moreover, if $\mathcal{D}$ has pullbacks, then the result dualizes to show that there is oplax/lax adjunctions $\mathbb{S}\mathrm{pan}(\mathcal{D}) \rightleftarrows \mathbb{D}$ whose left adjoint is opnormal and restricts to the identity on $\mathcal{D}$ precisely when $\mathbb{D}$ has 1-tabulators, conjoints, and companions.

The double categories mentioned above all have 1-cotabulators, companions, and conjoints, and the functors F and G are related to familiar constructions. In the double category of commutative rings (as well as, quantales), the functor F is given by the symmetric algebra on a bimodule and G is given by restriction of scalars. For categories (and posets), F is the collage construction. In the case of topological spaces, locales, and toposes, the functor F uses Artin-Wraith glueing.

The paper proceeds as follows. We begin in Section 2 with the double categories under consideration, followed by a review of companions and conjoints in Section 3. The notion of 1-tabulators (dually, 1-cotabulators) is then introduced in Section 4. After a brief discussion of oplax/lax adjunctions in Section 5, we present our characterization (Theorem 5.5) of those of the form $\mathbb{D} \rightleftarrows \mathbb{C}\mathrm{ospan}(\mathcal{D})$ such that right adjoint is normal and restricts to the identity on $\mathcal{D}$. Along the way, we obtain a possibly new characterization (Proposition 5.3) of double categories with companions and conjoints, in the case where the horizontal category $\mathcal{D}$ has pushouts, as those for which the identity functor on $\mathcal{D}$ extends to a normal lax functor $\mathbb{C}\mathrm{ospan}(\mathcal{D}) \longrightarrow \mathbb{D}$. We conclude with the dual (Corollary 5.6) classification of oplax/lax adjunctions $\mathbb{S}\mathrm{pan}(\mathcal{D}) \rightleftarrows \mathbb{D}$ whose left adjoint is opnormal and restricts to the identity on $\mathcal{D}$.

2. The Examples of Double Categories

Following Grandis-Paré [5, 12] and Shulman [13], we define a *double category* $\mathbb{D}$ to be a weak internal category

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{c} \mathbb{D}_1 \underset{d_1}{\overset{d_0}{\rightleftarrows}} \mathbb{D}_0$$

in CAT. It consists of objects (those of $\mathbb{D}_0$), two types of morphisms: horizontal (those of $\mathbb{D}_0$) and vertical (objects of $\mathbb{D}_1$ with domain and codomain given by d_0 and d_1), and cells

(morphisms of $\mathbb{D}_1$) denoted by

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle m}\downarrow & \varphi & \downarrow{\scriptstyle n} \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array} \qquad (\star)$$

Composition and identity morphisms are given horizontally in $\mathbb{D}_0$ and vertically via c and Δ, respectively.

The objects, horizontal morphisms, and *special* cells (i.e., ones in which the vertical morphisms are identities) form a 2-category called the *horizontal 2-category* of $\mathbb{D}$. Since $\mathbb{D}$ is a weak internal category in CAT, the associativity and identity axioms for vertical morphisms hold merely up to coherent isomorphism, and so we get an analogous *vertical bicategory*. When these isomorphisms are identities, we say that $\mathbb{D}$ is a *strict* double category.

The following double categories are of interest in this paper.

2.1. Example. $\mathbb{T}$op has topological spaces as objects and continuous maps as horizontal morphisms. Vertical morphisms $X_0 \mathrel{-\!\!\bullet\!\!\rightarrow} X_1$ are finite intersection-preserving maps $\mathcal{O}(X_0) \mathrel{-\!\!\bullet\!\!\rightarrow} \mathcal{O}(X_1)$ on the open set lattices, and there is a cell of the form $(\star)$ if and only if $f_1^{-1}n \subseteq mf_0^{-1}$.

2.2. Example. $\mathbb{L}$oc has locales as objects, locale morphisms (in the sense of [8]) as horizontal morphisms, and finite meet-preserving maps as vertical morphisms. There is a cell of the form $(\star)$ if and only if $f_1^*n \leq mf_0^*$.

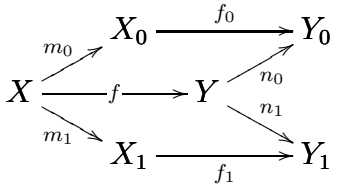
2.3. Example. $\mathbb{T}$opos has Grothendieck toposes as objects, geometric morphisms (in the sense of [9]) as horizontal morphisms, finite limit-preserving functors as vertical morphisms, and natural transformations $f_1^*n \longrightarrow mf_0^*$ as cells of the form $(\star)$.

2.4. Example. $\mathbb{C}$at has small categories as objects and functors as horizontal morphisms. Vertical morphisms $m\colon X_0 \mathrel{-\!\!\bullet\!\!\rightarrow} X_1$ are profunctors (also known as distributors and relators), i.e., functors $m\colon X_0^{op} \times X_1 \longrightarrow \mathrm{Sets}$, and natural transformations $m \longrightarrow n(f_0-, f_1-)$ are cells of the form $(\star)$.

2.5. Example. $\mathbb{P}$os has partially-ordered sets as objects and order-preserving maps as horizontal morphisms. Vertical morphisms $m\colon X_0 \mathrel{-\!\!\bullet\!\!\rightarrow} X_1$ are order ideals $m \subseteq X_0^{op} \times X_1$, and there is a cell of the form $(\star)$ if and only if $(x_0, x_1) \in m \Rightarrow (f_0(x_0), f_1(x_1)) \in n$.

2.6. Example. For a category $\mathcal{D}$ with pullbacks, the span double category $\mathbb{S}\mathrm{pan}(\mathcal{D})$ has objects and horizontal morphisms in $\mathcal{D}$, and vertical morphisms which are spans in $\mathcal{D}$, with composition defined via pullback and the identities $\mathrm{id}^\bullet\colon X \mathrel{-\!\!\bullet\!\!\rightarrow} X$ given by $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$.

The cells $m \longrightarrow n$ are commutative diagrams in $\mathcal{D}$ of the form



In particular, $\mathbb{S}$pan(Sets) is the double category $\mathbb{S}$et considered by Paré in [12], see also [1].

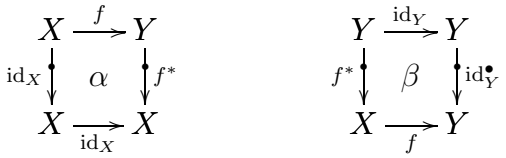
2.7. Example. $\mathbb{C}$ospan($\mathcal{D}$) is defined dually, for a category $\mathcal{D}$ with pushouts. In particular, $\mathbb{S}$pan(Top) is the double category used by Grandis [3, 4] in his study of 2-dimensional topological quantum field theory.

2.8. Example. For a symmetric monoidal category $\mathcal{V}$ with coequalizers, the double category $\mathbb{M}$od($\mathcal{V}$) has commutative monoids in $\mathcal{V}$ as objects and monoid homomorphisms as horizontal morphisms. Vertical morphisms from X_0 to X_1 are (X_0, X_1)-bimodules, with composition via tensor product, and cells are bimodule homomorphisms. Special cases include the double category $\mathbb{R}$ing of commutative rings with identity and the double category $\mathbb{Q}$uant of commutative unital quantales.

3. Companions and Conjoints

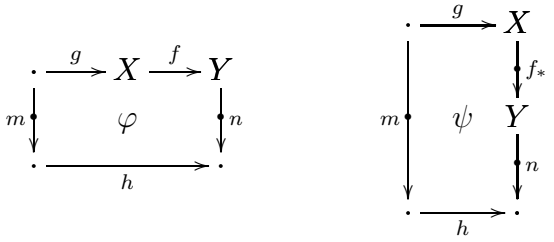
Recall [6] that companions and conjoints in a double category are defined as follows. Suppose $f\colon X \longrightarrow Y$ is a horizontal morphism. A *companion* for f is a vertical morphism $f_*\colon X \longrightarrow\!\!\!\!\bullet\; Y$ together with cells

whose horizontal and vertical compositions are identity cells. A *conjoint* for f is a vertical morphism morphism $f^*\colon Y \longrightarrow\!\!\!\!\bullet\; X$ together with cells

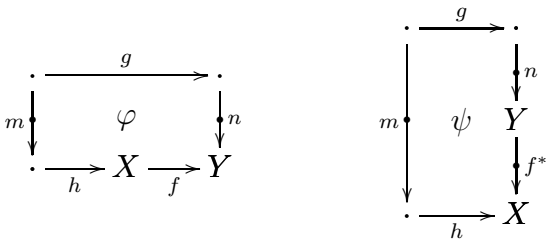


whose horizontal and vertical compositions are identity cells. We say $\mathbb{D}$ *has companions and conjoints* if every horizontal morphism has a companion and a conjoint. Such a double category is also known as a *framed bicategory* [13].

If f has a companion f_*, then one can show that there is a bijection between cells of the following form



Similarly, if f has a conjoint f^*, then there is a bijection between cells

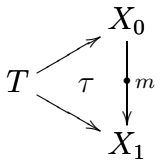


There are two other cases of this process (called *vertical flipping* in [6]) which we do not recall here as they will not be used in the following.

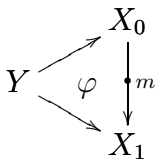
All of the double categories mentioned in the previous section have well known companions and conjoints. In $\mathbb{T}\mathrm{op}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{opos}$, the companion and conjoint of f are the usual maps denoted by f_* and f^*. For $\mathbb{C}\mathrm{at}$, they are the profunctors defined by $f_*(x,y) = Y(fx,y)$ and $f^*(y,x) = Y(y,fx)$, and analogously, for $\mathbb{P}\mathrm{os}$. If $\mathcal{V}$ is a symmetric monoidal category and $f\colon X \longrightarrow Y$ is a monoid homomorphism, then Y becomes an (X,Y)-bimodule and a (Y,X)-bimodule via f, and so Y is both a companion and conjoint for f. Finally, for $\mathbb{S}\mathrm{pan}(\mathcal{D})$ (respectively, $\mathbb{C}\mathrm{ospan}(\mathcal{D})$) the companion and conjoint of f are the span (respectively, cospan) with f as one leg and the appropriate identity morphism as the other.

4. 1-Tabulators and 1-Cotabulators

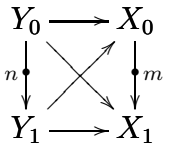
Tabulators in double categories were defined as follows in [5] (see also [12]). Suppose $\mathbb{D}$ is a double category and $m\colon X_0 \longrightarrow\!\!\!\!\!\bullet\; X_1$ is a vertical morphism in $\mathbb{D}$. A *tabulator* of m is an object T together with a cell



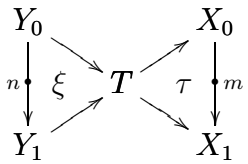
such that for any cell



there exists a unique morphism $f: Y \longrightarrow T$ such that $\tau f = \varphi$, and for any commutative tetrahedron of cells



there is a unique cell ξ such that



gives the tetrahedron in the obvious way.

Tabulators (and their duals cotabulators) arise in the next section, but we do not use the tetrahedron property in any of our proofs or constructions. Thus, we drop this condition in favor of a weaker notion which we call a *1-tabulator* (and dually, *1-cotabulator*) of a vertical morphism. When the tetrahedron condition holds, we call these tabulators (respectively, cotabulators) *strong*.

It is easy to show that the following proposition gives an alternative definition in terms of adjoint functors.

4.1. Proposition. *A double category $\mathbb{D}$ has 1-tabulators (respectively, 1-cotabulators) if and only if $\Delta: \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a right (respectively, left) adjoint which we denote by Σ (respectively, Γ).*

4.2. Corollary. $\mathbb{M}\mathrm{od}(\mathcal{V})$ *does not have 1-tabulators.*

Proof. Suppose $(\mathcal{V}, \otimes, I)$ is a symmetric monoidal category. Then I is an initial object $\mathbb{M}\mathrm{od}(\mathcal{V})_0$, which is the category of commutative monoids in $\mathcal{V}$. Since ΔI is not an initial object of $\mathbb{M}\mathrm{od}(\mathcal{V})_1$, we know Δ does not have a right adjoint, and so the result follows from Proposition 4.1. ∎

The eight examples under consideration have 1-cotabulators, and we know that 1-tabulators exist in all but one (namely, $\mathbb{M}\mathrm{od}(\mathcal{V})$). In fact, we will prove a general proposition that gives the existence of 1-tabulators from a property of 1-cotabulators (called 2-*glueing* in [11]) shared by $\mathbb{T}\mathrm{op}$, $\mathbb{L}\mathrm{oc}$, $\mathbb{T}\mathrm{opos}$, $\mathbb{C}\mathrm{at}$, and $\mathbb{P}\mathrm{os}$. We will also show that the 1-cotabulators in $\mathbb{R}\mathrm{ing}$ are not strong, and so consideration of only strong cotabulators would eliminate this example from consideration.

The cotabulator of $m\colon X_0 \nrightarrow X_1$ in $\mathbb{C}\mathrm{at}$ (and similarly, $\mathbb{P}\mathrm{os}$), also known as the *collage*, is the the category X over $\mathbb{2}$ whose fibers over 0 and 1 are X_0 and X_1, respectively, and morphism from objects of X_0 to those of X_1 are given by $m\colon X_0^{op} \times X_1 \longrightarrow \mathrm{Sets}$, with the obvious cell $m \longrightarrow \mathrm{id}^{\bullet}_X$.

Cotabulators in $\mathbb{T}\mathrm{opos}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{op}$ are constructed using Artin-Wraith glueing (see [9], [10], [11]). In particular, given $m\colon X_0 \nrightarrow X_1$ in $\mathbb{T}\mathrm{op}$, the points of Γm are given by the disjoint union of X_0 and X_1 with U *open* in Γm if and only if U_0 is open in X_0, U_1 is open in X_1, and $U_1 \subseteq m(U_0)$, where $U_i = U \cap X_i$.

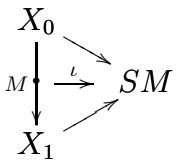
For $\mathbb{C}\mathrm{ospan}(\mathcal{D})$, the cotabulator of $X_0 \xrightarrow{c_0} X \xleftarrow{c_1} X_1$ is given by X with cell $(c_0, \mathrm{id}_X, c_1)$. If $\mathcal{D}$ has pullbacks and pushouts, then the cotabulator of $X_0 \xleftarrow{s_0} X \xrightarrow{s_1} X_1$ in $\mathbb{S}\mathrm{pan}(\mathcal{D})$ is the pushout of s_0 and s_1.

The situation in $\mathbb{M}\mathrm{od}(\mathcal{V})$ is more complicated. The following example (which generalizes to any $\mathcal{V}$ with a "symmetric algebra" functor) shows that $\mathbb{R}\mathrm{ing}$ has 1-cotabulators which do not satisfy the tetrahedron condition.

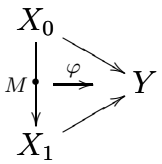
4.3. Example. Suppose $M\colon X_0 \nrightarrow X_1$ is a vertical morphism in $\mathbb{R}\mathrm{ing}$, i.e., M is an (X_0, X_1)-bimodule. Then M is an $X_0 \otimes X_1$-module, and so we can consider the symmetric $X_0 \otimes X_1$-algebra

$$SM = \bigoplus_{n \geq 0} M^{\otimes n}$$

together with the canonical inclusion $\iota\colon M \longrightarrow SM$, where $M^{\otimes 0} = X_0 \otimes X_1$ and $M^{\otimes n}$ is the tensor product over $X_0 \otimes X_1$ of n copies of M. We claim that the corresponding cell



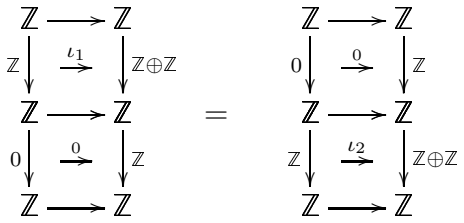
is a 1-cotabulator of $M\colon X_0 \nrightarrow X_1$. Since $X_0 \otimes X_1$ is the coproduct of X_0 and X_1 as commutative rings and S is left adjoint to the forgetful functor from $X_0 \otimes X_1$-algebras to $X_0 \otimes X_1$-modules, we know that any cell



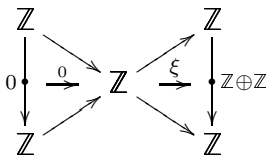
corresponds to an $X_0 \otimes X_1$-module homomorphism $\varphi\colon M \longrightarrow Y$, and hence, a unique $X_0 \otimes X_1$-algebra homomorphism $f\colon SM \longrightarrow Y$ such that $f\iota = \varphi$. Thus, SM has the desired structure of a 1-cotabulator in $\mathbb{R}\mathrm{ing}$.

To see that the tetrahedron condition does not hold, consider the 1-cotabulator of $0\colon \mathbb{Z} \nrightarrow \mathbb{Z}$, where $S0 \cong \mathbb{Z}$ and ι is the unique homomorphism $0 \longrightarrow \mathbb{Z}$. Taking $\iota_1(n) = (n, 0)$

and $\iota_2(n) = (0, n)$, the diagram



defines a commutative tetrahedron which does not factor as



for any homomorphism $\xi\colon \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$. Thus, the 1-cotabulators are not strong in $\mathbb{R}$ing.

Now, suppose $\mathbb{D}$ has 1-cotabulators and a terminal object 1, and let 2 denote the image under Γ of the vertical identity morphism on 1, where $\Gamma\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ is left adjoint to Δ (see Proposition 4.1). Then Γ induces a functor $\mathbb{D}_1 \longrightarrow \mathbb{D}_0/2$, which we also denote by Γ. If this functor is an equivalence of categories, then we say $\mathbb{D}$ *has 2-glueing*.

In $\mathbb{C}$at (and similarly, $\mathbb{P}$os), 2 is the category with two objects and one non-identity morphism. It is the Sierpinski space 2 in $\mathbb{T}$op, the Sierpinski locale $\mathcal{O}(2)$ in $\mathbb{L}$oc, and the Sierpinski topos S^2 in $\mathbb{T}$opos. That $\mathbb{C}$at has 2-glueing is Bénabou's equivalence cited in [14]. For $\mathbb{T}$op, $\mathbb{L}$oc, and $\mathbb{T}$opos, the equivalence follows from the glueing construction (see [9], [10], [11]). Note that in each of these cases, 2 is exponentiable in $\mathbb{D}_0$ (see [7, 9]), and so the functor $2^*\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_0/2$ has a right adjoint, usually denoted by Π_2.

4.4. Proposition. *If $\mathbb{D}$ has 2-glueing and 2 is exponentiable in $\mathbb{D}_0$, then $\mathbb{D}$ has 1-tabulators.*

Proof. Consider the composite $F\colon \mathbb{D}_0 \xrightarrow{2^*} \mathbb{D}_0/2 \simeq \mathbb{D}_1$. Since it is not difficulty to show that Γ takes the vertical identity on X to the projection $X \times 2 \longrightarrow 2$, it follows that $F = \Delta$, and so Δ has a right adjoint Σ, since 2^* does. Thus, $\mathbb{D}$ has 1-tabulators by Proposition 4.1. ∎

Applying Proposition 4.4, we see that $\mathbb{C}$at, $\mathbb{P}$os, $\mathbb{T}$op, $\mathbb{L}$oc, and $\mathbb{T}$opos have 1-tabulators (which can be shown to be strong). Unraveling the construction of Σ given in the proof above, one gets the following descriptions of 1-tabulators in $\mathbb{C}$at and $\mathbb{T}$op which can be shown to be strong.

Given $m\colon X_0 \nrightarrow X_1$ in $\mathbb{C}$at (and similarly $\mathbb{P}$os), the tabulator is the category of elements of m, i.e., the objects of Σm are of the form (x_0, x_1, α), where x_0 is an object of X_0, x_1 is an object of X_1, and $\alpha \in m(x_0, x_1)$. Morphisms from (x_0, x_1, α) to (x_0', x_1', α') in Σm are pairs $(x_0 \longrightarrow x_0', x_1 \longrightarrow x_1')$ of morphisms, compatible with α and α'.

The tabulator of $m: X_0 \nrightarrow X_1$ in $\mathbb{T}\mathrm{op}$ is the set

$$\Sigma m = \{(x_0, x_1) \mid \forall U_0 \in \mathcal{O}(X_0), x_1 \in m(U_0) \Rightarrow x_0 \in U_0\} \subseteq X_0 \times X_1$$

with the subspace topology. Note that one can directly see that this is the tabulator of m by showing that $(f_0, f_1): Y \longrightarrow X_0 \times X_1$ factors through Σm if and only if $f_1^{-1}m \subseteq f_0^{-1}$.

Although $\mathbb{S}\mathrm{pan}(\mathcal{D})$ and $\mathbb{C}\mathrm{ospan}(\mathcal{D})$ do not have 2-glueing (since $\Gamma(\mathrm{id}_1^\bullet) = 1$), and so Proposition [11] does not apply, the construction of their tabulators is dual to that of their cotabulators. Finally, $\mathbb{R}\mathrm{ing}$ and $\mathbb{Q}\mathrm{uant}$ do not have 1-tabulators, as they are special cases of Corollary 4.2.

5. The Adjunction

Recall from [5] that a *lax functor* $F: \mathbb{D} \longrightarrow \mathbb{E}$ consists of functors $F_i: \mathbb{D}_i \longrightarrow \mathbb{E}_i$, for $i = 0, 1$, compatible with d_0 and d_1; together with identity and composition comparison cells

$$\rho_X: \mathrm{id}^\bullet_{FX} \longrightarrow F(\mathrm{id}^\bullet_X) \qquad \text{and} \qquad \rho_{m,m'}: Fm' \bullet Fm \longrightarrow F(m' \bullet m)$$

for every object X and every vertical composite $m' \bullet m$ of $\mathbb{D}$, respectively; satisfying naturality and coherence conditions. If ρ_X is an isomorphism, for all X, we say that F is a *normal lax functor*. An *oplax functor* is defined dually with comparison cells in the opposite direction.

An *oplax/lax adjunction* consists of an oplax functor $F: \mathbb{D} \longrightarrow \mathbb{E}$ and a lax double functor $G: \mathbb{E} \longrightarrow \mathbb{D}$ together with double cells

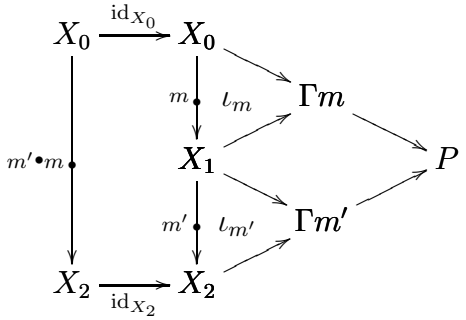
$$\begin{array}{ccc} X_0 & \xrightarrow{\eta_{X_0}} & GFX_0 \\ {\scriptstyle m}\downarrow & \eta_m & \downarrow{\scriptstyle GFm} \\ X_1 & \xrightarrow[\eta_{X_1}]{} & GFX_1 \end{array} \qquad\qquad \begin{array}{ccc} FGY_0 & \xrightarrow{\varepsilon_{Y_0}} & Y_0 \\ {\scriptstyle FGn}\downarrow & \varepsilon_n & \downarrow{\scriptstyle n} \\ FGY_1 & \xrightarrow[\varepsilon_{Y_1}]{} & Y_1 \end{array}$$

satisfying naturality and coherence conditions, as well as the usual adjunction identities (see [6]).

5.1. Example. Suppose $\mathbb{D}$ is a double category with 1-cotabulators and $\mathbb{D}_0$ has pushouts. Then, by Proposition 4.1, the functor $\Delta: \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a left adjoint (denoted by Γ), and so there is an oplax functor $F: \mathbb{D} \longrightarrow \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ which is the identity on objects and horizontal morphisms, and defined on vertical morphisms and cells by

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & X_0' \\ {\scriptstyle m}\downarrow & \varphi & \downarrow{\scriptstyle m'} \\ X_1 & \xrightarrow[f_1]{} & X_1' \end{array} \qquad \longmapsto \qquad \begin{array}{ccccc} X_0 & & \xrightarrow{f_0} & & X_0' \\ & {\scriptstyle i_0}\searrow & & \nearrow{\scriptstyle i_0'} & \\ & & \Gamma m & \xrightarrow{f} & \Gamma m' \\ & {\scriptstyle i_1}\nearrow & & \searrow{\scriptstyle i_1'} & \\ X_1 & & \xrightarrow[f_1]{} & & X_1' \end{array}$$

where f is induced by the universal property of the 1-cotabulator. The comparison cells $F(\mathrm{id}^{\bullet}_X) \longrightarrow \mathrm{id}^{\bullet}_{FX}$ and $F(m' \bullet m) \longrightarrow Fm' \bullet Fm$ also arise via the universal property, with the latter given by the horizontal morphism $\Gamma(m' \bullet m) \longrightarrow P$ corresponding to the diagram



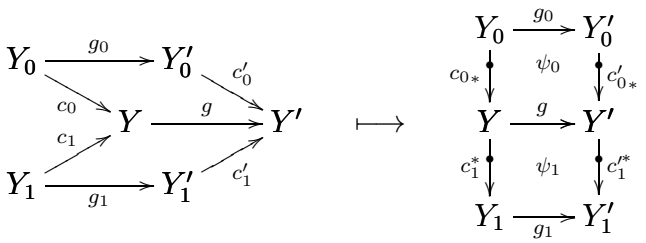
where P is a pushout and the large rectangle is the canonical cell.

Dually, we get:

5.2. Example. Suppose $\mathbb{D}$ is a double category with 1-tabulators and $\mathbb{D}_0$ has pullbacks. Then there is a lax functor $F\colon \mathbb{D} \longrightarrow \mathbb{S}\mathrm{pan}(\mathbb{D}_0)$ which is the identity on objects and horizontal morphisms and takes $m\colon X_0 \nrightarrow X_1$ to the span $X_0 \longleftarrow \Sigma m \longrightarrow X_1$, where Σ is the right adjoint to Δ.

5.3. Proposition. *Suppose $\mathbb{D}$ is a double category and $\mathbb{D}_0$ has pushouts. Then $\mathbb{D}$ has companions and conjoints if and only the identity functor on $\mathbb{D}_0$ extends to a normal lax functor $G\colon \mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \longrightarrow \mathbb{D}$.*

Proof. Suppose $\mathbb{D}$ has companions and conjoints. Then it is not difficult to show that there is a normal lax functor $G\colon \mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \longrightarrow \mathbb{D}$ which is the identity on objects and horizontal morphisms, and is defined on cells by

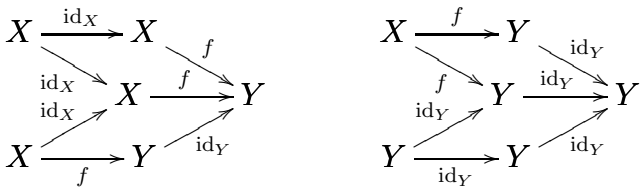


where ψ_0 and ψ_1 arise from the commutativity of the squares in the cospan cell, and the definitions of companion and conjoint.

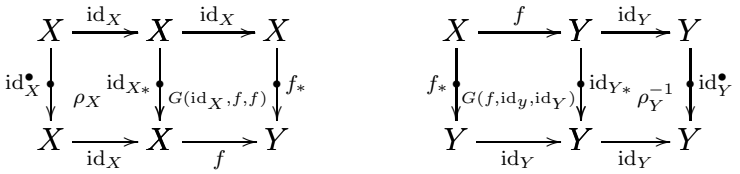
Conversely, suppose there is a normal lax functor $G\colon \mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \longrightarrow \mathbb{D}$ which is the identity on objects and horizontal morphisms. Then the companion and conjoint of $f\colon X \longrightarrow Y$ are defined as follows. Consider

$$f_* = G(X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y)\colon X \nrightarrow Y \qquad f^* = G(Y \xrightarrow{\mathrm{id}_Y} Y \xleftarrow{f} X)\colon Y \nrightarrow X$$

Applying G to the cospan diagrams

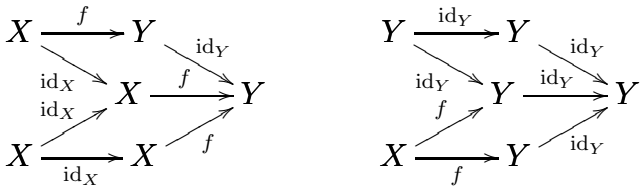


and composing with ρ_X and ρ_Y^{-1}, we get cells



which serve as η and ε, respectively, making f_* the companion of f. Note that the horizontal and vertical identities for η and ε follow from the normality and coherence axioms of G, respectively.

Similarly, the cells α and β for f^* arise from the cospan diagrams



and it follows that $\mathbb{D}$ has companions and conjoints. ∎

5.4. Corollary. *Suppose $\mathbb{D}$ is a double category and $\mathbb{D}_0$ has pullbacks. Then $\mathbb{D}$ has companions and conjoints if and only the identity functor on $\mathbb{D}_0$ extends to an opnormal oplax functor $\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \longrightarrow \mathbb{D}$.*

Proof. Apply Proposition 5.3 to $\mathbb{D}^{op}$. ∎

5.5. Theorem. *The following are equivalent for a double category $\mathbb{D}$ such that $\mathbb{D}_0$ has pushouts:*

(a) there is an oplax/lax adjunction $\mathbb{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ such that G is normal and restricts to the identity on $\mathbb{D}_0$;

(b) $\mathbb{D}$ has companions, conjoints, and 1-cotabulators;

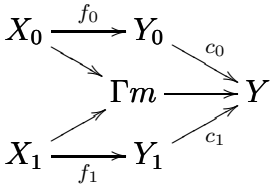
(c) $\mathbb{D}$ has companions and conjoints, and the induced normal lax functor $\mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \xrightarrow{G} \mathbb{D}$ has an oplax left adjoint;

(d) $\mathbb{D}$ *has companions, conjoints, and 1-cotabulators, and the induced oplax functor* $\mathbb{D} \xrightarrow{F} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ *is left adjoint to the induced normal lax functor* $\mathbb{C}\mathrm{ospan}(\mathbb{D}_0) \xrightarrow{G} \mathbb{D}$*;*

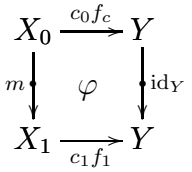
(e) $\mathbb{D}$ *has 1-cotabulators and the induced oplax functor* $\mathbb{D} \xrightarrow{F} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ *has a normal lax right adjoint.*

PROOF. (a)⇒(b) Given $F \dashv G$ such that G is normal and restricts to the identity on $\mathbb{D}_0$, we know $\mathbb{D}$ has companions and conjoints by Proposition 5.3. To see that $\mathbb{D}$ has 1-cotabulators, by Proposition 4.1, it suffices to show that $\Delta\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a left adjoint. Since Δ factors as $\mathbb{D}_0 \xrightarrow{\Delta} \mathrm{Cospan}(\mathbb{D}_0) \xrightarrow{G_1} \mathbb{D}_1$, by normality of G, and both these functors have left adjoints, the desired result follows.

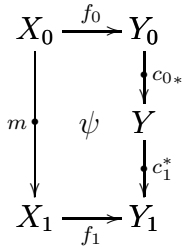
(b)⇒(c) Suppose $\mathbb{D}$ has companions, conjoints, and 1-cotabulators, and consider the induced functor $F\colon \mathbb{D} \longrightarrow \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$. Given $m\colon X_0 \nrightarrow X_1$ and $Y_0 \xrightarrow{c_0} Y \xleftarrow{c_1} Y_1$, applying the definition of 1-cotabulator, we know that every cell in $\mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$ of the form



corresponds to a unique cell



and hence, a unique cell



by vertical flipping, and it follows that $F \dashv G$.

(c)⇒(d) Suppose $\mathbb{D}$ has companions and conjoints, and the induced normal lax functor G has an oplax left adjoint F. As in the proof of (a)⇒(b), we know that $\mathbb{D}$ has 1-cotabulators, and so, it suffices to show that F is the induced functor. We know F takes $m\colon X_0 \nrightarrow X_1$ to a cospan of the form $X_0 \xrightarrow{c_0} \Gamma m \xleftarrow{c_1} X_1$, since F is the identity on objects and the left adjoint of $\Delta\colon \mathbb{D}_0 \longrightarrow \mathrm{Cospan}(\mathbb{D}_0)$ takes $X_0 \xrightarrow{c_0} \Gamma m \xleftarrow{c_1} X_1$ to X, and so the desired result easily follows.

(d)⇒(e) is clear.

(e)⇒(a) Suppose $\mathbb{D}$ has 1-cotabulators and the induced oplax functor F has a normal lax right adjoint G. Since F restricts to the identity on $\mathbb{D}_0$, then so does G, and the proof is complete. ∎

Note that this proof shows that if there is an oplax/lax adjunction

$$\mathbb{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{C}\mathrm{ospan}(\mathbb{D}_0)$$

such that G is normal and restricts to the identity on $\mathbb{D}_0$, then F is the induced by 1-cotabulators and G by companions and conjoints. Since the double categories in Examples 2.1–2.8 all have companions, conjoints, and 1-cotabulators, it follows that they each admits a unique (up to equivalence) oplax/lax adjunction of this form and it is induced in this manner.

Applying Theorem 5.5 to $\mathbb{D}^{op}$, we get:

5.6. Corollary. *The following are equivalent for a double category $\mathbb{D}$ such that $\mathbb{D}_0$ has pullbacks:*

(a) there is an oplax/lax adjunction $\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \underset{F}{\overset{G}{\rightleftarrows}} \mathbb{D}$ *such that G is opnormal and restricts to the identity on $\mathbb{D}_0$;*

(b) $\mathbb{D}$ has companions, conjoints, and 1-tabulators;

(c) $\mathbb{D}$ has companions and conjoints, and the induced opnormal oplax functor $\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \xrightarrow{G} \mathbb{D}$ *has a lax right adjoint;*

(d) $\mathbb{D}$ has companions, conjoints, and 1-tabulators, and the induced lax functor $\mathbb{D} \xrightarrow{F} \mathbb{S}\mathrm{pan}(\mathbb{D}_0)$ *is right adjoint to the induced opnormal oplax functor* $\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \xrightarrow{G} \mathbb{D}$*;*

(e) $\mathbb{D}$ has 1-tabulators and the induced lax functor $\mathbb{D} \xrightarrow{F} \mathbb{S}\mathrm{pan}(\mathbb{D}_0)$ *has an opnormal oplax left adjoint.*

As in the cospan case, if there is an oplax/lax adjunction

$$\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \underset{F}{\overset{G}{\rightleftarrows}} \mathbb{D}$$

such that G is opnormal and restricts to the identity on $\mathbb{D}_0$, then F is the induced by 1-tabulators and G is by companions and conjoints. Since the double categories in Examples 2.1–2.7 (i.e., all but $\mathbb{M}\mathrm{od}(\mathcal{V})$) have companions, conjoints, and 1-tabulators, it follows they each admits is a unique (up to equivalence) oplax/lax adjunction of this form, and it is induced by companions, conjoints, and 1-tabulators.

References

[1] R. J. MACG. Dawson, R. Paré, amd D. A. Pronk, The span construction, Theory Appl. Categ. 24 (2010), 302–377.

[2] Charles Ehresmann. Catégories structuées, Ann. Sci. Ecole Norm. Sup. 80 (1963) 349–426.

[3] Marco Grandis, Higher cospans and weak cubical categories (cospans in algebraic topology, I), Theory Appl. Categ. 18 (2007), 321–347.

[4] Marco Grandis, Collared cospans, cohomotopy and TQFT (cospans in algebraic topology, II), Theory Appl. Categ. 18 (2007), 602–630.

[5] Marco Grandis and Robert Paré, Limits in double categories, Cahiers de Top. et Géom. Diff. Catég. 40 (1999),162–220.

[6] Marco Grandis and Robert Paré, Adjoint for double categories, Cahiers de Top. et Géom. Diff. Catég. 45 (2004),193–240.

[7] J. M. E. Hyland, Function spaces in the category of locales, Springer Lecture Notes in Math. 871 (1981) 264–281.

[8] P. T. Johnstone, Stone Spaces, Cambridge University Press, 1982.

[9] P. T. Johnstone, Topos Theory, Academic Press, 1977.

[10] S. B. Niefield, Cartesian inclusions: locales and toposes, Comm. in Alg. 9(16) (1981), 1639–1671.

[11] S. B. Niefield, The glueing construction and double categories, J. Pure Appl. Alg. 216 (2012) 1827–1836.

[12] Robert Paré, Yoneda theory for double categories, Theory Appl. Categ. 25 (2011), 436–489.

[13] M. Shulman, Framed bicategories and monoidal fibrations, Theory Appl. Categ. 20 (2008), 650–738.

[14] R. Street, Powerful functors, expository note: www.maths.mq.edu.au/ ˜street/Pow.fun.pdf (2001).

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