IVANE JAVAKHISHVILI TBILISI STATE UNIVERSITY

ILIA VEKUA INSTITUTE OF APPLIED MATHEMATICS

GEORGIAN ACADEMY OF NATURAL SCIENCES

TBILISI INTERNATIONAL CENTRE OF

MATHEMATICS AND INFORMATICS

L E C T U R E   N O T E S

of

T I C M I

Volume 19, 2018

George Jaiani

PIEZOELECTRIC
VISCOELASTIC KELVIN-VOIGT
CUSPED PRISMATIC SHELLS

Tbilisi

LECTURE NOTES OF TICMI

Lecture Notes of TICMI publishes peer-reviewed texts of courses given at Advanced Courses and Workshops organized by TICMI (Tbilisi International Center of Mathematics and Informatics). The advanced courses cover the entire field of mathematics (especially of its applications to mechanics and natural sciences) and from informatics which are of interest to postgraduate and PhD students and young scientists.

DEDICATED TO THE

CENTENARY OF I.JAVAKHISHVILI TBILISI STATE UNIVERSITY

&

50th ANNIVERSARY OF I.VEKUA INSTITUTE OF APPLIED MATHEMATICS

Preface

The present work is intended to provide some materials for participants of the forthcoming TICMI Advance Courses (September 22-25, 2019) on "Mathematical Models of Piezoelectric Solids and Related Problems". This work is oriented mainly on the lecture course of the same name "Piezoelectric Viscoelastic Kelvin-Voigt Cusped Prismatic Shells", foreseen in the prospective programme of the above-mentioned Advance Courses. It mainly contains unpublished results of the author concerning piezoelectrics. Some auxiliary materials, which make the work self-contained, are provided as well.

The aim of the present work is also to draw the attention of scientists, particularly of young researchers, to problems to be solved, connected with cusped shell-like elastic and viscoelastic piezoelectric bodies with voids and with related nonclassical BVPs and IBVPs for partial differential equations with order and type degeneracy. The development of the corresponding numerical methods and numerical calculations on computers are especially challenging.

<div style="text-align: right">

George Jaiani
Tbilisi, Georgia
November, 2018

</div>

Contents

Abstract. The present work is devoted to construction of hierarchical models for piezoelectric nonhomogeneous porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories. Using I. Vekua's dimension reduction method, governing systems are derived and in the Nth approximation of hierarchical models boundary value problems (BVPs) and initial boundary value problems (IBVPs) are set. In the $N = 0$ approximation, considering, e.g., elastic, plates of a constant thickness, governing systems mathematically coincide with the governing systems of the plane strain corresponding to the basic three-dimensional (3D) linear theory up to a separate equation for the out of plane component of the displacement vector.

The ways of investigation of BVPs and IBVPs, including the case of cusped prismatic shells, are indicated and some preliminary results are presented. Antiplane deformation of piezoelectric nonhomogeneous materials in the three-dimensional formulation and in $N = 0$ approximation is analysed.

Well-posedness of Dirichlet and Keldysh type problems (BVP) are studied in the $N = 0$ order approximation of hierarchical models for cusped prismatic shells. Some BVPs are solved in explicit forms in concrete cases.

Key words and phrases: Hierarchical models, Piezoelectrics, Viscoelastic prismatic shells, Porous elastic prismatic shells, Materials with voids, Partial differential equations and Systems with order degeneracy.

2000 Mathematical Subject Classification: 74K20, 74K25, 74F15, 74D05, 35J70, 35J75, 35L80, 35L81, 74K10.

Author's Affiliation:
George Jaiani
I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, 2 University Street, 0186 Tbilisi, Georgia
email: george.jaiani@gmail.com

1   Introduction

In 1955 Ilia Vekua [1] published his models of elastic prismatic shells. In 1965 he offered analogous models for standard shells [2]. In both papers he considered a very important investigation of well-posedeness of boundary value problems (BVPs) of peculiar types which could arise in the case of cusped shells.

Cusped prismatic shells considered as 3D bodies may have non-Lipschitz surfaces as the boundaries and their thicknesses may vanish at the edge. Using I. Vekua's dimension reduction method, complexity of the 3D domain, occupied by the body will be transformed into the degeneracy of the order of the 2D governing equations of the constructed hierarchy of 2D models on the boundary of the 2D projection of the 3D bodies under consideration.

Consideration of BVPs and initial boundary value problems (IBVPs) within the framework of hierarchical models for elastic cusped prismatic shells leads to investigation of nonclassical BVPs and IBVPs for the governing elliptic and hyperbolic systems of equations of the second order with order degeneracy on the boundary of the domain under consideration in the case of two spatial variables [1]-[4]. We easily reduce this case to the case of systems of singular equations, i.e., to systems of equations with unbounded coefficients. Initial conditions (IC) for the so called weighted mathematical moments of displacements remain classical, while the boundary conditions (BC) for them are nonclassical, in general. It means that in certain cases the Dirichlet BCs should be replaced by the Keldysh BCs (i.e. some parts of the boundary, where the order of the equations degenerates, should be freed from the BCs) and in certain cases weighted BCs should be set (see [4]).

The present work is devoted to construction of hierarchical models for piezo-electric nonhomogeneous porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories [5]-[10]. Using I. Vekua's [1] (see also [2]) dimension reduction method, governing systems are derived and in the Nth approximation of hierarchical models BVPs and IBVPs are set. In the $N = 0$ approximation, considering, e.g., elastic plates of a constant thickness, governing systems mathematically coincide with the governing systems of the plane strain corresponding to the basic three-dimensional (3D) linear theory [1]-[4] up to a separate equation for the out of plane component of the displacement vector.

The ways of investigation of BVPs and IBVPs, including the case of cusped prismatic shells [4], are indicated and some preliminary results are presented. Antiplane deformation of piezoelectric nonhomogeneous materials in the three-dimensional formulation and in $N = 0$ approximation is analysed. Some BVPs are solved in explicit forms in concrete cases.

The aim of the present work is also to draw the attention of scientists, particularly of young researchers, to problems to be solved connected with cusped shell-like elastic and viscoelastic piezoelectric bodies with voids and with re-

lated nonclassical BVPs and IBVPs for partial differential equations with order and type degeneracy. The development of the corresponding numerical methods and numerical calculations on computers are especially challenging.

The work is organized as follows. Introduction is devoted to motivations of our research and the main targets of the work are indicated as well. Section 2 contains 3D field equations for nonhomogeneous piezoelectric Kelvin-Voight materials with voids in the case of the general anisotropy. In Section 3 the hierarchical models are constructed and in the Nth approximation BVPs and IBVPs are set in the case of noncusped prismatic shells. Section 4 deals with the analysis of Dirichlet and Keldysh type problems for the general governing system of the $N = 0$ approximation for nonhomogeneous piezoelectric Kelvin-Voight materials with voids and general anisotropy. To this end results of Section 8 are exploited. In Section 5 we consider transversely isotropic elastic piezoelectric nonhomogeneous bodies in the case when the poling axis coincides with one of the material symmetry axises. Namely, time-harmonic motion under conditions of anti-plane piezoelectric state is discussed. In Section 6 we study an antiplane deformation of piezoelectrics in $N = 0$ approximation of hierarchical models for prismatic shells, in particular, with cusped edges. In Section 7 we treat BVPs for porous isotropic elastic cusped prismatic shells. In Section 8 we examine well-posedeness of BVPs for systems of elliptic equations of the second order with an order degeneracy, covering systems of elliptic equations arising in previous sections. In Section 9 for the convenience of the reader we repeat the relevant material, concerning $\mathcal{H}$-weak solutions of BVPs for a single second order equation with an order degeneracy, from [15] with proofs in a slightly changed form, thus making our exposition of the present work self-contained. Section 10 provides some useful formulas for constructing the hierarchical models. Section 11 is devoted to conclusions, concerning mainly mechanical meaning.

2 Field Equations for Piezoelectric Kelvin-Voigt Materials with Voids

Let a piezoelectric solid occupy a reference configuration $\Omega \in \mathbb{R}^3$. Under the quasi-static conditions, when the rate of change of the magnetic field is small and there is no electric current, i.e., the electric field $\mathbf{E}$ and magnetic field $\mathbf{M}$ are curl free, the governing equations have the following form.

Motion Equations

$$X_{ji,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad (2.1)$$
$$t > t_0, \quad i = \overline{1,3};$$

$$H_{j,j} + H_0 + \mathcal{F} = \rho k \ddot{\varphi}, \tag{2.2}$$

$$D_{j,j} = f_e, \quad B_{j,j} = 0, \quad \Omega \times]0, T[, \tag{2.3}$$

where $X_{ij} \in C^1(\Omega)$ is the stress tensor; Φ_i are the volume force components; k is equilibrated inertia, ρ is the mass density; $\varphi := \nu - \nu_0 \in C^2(\Omega)$ is the change of the volume fraction from the matrix reference volume fraction ν_0 (clearly, the bulk density $\rho = \nu\gamma$, $0 < \nu \le 1$, here γ is the matrix density); $u_i \in C^2(\Omega)$ are the displacements; $H_j \in C^1(\Omega)$ is the component of the equilibrated stress vector, H_0 and $\mathcal{F}$ are the intrinsic and extrinsic equilibrated volume forces; Einstein's summation convention is used; indices after comma mean differentiation with respect to the corresponding variables of the Cartesian frame $Ox_1x_2x_3$ (throughout the work we assume existence of the indicated (continuous) derivatives unless otherwise stated); dots as superscripts of the symbols mean derivatives with respect to time t; $\chi : \Omega \times]0, T[\to \mathbb{R}^1$ and $\eta : \Omega \times]0, T[\to \mathbb{R}^1$ are electric and magnetic potentials, respectively, i.e., $\mathbf{E} = -grad\chi$, $\mathbf{M} = -grad\eta$, $f_e : \Omega \times]0, T[\to \mathbb{R}^1$ is electric charge density, p_{kij} are the piezoelectric coefficients, q_{kij} are the piezomagnetic coefficients, ς_{jl} and ξ_{jl} are the dielectric (permittivity) and magnetic permeability coefficients, respectively, $\tilde{a}_{jl}$ are the coupling coefficients connecting electric and magnetic fields. $\mathbf{D} := (D_1, D_2, D_3) : \Omega \times]0, T[\to \mathbb{R}^3$ is the electrical displacement vector, $\mathbf{B} := (B_1, B_2, B_3) : \Omega \times]0, T[\to \mathbb{R}^3$ is the magnetic induction vector.

Kinematic Relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = \overline{1,3}. \tag{2.4}$$

Constitutive Equations

$$X_{ji} = X_{ij} = E_{ijkl}e_{kl} + E^*_{ijkl}\dot{e}_{kl} + \tilde{b}_{ij}\varphi + b^*_{ij}\dot{\varphi} + d_{ijk}\varphi_{,k} + d^*_{ijk}\dot{\varphi}_{,k}$$
$$+ p_{kij}\chi_{,k} + p^*_{kij}\dot{\chi}_{,k} + q_{kij}\eta_{,k} + q^*_{kij}\dot{\eta}_{,k}, \quad i,j = \overline{1,3}, \tag{2.5}$$

$$H_j = d_{klj}e_{kl} + d^*_{klj}\dot{e}_{kl} + d_j\varphi + d^*_j\dot{\varphi} + \tilde{\alpha}_{ji}\varphi_{,i} + \alpha^*_{ji}\dot{\varphi}_{,i}, \quad j = \overline{1,3}, \tag{2.6}$$

$$H_0 = -\tilde{b}_{ij}e_{ij} - \tilde{\xi}\varphi - d_i\varphi_{,i} - b^*_{ij}\dot{e}_{ij} - \xi^*\dot{\varphi} - d^*_i\dot{\varphi}_{,i}, \tag{2.7}$$

$$D_j = p_{jkl}e_{kl} + p^*_{jkl}\dot{e}_{kl} - \varsigma_{jl}\chi_{,l} - \tilde{a}_{jl}\eta_{,l}, \quad j = \overline{1,3}, \tag{2.8}$$

$$B_j = q_{jkl}e_{kl} + q^*_{jkl}\dot{e}_{kl} - \tilde{a}_{jl}\chi_{,l} - \xi_{jl}\eta_{,l}, \quad j = \overline{1,3}, \tag{2.9}$$

where $e_{ij} \in C^1(\Omega)$ is the strain tensor; the constitutive coefficients E_{ijkl}, E^*_{ijkl}, $\tilde{b}_{ij}$, b^*_{ij}, d_{klj}, d^*_{klj}, d_i, d^*_i, $\tilde{\alpha}_{ji}$, α^*_{ji}, $\tilde{\xi}$, ξ^*, p_{kij}, p^*_{jkl}, q_{kij}, q^*_{jkl}, ς_{jl}, $\tilde{a}_{jl}$, ξ_{jl} satisfy the following relations

$$E_{ijkl} = E_{jikl} = E_{jilk} = E_{klij}; \quad E^*_{ijkl} = E^*_{jikl} = E^*_{jilk} = E^*_{klij};$$

9

$$\tilde{b}_{ij} = \tilde{b}_{ji}, \quad d_{ijk} = d_{jik}, \quad \tilde{\alpha}_{ij} = \tilde{\alpha}_{ji};$$

$$b_{ij}^* = b_{ji}^*, \quad d_{ijk}^* = d_{jik}^*, \quad \alpha_{ij}^* = \alpha_{ji}^*; \quad p_{jkl} = p_{jlk}, \quad q_{jkl} = q_{jlk}, \quad \varsigma_{jl} = \varsigma_{lj},$$

$$\tilde{a}_{jl} = \tilde{a}_{lj}, \quad \xi_{jl} = \xi_{lj}, \quad p_{jkl}^* = p_{jlk}^*, \quad q_{jkl}^* = q_{jlk}^*.$$

The constitutive equations also meet some other conditions, following from physical considerations (see [5], [8], and the references given there). With a view to apply I. Vekua's dimension reduction method, we require the constitutive coefficients to be independent of x_3.

Let us consider the general BVPs and IBVPs with the following mixed BCs

$$u_i = f_i \quad on \quad \Gamma_0, \quad X_{ij}n_j = g_i \quad on \quad \Gamma_1 = \partial\Omega \backslash \overline{\Gamma_0}, \quad i = \overline{1,3}, \tag{2.10}$$

$$\varphi = f^\varphi \quad on \quad \Gamma_0^\varphi, \quad H_j n_j = g^\varphi \quad on \quad \Gamma_1^\varphi = \partial\Omega \backslash \overline{\Gamma_0^\varphi}, \quad i = \overline{1,3}, \tag{2.11}$$

$$\chi = f^\chi \quad on \quad \Gamma_0^\chi, \quad D_j n_j = g^\chi \quad on \quad \Gamma_1^\chi = \partial\Omega \backslash \overline{\Gamma_0^\chi}, \quad i = \overline{1,3}, \tag{2.12}$$

$$\eta = f^\eta \quad on \quad \Gamma_0^\eta, \quad B_j n_j = g^\eta \quad on \quad \Gamma_1^\eta = \partial\Omega \backslash \overline{\Gamma_0^\eta}, \quad i = \overline{1,3}, \tag{2.13}$$

and the standard ICs in the case of dynamical problems

$$\mathbf{u}(x,0) = \mathbf{u}^0(x), \quad \dot{\mathbf{u}}(x,0) = \mathbf{u}^1(x), \quad \varphi(x,0) = \varphi^0(x), \quad \dot{\varphi}(x,0) = \varphi^1(x), \tag{2.14}$$
$$x \in \Omega;$$

here $\mathbf{n} := (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$, (f_1, f_2, f_3), f^φ, f^χ, f^η are the given displacement vector, volume fraction, electric and magnetic potentials, respectively, (g_1, g_2, g_3), g^φ, g^χ and g^η are the given stress vector, normal components of the equilibrated stress, electric displacement and magnetic induction vectors, respectively, while $\mathbf{u}^0$ and $\mathbf{u}^1$ are the initial mechanical displacement and velocity vectors, whereas φ^0 and φ^1 are the initial volume fraction distribution and its rate. Note that the sub-manifolds Γ_0, Γ_0^φ, Γ_0^χ, and Γ_0^η, of the boundary $\partial\Omega$ in boundary conditions (2.10)-(2.13) are different, in general, from each other and depending on the physical problem, some of them may be empty.

3 Construction of Hierarchical Models. Nth Approximation

Now, we construct hierarchical models for piezoelectric Kelvin-Voigt prismatic shells. First a few words about prismatic shells.

Let us consider prismatic shells (see, e.g., Figure 3.1 and [4], [11]), occupying 3D domain Ω with the projection ω (on the plane $x_3 = 0$) and the face surfaces

$$x_3 = \overset{(+)}{h}(x_1, x_2) \in C^2(\omega) \quad and \quad x_3 = \overset{(-)}{h}(x_1, x_2) \in C^2(\omega), \quad (x_1, x_2) \in \omega.$$
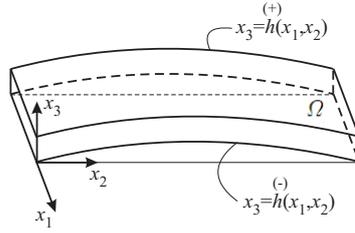
Figure 3.1: A prismatic shell of constant thickness. $\partial\Omega$ is a Lipschitz boundary
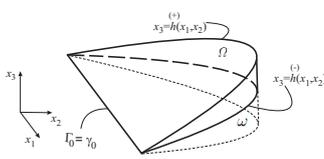
Figure 3.2: A sharp cusped prismatic shell with a semicircle projection. $\partial\Omega$ is a Lipschitz boundary
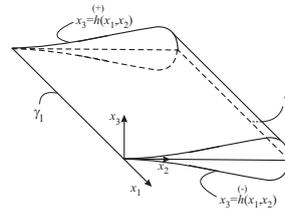
Figure 3.3: A cusped plate with sharp γ_1 and blunt γ_2 edges, $\gamma^0 := \gamma_1 \cup \gamma_2$. $\partial\Omega$ is a non-Lipschitz boundary

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) > 0, \quad (x_1, x_2) \in \omega,$$

is the thickness of the prismatic shell. A part of $\partial\omega$, where the thickness vanishes, i.e., $2h = 0$, is said to be a cusped edge. If $\partial\Omega$ contains it smoothly, we shall call it a blunt edge, otherwise, i.e., the points of the cusped edge are points of nonsmoothness of $\partial\Omega$, we shall call it a sharp edge (see Figures 3.2, 3.3).

Let

$$2\tilde{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2), \quad (x_1, x_2) \in \omega.$$
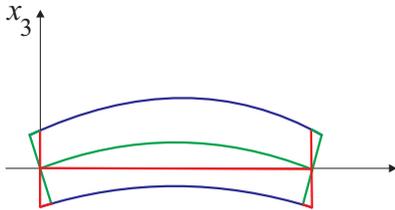
Figure 3.4: Comparison of cross-sections of prismatic and standard shells
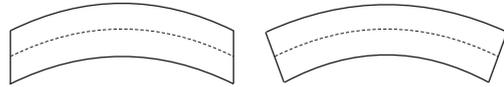
Figure 3.5: Cross-sections of a prismatic (left) and a standard shell with the same mid-surface

In the case of the symmetric prismatic shell, i.e., when

$$\overset{(-)}{h}(x_1, x_2) = -\overset{(+)}{h}(x_1, x_2),$$

evidently

$$2\tilde{h}(x_1, x_2) \equiv 0, \quad (x_1, x_2) \in \omega.$$

Distinctions between the prismatic shell of a constant thickness and the standard shell of a constant thickness are shown in the Figures 3.4, 3.5, where cross-sections of the prismatic shell of a constant thickness with its projection and of the standard shell of a constant thickness with its middle surface are given in red and green colors, respectively, with common parts in blue. In other words, the lateral boundary of the standard shell is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell's projection on $x_3 = 0$ (see [4]).

In particular, let ω be a domain bounded by a sufficiently smooth arc $(\partial\omega \setminus \overline{\gamma^0})$ lying in the half -plane $x_2 > 0$ and a segment $\overline{\gamma^0}$ of the x_1−axis ($x_2 = 0$). Let the thickness look like (see Figures 3.2, 3.3)

$$2h(x_1, x_2) = 2h_0 x_2^\kappa, \qquad h_0, \ \kappa = const > 0, \tag{3.1}$$

which corresponds to the case

$$\overset{(\pm)}{h}(x_1, x_2) = \overset{(\pm)}{h_0} x_2^\kappa, \quad \overset{(\pm)}{h_0} = const, \quad \overset{(+)}{h_0} > \overset{(-)}{h_0}, \quad 2h_0 := \overset{(+)}{h_0} - \overset{(-)}{h_0}.$$

In this case we have to do with a blunt edge for $\kappa < 1$ and with a sharp edge for $\kappa \geq 1$, respectively.

In Figures 3.6-3.20 ($\hat{\varphi}$ is the angle at the cusp between tangents $\overset{(+)}{T}$ and $\overset{(-)}{T}$, ν is an inward normal at O to $\partial\omega$) we show some characteristic (typical) profiles (cross-sections) of cusped prismatic shells (see also figures in [1], [11], [4]).

First we consider the general case of ω and of positive thickness. In such a case the prismatic shell under consideration has not cusped edges.

Figure 3.6: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} = \frac{\pi}{2}$). It has a Lipschitz boundary

Figure 3.7: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} \in]0, \frac{\pi}{2}[$). It has a Lipschitz boundary

Figure 3.8: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} = 0$). It has a non-Lipschitz boundary
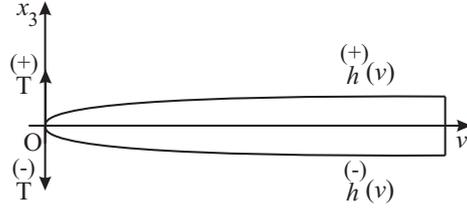
Figure 3.9: A cross-section of a blunt cusped plate ($\hat{\varphi} = \pi$). It has a Lipschitz boundary
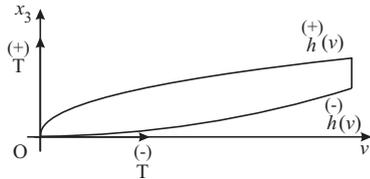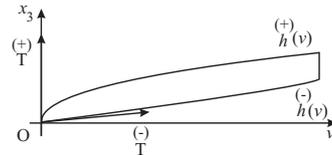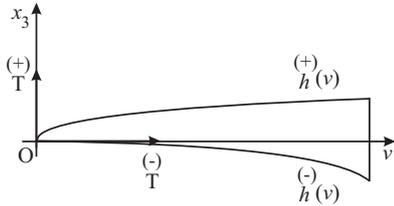

Figure 3.10: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} = \frac{\pi}{2}$). It has a Lipschitz boundary
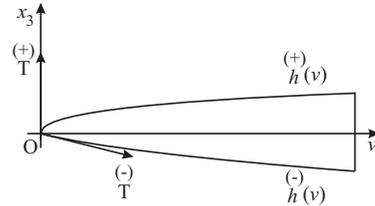
Figure 3.11: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} \in]\frac{\pi}{2}, \pi[$). It has a Lipschitz boundary
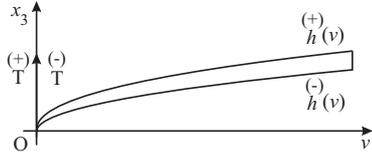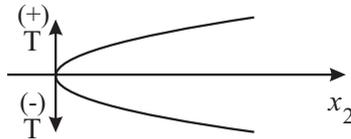

Figure 3.12: $\hat{\varphi} = \pi$
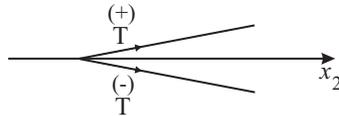
Figure 3.13: Wedge, $\hat{\varphi} \in]0, \pi[$
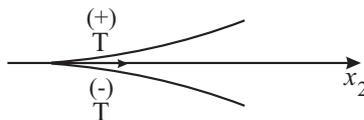
Figure 3.14: $\hat{\varphi} = 0$
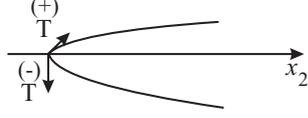
Figure 3.15: $\frac{\pi}{2} < \hat{\varphi} < \pi$

Figure 3.16: $\frac{\pi}{2} < \hat{\varphi} < \pi$
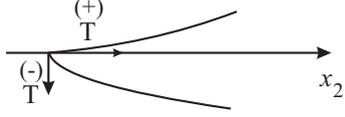
Figure 3.17: $\hat{\varphi} = \frac{\pi}{2}$

Figure 3.18: $0 < \hat{\varphi} < \frac{\pi}{2}$
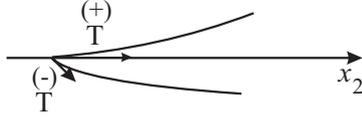
Figure 3.19: $0 < \hat{\varphi} < \frac{\pi}{2}$
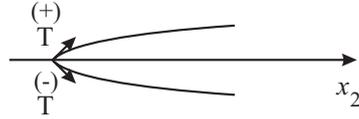
Figure 3.20: $0 < \hat{\varphi} < \pi$

rth order moments of the following quantities are defined as the integrals

$$\Big(u_{ir},\, X_{ijr},\, e_{ijr},\, \Phi_{jr},\, H_{ir},\, H_{0r},\, \varphi_r,\, \mathcal{F}_r,\, D_{jr},\, B_{jr}, \chi_r, \eta_r, f_{er} \Big)(x_1, x_2, t)$$

$$:= \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \Big(u_i,\, X_{ij},\, e_{ij},\, \Phi_j,\, H_i,\, H_0, \varphi, \mathcal{F}, D_j, B_j, \chi, \eta, f_e \Big)(x_1, x_2, x_3, t)$$

$$\times P_r(ax_3 - b)\ dx_3, \qquad i,j = \overline{1,3}, \quad (3.2)$$

where

$$P_r(ax_3 - b) \quad \left(a(x_1,x_2) := \frac{2}{\overset{(+)}{h} - \overset{(-)}{h}} = \frac{1}{h}, \ \ b(x_1,x_2) := \frac{\overset{(+)}{h} + \overset{(-)}{h}}{\overset{(+)}{h} - \overset{(-)}{h}} = \frac{\tilde{h}}{h} \right),$$

$$r = 0, 1, \cdots,$$

are the rth order Legendre polynomials.

Under the well-know restrictions (see, e.g., [1]) the following Fourier-Legendre series

$$\Big(u_i, X_{ij}, e_{ij}, \Phi_j, H_i, H_0, \varphi, \mathcal{F}, D_j, B_j, \chi, \eta, f_e \Big)(x_1, x_2, x_3, t) = \sum_{r=0}^{\infty} a\Big(r + \frac{1}{2}\Big)$$

$$\times \Big(u_{ir}, X_{ijr}, e_{ijr}, \Phi_{jr}, H_{ir}, H_{0r}, \varphi_r, \mathcal{F}_r, D_{jr}, B_{jr}, \chi_r, \eta_r, f_{er} \Big)(x_1, x_2, t) \qquad (3.3)$$

$$\times P_r(ax_3 - b)$$

14

are convergent.

Therefore on the upper and lower face surfaces of the prismatic shell under consideration

$$\overset{(\pm)}{u}_i := u_i(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2), t) = \sum_{s=0}^{\infty} a\left(s + \frac{1}{2}\right) u_{is}(\pm 1)^s$$

$$= \sum_{s=0}^{\infty} \frac{(\pm 1)^s (2s+1)}{2h} u_{is},$$

whence

$$\overset{(+)}{u}_i - (-1)^r \overset{(-)}{u}_i = -\sum_{s=0}^{\infty} \overset{r}{a}_{3s} u_{is}, \quad i = \overline{1,3},$$

$$\overset{(+)}{u}_i \overset{(+)}{h}_{,\alpha} - (-1)^r \overset{(-)}{u}_i \overset{(-)}{h}_{,\alpha} = \sum_{s=0}^{\infty} \overset{r}{a}^*_{\alpha s} u_{is}, \quad i = \overline{1,3}, \quad \alpha = 1,2,$$

where

$$\overset{r}{a}^*_{\alpha s} = \overset{r}{a}_{\alpha s}, \quad s \neq r, \quad \overset{r}{a}^*_{\alpha \underline{r}} = (2r+1)\frac{h_{,\alpha}}{h},$$

$$\overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r, \quad \overset{r}{a}_{\alpha \underline{r}} := r\frac{h_{,\alpha}}{h},$$

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}.$$

$$\Bigg(\overset{r}{b}_{js} := -\overset{r}{a}_{js}, \quad s > r; \quad \overset{r}{b}_{js} = 0, \quad s < r;$$

$$\overset{r}{b}_{\alpha \underline{r}} := \overset{r}{a}_{\alpha \underline{r}} - \overset{r}{a}^*_{\alpha \underline{r}} = -(r+1)\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2h}, \quad \overset{r}{b}_{3\underline{r}} = 0 \Bigg).$$

Using (see formulas (10.4), (10.5) of Section 10)

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b)f_{,\alpha}\, dx_3 = f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,\alpha},$$

$$\alpha = 1,2,$$

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b)f_{,3}\, dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s + \overset{(+)}{f} - (-1)^r \overset{(-)}{f},$$

15

from (2.1)-(2.3), after multiplying them by $P_r(ax_3 - b)$ for $r = 0, 1, \cdots$, and then integrating within the limits $\overset{(-)}{h}(x_1, x_2)$ and $\overset{(+)}{h}(x_1, x_2)$ with respect to the thickness variable x_3, we obtain the following formulas in ω:

$$X_{\alpha i r, \alpha} + \sum_{s=0}^{r} \overset{r}{a}_{js} X_{jis} + \overset{r}{X}_i = \rho \frac{\partial^2 u_{ir}}{\partial t^2}, \quad i = \overline{1,3}, \ r = 0, 1, \cdots, \tag{3.4}$$

$$H_{\alpha r, \alpha} + \sum_{s=0}^{r} \overset{r}{a}_{is} H_{is} + H_{0r} + \overset{r}{H} = \rho k \frac{\partial^2 \varphi_r}{\partial t^2}, \quad r = 0, 1, \cdots, \tag{3.5}$$

$$D_{\alpha r, \alpha} + \sum_{s=0}^{r} \overset{r}{a}_{is} D_{is} + \overset{r}{D} = f_{er}, \quad r = 0, 1, \cdots, \tag{3.6}$$

$$B_{\alpha r, \alpha} + \sum_{s=0}^{r} \overset{r}{a}_{is} B_{is} + \overset{r}{B} = 0, \quad r = 0, 1, \cdots, \tag{3.7}$$

where

$$\overset{r}{X}_j := \overset{(+)}{X}_{3j} - \overset{(+)}{X}_{\alpha j} \overset{(+)}{h}_{,\alpha} + (-1)^r \left[-\overset{(-)}{X}_{3j} + \overset{(-)}{X}_{\alpha j} \overset{(-)}{h}_{,\alpha} \right] + \Phi_{jr}$$

$$= X_{\overset{(+)}{n}_j} \sqrt{1 + \left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2} + (-1)^r X_{\overset{(-)}{n}_j} \sqrt{1 + \left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2} + \Phi_{jr},$$

$$j = \overline{1,3}, \ r = 0, 1, 2, \cdots,$$

$X_{\overset{(+)}{n}_j}$ and $X_{\overset{(-)}{n}_j}$ are components of the stress vectors acting on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively,

$$\overset{r}{H} := \overset{(+)}{H}_3 - \overset{(+)}{H}_\alpha \overset{(+)}{h}_{,\alpha} + (-1)^r \left[-\overset{(-)}{H}_3 + \overset{(-)}{H}_\alpha \overset{(-)}{h}_{,\alpha} \right] + \mathcal{F}_r$$

$$= \overset{(+)}{H}_i \overset{(+)}{n}_i \sqrt{1 + \left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2} + (-1)^r \overset{(-)}{H}_i \overset{(-)}{n}_i \sqrt{1 + \left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2} + \mathcal{F}_r,$$

$$r = 0, 1, \cdots,$$

$\overset{(+)}{H}_j$ and $\overset{(-)}{H}_j$ are components of the equilibrated stress vectors on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively,

$$\overset{r}{D} := \overset{(+)}{D}_3 - \overset{(+)}{D}_\gamma \overset{(+)}{h}_{,\gamma} + (-1)^r \left[-\overset{(-)}{D}_3 + \overset{(-)}{D}_\gamma \overset{(-)}{h}_\gamma \right]$$

$$= \overset{(+)}{D}_i \overset{(+)}{n}_i \sqrt{1 + (\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2} + \overset{(-)}{D}_i \overset{(-)}{n}_i \sqrt{1 + (\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2},$$

16

$$\overset{r}{B} := \overset{(+)}{B_3} - \overset{(+)}{B_\gamma}\overset{(+)}{h_{,\gamma}} + (-1)^r\Big[-\overset{(-)}{B_3} + \overset{(-)}{B_\gamma}\overset{(-)}{h_\gamma}\Big]$$

$$= \overset{(+)}{B_i}\overset{(+)}{n_i}\sqrt{1 + (\overset{(+)}{h_{,1}})^2 + (\overset{(+)}{h_{,2}})^2} + \overset{(-)}{B_i}\overset{(-)}{n_i}\sqrt{1 + (\overset{(-)}{h_{,1}})^2 + (\overset{(-)}{h_{,2}})^2}$$

(in the above calculations we have used formulas (10.13) and (10.14) of Section 10 corresponding for $X_{\alpha i}$ and X_{3i}, H_α and H_3, D_α and D_3, B_α and B_3 instead of f).

Using (10.5), (10.6) of Section 10 for u_i instead of f and the Fourier-Legendre expansions of u_i on the upper and lower face surfaces

$$\overset{(\pm)}{u}_i = \sum_{s=0}^{\infty}\frac{(\pm 1)^s(2s+1)}{2h}u_{is},\quad i = \overline{1,3},$$

from (2.4), similarly to (10.10), (10.11), we obtain

$$e_{ijr} = \frac{1}{2}\Big(u_{ir,j} + u_{jr,i}\Big) + \frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b_{is}}u_{js} + \frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b_{js}}u_{is}, \tag{3.8}$$

$$i,j = \overline{1,3},\quad r = 0,1,\cdots,$$

$$\theta_r := e_{iir} = u_{\gamma r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b_{ks}}u_{ks},\quad r = 0,1,\cdots.$$

In view of

$$\overset{r}{b_{3r}} = 0,\quad h^{r+1}(h^{-r-1})_{,\alpha} = \overset{r}{b_{\alpha\underline{r}}},\quad \alpha = 1,2,$$

we can rewrite (3.8) for

$$v_{ir} := h^{-r-1}u_{ir}$$

as follows

$$e_{ijr} = \frac{1}{2}h^{r+1}\Big(v_{ir,j} + v_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{\infty}h^{s+1}\Big(\overset{r}{b_{is}}v_{js} + \overset{r}{b_{js}}v_{is}\Big), \tag{3.9}$$

$$i,j = \overline{1,3},\ r = 0,1,\cdots,$$

$$\theta_r := e_{iir} = h^{r+1}v_{\gamma r,\gamma} + \sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b_{ks}}v_{ks},\quad r = 0,1,\cdots. \tag{3.10}$$

If we apply formulas (10.11), (10.12) of Section 10, using the Fourier-Legendre expansions of χ and η on the upper and lower face surfaces

$$\overset{(\pm)}{\chi} = \sum_{s=0}^{\infty}\frac{(\pm 1)^s(2s+1)}{2h}\chi_s,$$

17

$$\overset{(\pm)}{\eta} = \sum_{s=0}^{\infty} \frac{(\pm 1)^s (2s+1)}{2h} \eta_s,$$

from (2.5) we obtain

$$X_{ijr} = E_{ijkl} e_{klr} + E_{ijkl}^* \dot{e}_{klr} + \tilde{b}_{ij}\varphi_r + b_{ij}^*\dot{\varphi}_r + d_{ij\gamma}\Big(\varphi_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\varphi_s\Big)$$

$$-d_{ij3}\sum_{s=r+1}^{\infty} \overset{r}{a}_{\gamma s}\varphi_s + d_{ij\gamma}^*\Big(\dot{\varphi}_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\dot{\varphi}_s\Big) - d_{ij3}^*\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\dot{\varphi}_s$$

$$+p_{\gamma ij}(\chi_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\chi_s) - p_{3ij}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\chi_s + p_{\gamma ij}^*\Big(\dot{\chi}_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\dot{\chi}_s\Big)$$

$$-p_{3ij}^*\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\dot{\chi}_s + q_{\gamma ij}(\eta_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\eta_s) - q_{3ij}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\eta_s$$

$$+q_{\gamma ij}^*(\dot{\eta}_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\dot{\eta}_s) - q_{3ij}^*\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\dot{\eta}_s, \quad i,j=\overline{1,3}, \quad r=0,1,\cdots. \quad (3.11)$$

Therefore, by virtue of (3.8),

$$X_{ijr} = \frac{1}{2}E_{ijkl}\Big(u_{kr,l} + u_{lr,k}\Big) + \frac{1}{2}E_{ijkl}\sum_{s=r}^{\infty}\Big(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks}\Big)$$

$$+\frac{1}{2}E_{ijkl}^*\Big(\dot{u}_{kr,l} + \dot{u}_{lr,k}\Big) + \frac{1}{2}E_{ijkl}^*\sum_{s=r}^{\infty}\Big(\overset{r}{b}_{ks}\dot{u}_{ls} + \overset{r}{b}_{ls}\dot{u}_{ks}\Big)$$

$$+\tilde{b}_{ij}\varphi_r + b_{ij}^*\dot{\varphi}_r + d_{ij\gamma}\Big(\varphi_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\varphi_s\Big)$$

$$-d_{ij3}\sum_{s=r+1}^{\infty} \overset{r}{a}_{\gamma s}\varphi_s + d_{ij\gamma}^*\Big(\dot{\varphi}_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\dot{\varphi}_s\Big) - d_{ij3}^*\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\dot{\varphi}_s$$

$$+p_{\gamma ij}\Big(\chi_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\chi_s\Big) - p_{3ij}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\chi_s + p_{\gamma ij}^*\Big(\dot{\chi}_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\dot{\chi}_s\Big)$$

$$-p_{3ij}^*\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\dot{\chi}_s + q_{\gamma ij}\Big(\eta_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\eta_s\Big) - q_{3ij}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\eta_s$$

$$+q_{\gamma ij}^*(\dot{\eta}_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\dot{\eta}_s) - q_{3ij}^*\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\dot{\eta}_s, \quad i,j=\overline{1,3}, \quad r=0,1,\cdots. \quad (3.12)$$

Let

$$v_r := \frac{u_r}{h^{r+1}}, \quad \psi_r := \frac{\varphi_r}{h^{r+1}}, \quad \widetilde{\chi}_r := \frac{\chi_r}{h^{r+1}}, \quad \widetilde{\eta}_r := \frac{\eta_r}{h^{r+1}}. \quad (3.13)$$

18

Substituting (3.9) into (3.11), and taking into account (3.13), it follows that

$$X_{ijr} = \frac{1}{2}E_{ijkl}h^{r+1}\Big(v_{kr,l} + v_{lr,k}\Big) + \frac{1}{2}E_{ijkl}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{ks}v_{ls} + \overset{r}{b}_{ls}v_{ks}\Big)$$

$$+\frac{1}{2}E_{ijkl}^{*}h^{r+1}\Big(\dot{v}_{kr,l} + \dot{v}_{lr,k}\Big) + \frac{1}{2}E_{ijkl}^{*}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{ks}\dot{v}_{ls} + \overset{r}{b}_{ls}\dot{v}_{ks}\Big)$$

$$+\tilde{b}_{ij}h^{r+1}\psi_r + b_{ij}^{*}h^{r+1}\dot{\psi}_r + d_{ij\gamma}h^{r+1}\psi_{r,\gamma} + d_{ijk}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\psi_s + d_{ij\gamma}^{*}h^{r+1}\dot{\psi}_{r,\gamma}$$

$$+d_{ijk}^{*}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\dot{\psi}_s + p_{\gamma ij}h^{r+1}\widetilde{\chi}_{r,\gamma} + p_{kij}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\widetilde{\chi}_s$$

$$+p_{\gamma ij}^{*}h^{r+1}\dot{\widetilde{\chi}}_{r,\gamma} + p_{kij}^{*}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\dot{\widetilde{\chi}}_s + q_{\gamma ij}h^{r+1}\widetilde{\eta}_{r,\gamma} + q_{kij}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\widetilde{\eta}_s$$

$$+q_{\gamma ij}^{*}h^{r+1}\dot{\widetilde{\eta}}_{r,\gamma} + q_{kij}^{*}\sum_{s=r+1}^{\infty} h^{s+1}\,\overset{r}{b}_{ks}\dot{\widetilde{\eta}}_s, \quad i,j = \overline{1,3}, \quad r = 0,1,\cdots \tag{3.14}$$

(because of

$$(h^{r+1}\tilde{\chi}_r)_{,\gamma} - h^{r+1}(r+1)\frac{h_{,\alpha}}{h}\tilde{\chi}_r = h^{r+1}\tilde{\chi}_{r,\gamma},$$

and the similar formulas for ψ and $\tilde{\eta}$).

Analogously, from (2.6) we have (compare with (10.11), (10.12))

$$H_{jr} = \frac{1}{2}d_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}d_{klj}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks})$$

$$+\frac{1}{2}d_{klj}^{*}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}d_{klj}^{*}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks}) + d_j\varphi_r + d_j^{*}\dot{\varphi}_r$$

$$+\tilde{\alpha}_{j\beta}\Big[\varphi_{r,\beta} + \sum_{s=0}^{r} \overset{r}{a}_{\beta s}\varphi_s - \overset{(+)}{\varphi}\overset{(+)}{h}_{,\beta} + (-1)^r\overset{(-)}{\varphi}\overset{(-)}{h}_{,\beta}\Big]$$

$$+\tilde{\alpha}_{j3}\Big[\sum_{s=0}^{r} \overset{r}{a}_{3s}\varphi_s + \overset{(+)}{\varphi} - (-1)^r\overset{(-)}{\varphi}\Big]$$

$$+\alpha_{j\beta}^{*}\Big[\dot{\varphi}_{r,\beta} + \sum_{s=0}^{r} \overset{r}{a}_{\beta s}\dot{\varphi}_s - \overset{(+)}{\dot{\varphi}}\overset{(+)}{h}_{,\alpha} - (-1)^r\overset{(-)}{\dot{\varphi}}\overset{(-)}{h}_{,\alpha}\Big]$$

$$+\alpha_{j3}^{*}\Big[\sum_{s=0}^{r} \overset{r}{a}_{3s}\dot{\varphi}_s + \overset{(+)}{\dot{\varphi}} - (-1)^r\overset{(-)}{\dot{\varphi}}\Big], \quad j = \overline{1,3},$$

and substituting here the corresponding Fourier-Legendre expansions of φ on

the upper and lower face surfaces

$$\overset{(\pm)}{\varphi} = \sum_{s=0}^{\infty} \frac{(\pm 1)^s (2s+1)}{2h} \varphi_s,$$

we get

$$H_{jr} = \frac{1}{2} d_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2} d_{klj} \sum_{s=r}^{\infty} (\overset{r}{b}_{ks} u_{ls} + \overset{r}{b}_{ls} u_{ks})$$

$$+ \frac{1}{2} d^*_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2} d^*_{klj} \sum_{s=r}^{\infty} (\overset{r}{b}_{ks} u_{ls} + \overset{r}{b}_{ls} u_{ks}) + d_j \varphi_r + d^*_j \dot{\varphi}_r$$

$$+ \tilde{\alpha}_{jk} \Big(\varphi_{r,k} + \sum_{s=r}^{\infty} \overset{r}{b}_{ks} \varphi_s \Big) + \alpha^*_{jk} \Big(\dot{\varphi}_{r,k} + \sum_{s=r}^{\infty} \overset{r}{b}_{ks} \dot{\varphi}_s \Big), \ j = \overline{1,3}, \ r = 0, 1, \cdots, \ (3.15)$$

i.e. (see (3.13))

$$H_{jr} = \frac{1}{2} d_{klj} h^{r+1} (v_{kr,l} + v_{lr,k}) + \frac{1}{2} d_{klj} \sum_{s=r+1}^{\infty} h^{s+1} (\overset{r}{b}_{ks} v_{hs} + \overset{r}{b}_{ls} v_{ks})$$

$$+ \frac{1}{2} d^*_{klj} h^{r+1} (\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2} d^*_{klj} \sum_{s=r+1}^{\infty} h^{s+1} (\overset{r}{b}_{ks} \dot{v}_{hs} + \overset{r}{b}_{ls} \dot{v}_{ks})$$

$$+ d_j h^{r+1} \psi_r + d^*_j h^{r+1} \dot{\psi}_r$$

$$+ \tilde{\alpha}_{ji} \Big(h^{r+1} \psi_{r,i} + \sum_{s=r+1}^{\infty} h^{s+1} \overset{r}{b}_{is} \psi_s \Big) + \alpha^*_{ji} \Big(h^{r+1} \dot{\psi}_{r,i} + \sum_{s=r+1}^{\infty} h^{s+1} \overset{r}{b}_{is} \dot{\psi}_s \Big), \ (3.16)$$

$$j = \overline{1,3}, \ r = 0, 1 \cdots.$$

From (2.7), on account of combined (10.11), (10.12), evidently,

$$H_{0r} = -d_i (\varphi_{r,i} + \sum_{s=r}^{\infty} \overset{r}{b}_{is} \varphi_s) - \tilde{b}_{ij} e_{ijr} - \tilde{\xi} \varphi_r$$

$$- d^* (\dot{\varphi}_{r,i} + \sum_{s=r}^{\infty} \overset{r}{b}_{is} \dot{\varphi}_s) - b^*_{ij} \dot{e}_{ijr} - \xi^* \dot{\varphi}_r, \quad r = 0, 1, \cdots$$

and, in view of (3.8),

$$H_{0r} = -d_i (\varphi_{r,i} + \sum_{s=r}^{\infty} \overset{r}{b}_{is} \varphi_s)$$

$$- \tilde{b}_{ij} \Big[\frac{1}{2} \Big(u_{ir,j} + u_{jr,i} \Big) + \frac{1}{2} \sum_{s=r}^{\infty} \overset{r}{b}_{is} u_{js} + \frac{1}{2} \sum_{s=r}^{\infty} \overset{r}{b}_{js} u_{is} \Big]$$

$$-\tilde{\xi}\varphi_r - d^*\Big(\dot{\varphi}_{r,i} + \sum_{s=r}^{\infty} \overset{r}{b}_{is}\dot{\varphi}_s\Big)$$

$$-b_{ij}^*\Big[\frac{1}{2}\Big(\dot{u}_{ir,j} + \dot{u}_{jr,i}\Big) + \frac{1}{2}\sum_{s=r}^{\infty} \overset{r}{b}_{is}\dot{u}_{js} + \frac{1}{2}\sum_{s=r}^{\infty} \overset{r}{b}_{js}\dot{u}_{is}\Big] - \xi^*\dot{\varphi}_r, \qquad (3.17)$$

while, by virtue of (3.9) and (3.13),

$$H_{0r} = -\tilde{b}_{ij}\Big[\frac{1}{2}h^{r+1}\Big(v_{ir,j} + v_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{is}v_{js} + \overset{r}{b}_{js}v_{is}\Big)\Big]$$

$$-\tilde{\xi}h^{r+1}\psi_r - d_i\Big(h^{r+1}\psi_{r,i} + \sum_{s=r+1}^{\infty} \overset{r}{b}_{is}\psi_s\Big)$$

$$-b_{ij}^*\Big[\frac{1}{2}h^{r+1}\Big(\dot{v}_{ir,j} + \dot{v}_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{is}\dot{v}_{js} + \overset{r}{b}_{js}\dot{v}_{is}\Big)\Big]$$

$$-\xi^*h^{r+1}\dot{\psi}_r - d_i^*\Big(h^{r+1}\dot{\psi}_{r,i} + \sum_{s=r+1}^{\infty} \overset{r}{b}_{is}\dot{\psi}_s\Big), \quad r = 0, 1, \cdots. \qquad (3.18)$$

Similarly, from (2.8) it follows

$$D_{jr} = p_{jkl}e_{klr} + p_{jkl}^*\dot{e}_{klr} - \varsigma_{j\gamma}\Big(\chi_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\chi_s\Big) + \varsigma_{j3}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\chi_s$$

$$-\tilde{a}_{j\gamma}\Big(\eta_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\eta_s\Big) + \tilde{a}_{j3}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\eta_s, \quad j = \overline{1,3}, \quad r = 0, 1, \cdots,$$

i.e., in view of (3.8),

$$D_{jr} = \frac{1}{2}p_{jkl}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}p_{jkl}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks})$$

$$+\frac{1}{2}p_{jkl}^*(\dot{u}_{kr,l} + \dot{u}_{lr,k}) + \frac{1}{2}p_{jkl}^*\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}\dot{u}_{ls} + \overset{r}{b}_{ls}\dot{u}_{ks})$$

$$-\varsigma_{j\gamma}\Big(\chi_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\chi_s\Big) + \varsigma_{j3}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\chi_s$$

$$-\tilde{a}_{j\gamma}\Big(\eta_{r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{\gamma s}\eta_s\Big) + \tilde{a}_{j3}\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s}\eta_s, \quad j = \overline{1,3}, \quad r = 0, 1, \cdots, (3.19)$$

while by virtue of (3.9) and (3.13), we have

$$D_{jr} = \frac{1}{2}p_{jkl}h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}p_{jkl}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}v_{ls} + \overset{r}{b}_{ls}v_{ks})$$

$$+\frac{1}{2}p_{jkl}^{*}h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}p_{jkl}^{*}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{ls} + \overset{r}{b}_{ls}\dot{v}_{ks})$$

$$-\varsigma_{j\gamma}\Big(h^{r+1}\widetilde{\mathcal{X}}_{r,\gamma} + \sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\mathcal{X}}_{s}\Big) + \varsigma_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\mathcal{X}}_{s}$$

$$-\tilde{a}_{j\gamma}\Big(h^{r+1}\widetilde{\eta}_{r,\gamma} + \sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\eta}_{s}\Big) + \tilde{a}_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\eta}_{s}, \qquad (3.20)$$

$$j = \overline{1,3}, \quad r = 0, 1, \cdots.$$

In the same way from (2.9) we get

$$B_{jr} = \frac{1}{2}q_{jkl}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}q_{jkl}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks})$$

$$+\frac{1}{2}q_{jkl}^{*}(\dot{u}_{kr,l} + \dot{u}_{lr,k}) + \frac{1}{2}q_{jkl}^{*}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}\dot{u}_{ls} + \overset{r}{b}_{ls}\dot{u}_{ks})$$

$$-\tilde{a}_{j\gamma}\Big(\mathcal{X}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\mathcal{X}_{s}\Big) + \tilde{a}_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\mathcal{X}_{s}$$

$$-\xi_{j\gamma}\Big(\eta_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_{s}\Big) + \xi_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_{s}, \quad j = \overline{1,3}, \quad r = 0, 1, \cdots, \quad (3.21)$$

i.e.

$$B_{jr} = \frac{1}{2}q_{jkl}h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}q_{jkl}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}v_{ls} + \overset{r}{b}_{ls}v_{ks})$$

$$+\frac{1}{2}q_{jkl}^{*}h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}q_{jkl}^{*}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{ls} + \overset{r}{b}_{ls}\dot{v}_{ks})$$

$$-\tilde{a}_{j\gamma}\Big(h^{r+1}\widetilde{\mathcal{X}}_{r,\gamma} + \sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\mathcal{X}}_{s}\Big) + \tilde{a}_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\mathcal{X}}_{s}$$

$$-\xi_{j\gamma}\Big(h^{r+1}\widetilde{\eta}_{r,\gamma} + \sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\eta}_{s}\Big) + \xi_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\eta}_{s}, \qquad (3.22)$$

$$j = \overline{1,3}, \ r = 0, 1, \cdots;$$

We remind that

$$\overset{r}{b}_{\alpha\underline{r}} := -(r+1)\frac{h_{,\alpha}}{h}, \quad \overset{r}{b}_{3\underline{r}} = 0,$$

22

$$
\overset{r}{b}_{js} := \begin{cases} 0, & s < r, \\[2mm] -\overset{r}{a}_{\alpha s} = -(2s+1)\dfrac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+\underline{s}}\overset{(-)}{h}_{,\alpha}}{2h}, & j = \alpha, \ s > r, \\[4mm] -\overset{r}{a}_{3s} = (2s+1)\dfrac{1-(-1)^{\underline{s}+r}}{2h}, & j = 3, \ s > r, \end{cases}
$$

$$
\alpha = 1,2, \quad j = \overline{1,3}, \quad r,s = 0,1,2,\cdots .
$$

So, we get the equivalent[i] to (2.1)-(2.9), infinite system (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21), (3.8) with respect to the so called r-th order moments X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, φ_r, χ_r, η_r. Then, substituting (3.12) into (3.4), expressions (3.15) and (3.17) into (3.5); expressions (3.19) and (3.21) into (3.6) and (3.7), respectively, we construct an equivalent infinite system with respect to the r-th order moments u_{ir}, φ_r, χ_r, η_r. Namely,

$$
\frac{1}{2}\Big(E_{\alpha ik\delta}u_{kr,\delta}\Big)_{,\alpha} + \frac{1}{2}\Big(E_{\alpha i\gamma l}u_{lr,\gamma}\Big)_{,\alpha} + \frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}u_{ls}\Big)_{,\alpha}
$$

$$
+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r}^{\infty}\overset{r}{b}_{ls}u_{ks}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ik\delta}\dot{u}_{kr,\delta}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha i\gamma l}\dot{u}_{lr,\gamma}\Big)_{,\alpha}
$$

$$
+\frac{1}{2}\Big(E^*_{\alpha ikl}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\dot{u}_{ls}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ikl}\sum_{s=r}^{\infty}\overset{r}{b}_{ls}\dot{u}_{ks}\Big)_{,\alpha} + \Big(\tilde{b}_{\alpha i}\varphi_r\Big)_{,\alpha}
$$

$$
+\Big(b^*_{\alpha i}\dot{\varphi}_r\Big)_{,\alpha} + \Big(d_{\alpha i\gamma}\varphi_{r,\gamma}\Big)_{,\alpha} + \Big(d_{\alpha i\gamma}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\varphi_s\Big)_{,\alpha} - \Big(d_{\alpha i3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\varphi_s\Big)_{,\alpha}
$$

$$
+\Big(d^*_{\alpha i\gamma}\dot{\varphi}_{r,\gamma}\Big)_{,\alpha} + \Big(d^*_{\alpha i\gamma}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\varphi}_s\Big)_{,\alpha} - \Big(d^*_{\alpha i3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\varphi}_s\Big)_{,\alpha}
$$

$$
+\Big(p_{\gamma\alpha i}\chi_{r,\gamma}\Big)_{,\alpha} + \Big(p_{\gamma\alpha i}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big)_{,\alpha} - \Big(p_{3\alpha i}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s\Big)_{,\alpha}
$$

$$
+\Big(p^*_{\gamma\alpha i}\dot{\chi}_{r,\gamma}\Big)_{,\alpha} + \Big(p^*_{\gamma\alpha i}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\chi}_s\Big)_{,\alpha} - \Big(p^*_{3\alpha i}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\chi}_s\Big)_{,\alpha}
$$

$$
+\Big(q_{\gamma\alpha i}\eta_{r,\gamma}\Big)_{,\alpha} + \Big(q_{\gamma\alpha i}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big)_{,\alpha} - \Big(q_{3\alpha i}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s\Big)_{,\alpha}
$$

$$
+\Big(q^*_{\gamma\alpha i}\dot{\eta}_{r,\gamma}\Big)_{,\alpha} + \Big(q^*_{\gamma\alpha i}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\eta}_s\Big)_{,\alpha} - \Big(q^*_{3\alpha i}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\eta}_s\Big)_{,\alpha}
$$

---

[i]in the following sense: if X_{ij}, e_{ij}, u_i, H_i, H_0, D_j, B_j, χ, η, and φ satisfy the relations (2.1)-(2.9), then constructed by (3.2) functions X_{ijr}, e_{ijr}, u_{ir}, H_{ir}, H_{0r}, D_{jr}, B_{jr}, χ_r, η_r, and φ_r will satisfy the infinite relations (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21), (3.8) and, vice versa, if X_{ijr}, e_{ijr}, u_{ir}, H_{ir}, H_{0r}, D_{jr}, B_{jr}, χ_r, η_r, and φ_r satisfy the infinite relations (3.4)-(3.8), (3.12), (3.15), (3.17), (3.19), (3.21), (3.8), then constructed by means of (3.3) functions X_{ij}, e_{ij}, u_i, H_i, H_0, D_j, B_j, χ, η and φ will satisfy the relations (2.1)-(2.9).

$$+\sum_{s=0}^{r} \overset{r}{a}_{js}\left[\frac{1}{2}E_{jikl}\Big(u_{ks,l}+u_{ls,k}\Big)+\frac{1}{2}E_{jikl}\sum_{s'=s}^{\infty}\Big(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'}\Big)\right.$$

$$+\frac{1}{2}E_{jikl}^{*}\Big(\dot{u}_{ks,l}+\dot{u}_{ls,k}\Big)+\frac{1}{2}E_{jikl}^{*}\sum_{s'=s}^{\infty}\Big(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'}\Big)$$

$$+\tilde{b}_{ij}\varphi_{s}+b_{ij}^{*}\dot{\varphi}_{s}+d_{ji\gamma}\Big(\varphi_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\varphi_{s'}\Big)-d_{ji3}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\varphi_{s'}$$

$$+d_{ji\gamma}^{*}\Big(\dot{\varphi}_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\dot{\varphi}_{s'}\Big)-d_{ji3}^{*}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\dot{\varphi}_{s'}$$

$$+p_{\gamma ji}\Big(\chi_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\chi_{s'}\Big)-p_{3ij}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\chi_{s'}$$

$$+p_{\gamma ji}^{*}\Big(\dot{\chi}_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\dot{\chi}_{s'}\Big)-p_{3ij}^{*}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\dot{\chi}_{s'}$$

$$+q_{\gamma ji}\Big(\eta_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\eta_{s'}\Big)-q_{3ji}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\eta_{s'}$$

$$\left.+q_{\gamma ji}^{*}\Big(\dot{\eta}_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\dot{\eta}_{s'}\Big)-q_{3ji}^{*}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\dot{\eta}_{s'}\right]+\overset{r}{X}_{i}=\rho\frac{\partial^{2}u_{ir}}{\partial t^{2}},$$

$$i=\overline{1,3},\ \ r=0,1,\cdots,$$

$$\frac{1}{2}\Big(d_{kl\alpha}u_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}u_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}u_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r}^{\infty}\overset{r}{b}_{ls}u_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}^{*}\dot{u}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}^{*}\dot{u}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(d_{kl\alpha}^{*}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\dot{u}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}^{*}\sum_{s=r}^{\infty}\overset{r}{b}_{ls}\dot{u}_{ks}\Big)_{,\alpha}+\Big(d_{\alpha}\varphi_{r}\Big)_{,\alpha}+\Big(d_{\alpha}^{*}\dot{\varphi}_{r}\Big)_{,\alpha}$$

$$+\Big(\tilde{\alpha}_{\alpha k}\varphi_{r,k}\Big)_{,,\alpha}+\Big(\tilde{\alpha}_{\alpha k}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\varphi_{s}\Big)_{,\alpha}+\Big(\alpha_{\alpha k}^{*}\dot{\varphi}_{r,k}\Big)_{,\alpha}+\Big(\alpha_{\alpha k}^{*}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\dot{\varphi}_{s}\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\left[\frac{1}{2}d_{kli}\Big(u_{ks,l}+u_{ls,k}\Big)+\frac{1}{2}d_{kli}\sum_{s'=s}^{\infty}\Big(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'}\Big)\right.$$

$$+\frac{1}{2}d_{kli}^{*}\Big(\dot{u}_{ks,l}+\dot{u}_{ls,k}\Big)+\frac{1}{2}d_{kli}^{*}\sum_{s'=s}^{\infty}\Big(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'}\Big)+d_{i}\varphi_{s}+d_{i}^{*}\dot{\varphi}_{s}$$

$$\left.+\tilde{\alpha}_{ik}\Big(\varphi_{s,k}+\sum_{s'=s}^{\infty}\overset{s}{b}_{ks'}\varphi_{s'}\Big)+\alpha_{ik}^{*}\Big(\dot{\varphi}_{s,k}+\sum_{s'=s}^{\infty}\overset{s}{b}_{ks'}\dot{\varphi}_{s'}\Big)\right]$$

$$-\tilde{b}_{ij}\Big[\frac{1}{2}\Big(u_{ir,j}+u_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{is}u_{js}+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{js}u_{is}\Big]-\tilde{\xi}\varphi_r$$

$$-d_i\Big(\varphi_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\varphi_s\Big)-b_{ij}^*\Big[\frac{1}{2}\Big(\dot{u}_{ir,j}+\dot{u}_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{is}\dot{u}_{js}+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{js}\dot{u}_{is}\Big]$$

$$-\xi^*\dot{\varphi}_r-d_i^*\Big(\dot{\varphi}_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\dot{\varphi}_s\Big)+\overset{r}{H}=\rho k\frac{\partial^2\varphi_r}{\partial t^2},\quad r=0,1,\cdots,$$

$$\frac{1}{2}\Big(p_{\alpha kl}u_{kr,l}\Big){}_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}u_{lr,k}\Big){}_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}u_{ls}\Big){}_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r}^{\infty}\overset{r}{b}_{ls}u_{ks}\Big){}_{,\alpha}$$

$$+\frac{1}{2}\Big(p_{\alpha kl}^*\dot{u}_{kr,l}\Big){}_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}^*\dot{u}_{lr,k}\Big){}_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}^*\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\dot{u}_{ls}\Big){}_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}^*\sum_{s=r}^{\infty}\overset{r}{b}_{ls}\dot{u}_{ks}\Big){}_{,\alpha}$$

$$-\Big(\varsigma_{\alpha\gamma}\chi_{r,\gamma}\Big){}_{,\alpha}-\Big(\varsigma_{\alpha\gamma}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big){}_{,\alpha}+\Big(\varsigma_{\alpha 3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s\Big){}_{,\alpha}$$

$$-\Big(\tilde{a}_{\alpha\gamma}\eta_{r,\gamma}\Big){}_{,\alpha}-\Big(\tilde{a}_{\alpha\gamma}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big){}_{,\alpha}+\Big(\tilde{a}_{\alpha 3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s\Big){}_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}p_{ikl}(u_{ks,l}+u_{ls,k})+\frac{1}{2}p_{ikl}\sum_{s'=s}^{\infty}(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'})$$

$$+\frac{1}{2}p_{ikl}^*(\dot{u}_{ks,l}+\dot{u}_{ls,k})+\frac{1}{2}p_{ikl}^*\sum_{s'=s}^{\infty}(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'})$$

$$-\varsigma_{i\gamma}\Big(\chi_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\chi_{s'}\Big)+\varsigma_{i3}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\chi_{s'}$$

$$-\tilde{a}_{i\gamma}\Big(\eta_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\eta_{s'}\Big)+\tilde{a}_{i3}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\eta_{s'}\Big]+\overset{r}{D}=f_{er},\quad r=0,1,\cdots,$$

$$\frac{1}{2}\Big(q_{\alpha kl}u_{kr,l}\Big){}_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}u_{lr,k}\Big){}_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}\sum_{s=r}^{\infty}\overset{r}{b}_{ks}u_{ls}\Big){}_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}\sum_{s=r}^{\infty}\overset{r}{b}_{ls}u_{ks}\Big){}_{,\alpha}$$

$$+\frac{1}{2}\Big(q_{\alpha kl}^*\dot{u}_{kr,l}\Big){}_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}^*\dot{u}_{lr,k}\Big){}_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}^*\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\dot{u}_{ls}\Big){}_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}^*\sum_{s=r}^{\infty}\overset{r}{b}_{ls}\dot{u}_{ks}\Big){}_{,\alpha}$$

$$-\Big(\tilde{a}_{\alpha\gamma}\chi_{r,\gamma}\Big){}_{,\alpha}-\Big(\tilde{a}_{\alpha\gamma}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big){}_{,\alpha}+\Big(\tilde{a}_{\alpha 3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s\Big){}_{,\alpha}$$

$$-\Big(\xi_{\alpha\gamma}\eta_{r,\gamma}\Big){}_{,\alpha}-\Big(\xi_{\alpha\gamma}\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big){}_{,\alpha}+\Big(\xi_{\alpha 3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s\Big){}_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}q_{ikl}(u_{ks,l}+u_{ls,k})+\frac{1}{2}q_{ikl}\sum_{s'=s}^{\infty}(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'})$$

$$+\frac{1}{2}q_{ikl}^{*}(\dot{u}_{ks,l}+\dot{u}_{ls,k})+\frac{1}{2}q_{ikl}^{*}\sum_{s'=s}^{\infty}(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'})$$

$$-\tilde{a}_{i\gamma}\Big(\chi_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\chi_{s'}\Big)+\tilde{a}_{i3}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\chi_{s'}$$

$$-\xi_{i\gamma}\Big(\eta_{s,\gamma}+\sum_{s'=s}^{\infty}\overset{s}{b}_{\gamma s'}\eta_{s'}\Big)+\xi_{i3}\sum_{s'=s+1}^{\infty}\overset{s}{a}_{3s'}\eta_{s'}\Big]+\overset{r}{B}=0,\quad r=0,1,\cdots.$$

If we suppose that the moments, whose subscripts, indicating moments' order, are greater than N, equal to zero and consider only the first $N+1$ equations $(r=\overline{0,N})$ for each $i=1,2,3$, φ_r, χ_r, η_r, from the obtained infinite system of equations with respect to

$$u_{ir},\quad i=\overline{1,3},\quad \varphi_r,\quad \chi_r,\quad \eta_r\quad r=0,1,\cdots,$$

we obtain the Nth order approximation (hierarchical model) governing system consisting of $6N+6$ equations with respect to $6N+6$ unknown functions

$$\overset{N}{u}_{ir},\quad \overset{N}{\varphi}_r,\quad \overset{N}{\chi}_r,\quad \overset{N}{\eta}_r,\quad i=\overline{1,3},\quad r=\overline{0,N}$$

(roughly speaking $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$, $\overset{N}{\chi}_r$, $\overset{N}{\eta}_r$ are "approximate values" of u_{ir}, φ_r, χ, η since $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$, $\overset{N}{\chi}$, $\overset{N}{\eta}$ are solutions of the derived finite system; below superscript N is omitted in order to avoid overloading the symbols):

$$\frac{1}{2}\Big(E_{\alpha ikl}u_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha ikl}u_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r}^{N}\overset{r}{b}_{ks}u_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r}^{N}\overset{r}{b}_{ls}u_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha ikl}^{*}\dot{u}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha ikl}^{*}\dot{u}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(E_{\alpha ikl}^{*}\sum_{s=r}^{N}\overset{r}{b}_{ks}\dot{u}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha ikl}^{*}\sum_{s=r}^{N}\overset{r}{b}_{ls}\dot{u}_{ks}\Big)_{,\alpha}+\Big(\tilde{b}_{\alpha i}\varphi_r\Big)_{,\alpha}$$

$$+\Big(b_{\alpha i}^{*}\dot{\varphi}_r\Big)_{,\alpha}+\Big(d_{\alpha i\gamma}\varphi_{r,\gamma}\Big)_{,\alpha}+\Big(d_{\alpha i\gamma}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\varphi_s\Big)_{,\alpha}-\Big(d_{\alpha i3}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\varphi_s\Big)_{,\alpha}$$

$$+\Big(d_{\alpha i\gamma}^{*}\dot{\varphi}_{r,\gamma}\Big)_{,\alpha}+\Big(d_{\alpha i\gamma}^{*}\dot{\varphi}_{r,\gamma}\Big)_{,\alpha}-\Big(d_{\alpha i3}^{*}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\dot{\varphi}_s\Big)_{,\alpha}$$

$$+\left(p_{\gamma\alpha i}\chi_{r,\gamma}\right)_{,\alpha}+\left(p_{\gamma\alpha i}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\chi_{s}\right)_{,\alpha}-\left(p_{3\alpha i}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\chi_{s}\right)_{,\alpha}$$

$$+\left(p_{\gamma\alpha i}^{*}\dot{\chi}_{r,\gamma}\right)_{,\alpha}+\left(p_{\gamma\alpha i}^{*}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\dot{\chi}_{s}\right)_{,\alpha}-\left(p_{3\alpha i}^{*}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\dot{\chi}_{s}\right)_{,\alpha}$$

$$+\left(q_{\gamma\alpha i}\eta_{r,\gamma}\right)_{,\alpha}+\left(q_{\gamma\alpha i}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\eta_{s}\right)_{,\alpha}-\left(q_{3\alpha i}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\eta_{s}\right)_{,\alpha}$$

$$+\left(q_{\gamma\alpha i}^{*}\dot{\eta}_{r,\gamma}\right)_{,\alpha}+\left(q_{\gamma\alpha i}^{*}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\dot{\eta}_{s}\right)_{,\alpha}-\left(q_{3\alpha i}^{*}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\dot{\eta}_{s}\right)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{js}\left[\frac{1}{2}E_{jikl}\left(u_{ks,l}+u_{ls,k}\right)+\frac{1}{2}E_{jikl}\sum_{s'=s}^{N}\left(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'}\right)\right.$$

$$+\frac{1}{2}E_{jikl}^{*}\left(\dot{u}_{ks,l}+\dot{u}_{ls,k}\right)+\frac{1}{2}E_{jikl}^{*}\sum_{s'=s}^{N}\left(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'}\right)$$

$$+\tilde{b}_{ij}\varphi_{s}+b_{ij}^{*}\dot{\varphi}_{s}+d_{ji\gamma}\left(\varphi_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\varphi_{s'}\right)-d_{ji3}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\varphi_{s'}$$

$$+d_{ji\gamma}^{*}\left(\dot{\varphi}_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\dot{\varphi}_{s'}\right)-d_{ji3}^{*}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\dot{\varphi}_{s'}$$

$$+p_{\gamma ji}\left(\chi_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\chi_{s'}\right)-p_{3ij}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\chi_{s'}$$

$$+p_{\gamma ji}^{*}\left(\dot{\chi}_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\dot{\chi}_{s'}\right)-p_{3ij}^{*}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\dot{\chi}_{s'}$$

$$+q_{\gamma ji}\left(\eta_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\eta_{s'}\right)-q_{3ji}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\eta_{s'}$$

$$+q_{\gamma ji}^{*}\left(\dot{\eta}_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\dot{\eta}_{s'}\right)-q_{3ji}^{*}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\dot{\eta}_{s'}\left.\right]+\overset{r}{X}_{i}=\rho\frac{\partial^{2}u_{ir}}{\partial t^{2}},$$

$$i=\overline{1,3},\ \ r=\overline{0,N},$$

$$\frac{1}{2}\left(d_{kl\alpha}u_{kr,l}\right)_{,\alpha}+\frac{1}{2}\left(d_{kl\alpha}u_{lr,k}\right)_{,\alpha}+\frac{1}{2}\left(d_{kl\alpha}\sum_{s=r}^{N}\overset{r}{b}_{ks}u_{ls}\right)_{,\alpha}$$

$$+\frac{1}{2}\left(d_{kl\alpha}\sum_{s=r}^{N}\overset{r}{b}_{ls}u_{ks}\right)_{,\alpha}+\frac{1}{2}\left(d_{kl\alpha}^{*}\dot{u}_{kr,l}\right)_{,\alpha}+\frac{1}{2}\left(d_{kl\alpha}^{*}\dot{u}_{lr,k}\right)_{,\alpha}$$

$$+\frac{1}{2}\Big(d^*_{kl\alpha}\sum_{s=r}^{N}\overset{r}{b}_{ks}\dot{u}_{ls}\Big),_\alpha+\frac{1}{2}\Big(d^*_{kl\alpha}\sum_{s=r}^{N}\overset{r}{b}_{ls}\dot{u}_{ks}\Big),_\alpha+\Big(d_\alpha\varphi_r\Big),_\alpha+\Big(d^*_\alpha\dot{\varphi}_r\Big),_\alpha$$

$$+\Big(\tilde{\alpha}_{\alpha k}\varphi_{r,k}\Big),,_\alpha+\Big(\tilde{\alpha}_{\alpha k}\sum_{s=r}^{N}\overset{r}{b}_{ks}\varphi_s\Big),_\alpha+\Big(\alpha^*_{\alpha k}\dot{\varphi}_{r,k}\Big),_\alpha+\Big(\alpha^*_{\alpha k}\sum_{s=r}^{N}\overset{r}{b}_{ks}\dot{\varphi}_s\Big),_\alpha$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}d_{kli}\Big(u_{ks,l}+u_{ls,k}\Big)+\frac{1}{2}d_{kli}\sum_{s'=s}^{N}\Big(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'}\Big)$$

$$+\frac{1}{2}d^*_{kli}\Big(\dot{u}_{ks,l}+\dot{u}_{ls,k}\Big)+\frac{1}{2}d^*_{kli}\sum_{s'=s}^{N}\Big(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'}\Big)+d_i\varphi_s+d^*_i\dot{\varphi}_s$$

$$+\tilde{\alpha}_{ik}\Big(\varphi_{s,k}+\sum_{s'=s}^{N}\overset{s}{b}_{ks'}\varphi_{s'}\Big)+\alpha^*_{ik}\Big(\dot{\varphi}_{s,k}+\sum_{s'=s}^{N}\overset{s}{b}_{ks'}\dot{\varphi}_{s'}\Big)\Big]$$

$$-\tilde{b}_{ij}\Big[\frac{1}{2}\Big(u_{ir,j}+u_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{N}\overset{r}{b}_{is}u_{js}+\frac{1}{2}\sum_{s=r}^{N}\overset{r}{b}_{js}u_{is}\Big]-\tilde{\xi}\varphi_r$$

$$-d_i\Big(\varphi_{r,i}+\sum_{s=r}^{N}\overset{r}{b}_{is}\varphi_s\Big)-b^*_{ij}\Big[\frac{1}{2}\Big(\dot{u}_{ir,j}+\dot{u}_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{N}\overset{r}{b}_{is}\dot{u}_{js}+\frac{1}{2}\sum_{s=r}^{N}\overset{r}{b}_{js}\dot{u}_{is}\Big]$$

$$-\xi^*\dot{\varphi}_r-d^*_i\Big(\dot{\varphi}_{r,i}+\sum_{s=r}^{N}\overset{r}{b}_{is}\dot{\varphi}_s\Big)+\overset{r}{H}=\rho k\frac{\partial^2\varphi_r}{\partial t^2},\quad r=\overline{0,N},$$

$$\frac{1}{2}\Big(p_{\alpha kl}u_{kr,l}\Big),_\alpha+\frac{1}{2}\Big(p_{\alpha kl}u_{lr,k}\Big),_\alpha+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ks}u_{ls}\Big),_\alpha+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ls}u_{ks}\Big),_\alpha$$

$$+\frac{1}{2}\Big(p^*_{\alpha kl}\dot{u}_{kr,l}\Big),_\alpha+\frac{1}{2}\Big(p^*_{\alpha kl}\dot{u}_{lr,k}\Big),_\alpha+\frac{1}{2}\Big(p^*_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ks}\dot{u}_{ls}\Big),_\alpha+\frac{1}{2}\Big(p^*_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ls}\dot{u}_{ks}\Big),_\alpha$$

$$-\Big(\varsigma_{\alpha\gamma}\chi_{r,\gamma}\Big),_\alpha-\Big(\varsigma_{\alpha\gamma}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\chi_s\Big),_\alpha+\Big(\varsigma_{\alpha 3}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\chi_s\Big),_\alpha$$

$$-\Big(\tilde{a}_{\alpha\gamma}\eta_{r,\gamma}\Big),_\alpha-\Big(\tilde{a}_{\alpha\gamma}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\eta_s\Big),_\alpha+\Big(\tilde{a}_{\alpha 3}\sum_{s=r+1}^{N}\overset{r}{a}_{3s}\eta_s\Big),_\alpha$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}p_{ikl}(u_{ks,l}+u_{ls,k})+\frac{1}{2}p_{ikl}\sum_{s'=s}^{N}(\overset{s}{b}_{ks'}u_{ls'}+\overset{s}{b}_{ls'}u_{ks'})$$

$$+\frac{1}{2}p^*_{ikl}(\dot{u}_{ks,l}+\dot{u}_{ls,k})+\frac{1}{2}p^*_{ikl}\sum_{s'=s}^{N}(\overset{s}{b}_{ks'}\dot{u}_{ls'}+\overset{s}{b}_{ls'}\dot{u}_{ks'})$$

$$-\varsigma_{i\gamma}\Big(\chi_{s,\gamma}+\sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\chi_{s'}\Big)+\varsigma_{i3}\sum_{s'=s+1}^{N}\overset{s}{a}_{3s'}\chi_{s'}$$

28

$$-\tilde{a}_{i\gamma}\left(\eta_{s,\gamma} + \sum_{s'=s}^{N} \overset{s}{b}_{\gamma s'}\eta_{s'}\right) + \tilde{a}_{i3}\sum_{s'=s+1}^{N} \overset{s}{\mathring{a}}_{3s'}\eta_{s'}\Bigg] + \overset{r}{D} = f_{er}, \quad r = \overline{0,N},$$

$$\frac{1}{2}\left(q_{\alpha kl}u_{kr,l}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}u_{lr,k}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ks}u_{ls}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ls}u_{ks}\right)_{,\alpha}$$

$$+\frac{1}{2}\left(q^{*}_{\alpha kl}\dot{u}_{kr,l}\right)_{,\alpha} + \frac{1}{2}\left(q^{*}_{\alpha kl}\dot{u}_{lr,k}\right)_{,\alpha} + \frac{1}{2}\left(q^{*}_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ks}\dot{u}_{ls}\right)_{,\alpha} + \frac{1}{2}\left(q^{*}_{\alpha kl}\sum_{s=r}^{N}\overset{r}{b}_{ls}\dot{u}_{ks}\right)_{,\alpha}$$

$$-\left(\tilde{a}_{\alpha\gamma}\chi_{r,\gamma}\right)_{,\alpha} - \left(\tilde{a}_{\alpha\gamma}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\chi_s\right)_{,\alpha} + \left(\tilde{a}_{\alpha 3}\sum_{s=r+1}^{N}\overset{r}{\mathring{a}}_{3s}\chi_s\right)_{,\alpha}$$

$$-\left(\xi_{\alpha\gamma}\eta_{r,\gamma}\right)_{,\alpha} - \left(\xi_{\alpha\gamma}\sum_{s=r}^{N}\overset{r}{b}_{\gamma s}\eta_s\right)_{,\alpha} + \left(\xi_{\alpha 3}\sum_{s=r+1}^{N}\overset{r}{\mathring{a}}_{3s}\eta_s\right)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{\mathring{a}}_{is}\Bigg[\frac{1}{2}q_{ikl}(u_{ks,l} + u_{ls,k}) + \frac{1}{2}q_{ikl}\sum_{s'=s}^{N}(\overset{s}{b}_{ks'}u_{ls'} + \overset{s}{b}_{ls'}u_{ks'})$$

$$+\frac{1}{2}q^{*}_{ikl}(\dot{u}_{ks,l} + \dot{u}_{ls,k}) + \frac{1}{2}q^{*}_{ikl}\sum_{s'=s}^{N}(\overset{s}{b}_{ks'}\dot{u}_{ls'} + \overset{s}{b}_{ls'}\dot{u}_{ks'})$$

$$-\tilde{a}_{i\gamma}\left(\chi_{s,\gamma} + \sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\chi_{s'}\right) + \tilde{a}_{i3}\sum_{s'=s+1}^{N}\overset{s}{\mathring{a}}_{3s'}\chi_{s'}$$

$$-\xi_{i\gamma}\left(\eta_{s,\gamma} + \sum_{s'=s}^{N}\overset{s}{b}_{\gamma s'}\eta_{s'}\right) + \xi_{i3}\sum_{s'=s+1}^{N}\overset{s}{\mathring{a}}_{3s'}\eta_{s'}\Bigg] + \overset{r}{B} = 0, \quad r = \overline{0,N}.$$

In the Nth approximation (hierarchical model)

$$(u_i, \varphi, \chi, \eta)(x_1, x_2, x_3, t)$$
$$\cong \sum_{r=0}^{N} a\left(r + \frac{1}{2}\right)\left(\overset{N}{u}_{ir}, \overset{N}{\varphi}_{ir}, \overset{N}{\chi}_{ir}, \overset{N}{\eta}_{ir}\right)(x_1, x_2, t)P_r(ax_3 - b),$$

where $\left(\overset{N}{u}_{ir}, \overset{N}{\varphi}_{ir}, \overset{N}{\chi}_{ir}, \overset{N}{\eta}_{ir}\right)$ is a solution of the above system.

Remark 3.1. Note that solutions

$$\overset{N}{u}_{ir}, \quad i = \overline{1,3}, \quad \overset{N}{\phi}_r, \quad \overset{N}{\chi}_r, \quad \overset{N}{\eta}_r, \quad i = \overline{1,3}, \quad r = \overline{0,N}$$

of the governing system of the Nth approximation are not rth order moments in sense of (3.2) any more, in general.

It should be also mentioned that by deriving the governing system of the Nth approximation we use the following expressions prescribed on the face surfaces of the prismatic shell:

$$\overset{(\pm)}{X}_{ni} = X_{ji}\overset{(\pm)}{n}_j, \quad i = \overline{1,3};$$

$$\overset{(\pm)}{H}_n = (H, \overset{(\pm)}{n}) = H_j\overset{(\pm)}{n}_j;$$

$$\overset{(\pm)}{D}_n = (D, \overset{(\pm)}{n}) = D_j\overset{(\pm)}{n}_j;$$

$$\overset{(\pm)}{B}_n = (B, \overset{(\pm)}{n}) = B_j\overset{(\pm)}{n}_j.$$

In the Nth order approximation BCs (2.10)-(2.13) on the lateral boundary of the prismatic shell and ICs (2.14) should be written in terms of r-the order, $r = \overline{0, N}$, moments for functions participating in BCs.

BCs (on the lateral surface of the prismatic shell) and ICs we easily get from BCs (2.10)-(2.13) and ICs (2.14) after multiplying them by $P_r(ax_3 - b)$ and integrating the obtained within the limits $\overset{(-)}{h}(x_1, x_2)$ and $\overset{(+)}{h}(x_1, x_2)$ with respect to x_3, provided the constitutive coefficients and the thickness do not vanish on $\bar{\omega}$:

$$u_{ir} = f_{ir} \quad on \quad \gamma_0, \quad X_{jir}n_j = g_{ir} \quad on \quad \gamma_1 = \partial\omega\backslash\overline{\gamma_0}, \quad i = \overline{1,3}, \qquad (3.23)$$

$$\varphi_r = f_r^\varphi \quad on \quad \gamma_0^\varphi, \quad H_{jr}n_j = g_r^\varphi \quad on \quad \gamma_1^\varphi = \partial\omega\backslash\overline{\gamma_0^\varphi}, \quad i = \overline{1,3}, \qquad (3.24)$$

$$\chi_r = f_r^\chi \quad on \quad \gamma_0^\chi, \quad D_{jr}n_j = g_r^\chi \quad on \quad \gamma_1^\chi = \partial\omega\backslash\overline{\gamma_0^\chi}, \quad i = \overline{1,3}, \qquad (3.25)$$

$$\eta_r = f_r^\eta \quad on \quad \gamma_0^\eta, \quad B_{jr}n_j = g_r^\eta \quad on \quad \gamma_1^\eta = \partial\omega\backslash\overline{\gamma_0^\eta}, \quad i = \overline{1,3}, \qquad (3.26)$$

and the standard ICs in dynamical problems

$$\mathbf{u}_r(x, 0) = \mathbf{u}_r^0(x), \quad \dot{\mathbf{u}}_r(x, 0) = \mathbf{u}_r^1(x),$$

$$\varphi_r(x, 0) = \varphi_r^0(x), \quad \dot{\varphi}_r(x, 0) = \varphi_r^1(x), \quad x \in \Omega;$$

here $\mathbf{n} := (n_1, n_2)$ is the outward unit normal vector to $\partial\omega$, (f_{1r}, f_{2r}, f_{3r}), f_r^φ, f_r^χ, f_r^η are the rth order moments of the given displacement vector, volume fraction, electric and magnetic potentials, respectively, (g_{1r}, g_{2r}, g_{3r}), g_r^φ, g_r^χ and g^η are the rth order moments of the given stress vector, normal components of the equilibrated stress, electric displacement and magnetic induction vectors, respectively, while $\mathbf{u}_r^0$ and $\mathbf{u}_r^1$ are the rth order moments of the initial mechanical displacement and velocity vectors, whereas φ_r^0 and φ_r^1 are the initial volume fraction distribution and its rate of change. Note that the curves γ_0, γ_0^φ, γ_0^χ, and γ_0^η, which represent projections on plane $x_3 = 0$ of the corresponding parts of the lateral boundary of the prismatic shell, in BCs (3.23)-(3.26) are different, in general, from each other and depending on the physical problem some of them may be empty.

Remark 3.2. In case of cusped prismatic shells the setting of BCs is not classical, in general, and is depending on the character of tapering: either the Dirichlet problem should be replaced by the Keldysh problem or BCs should be weighted ones, in general. On the parts of $\partial\omega$, where the thickness is positive (i.e., it does not vanish) the BCs have the form (3.23)-(3.26).

Multiplying equality (3.4) by h^r and, taking into account that

$$\overset{r}{a}_{\alpha\underline{r}} = rh^{-1}h_{,\alpha}\,,$$

we get

$$(h^r X_{\alpha ir})_{,\alpha} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{js} X_{jis} + h^r \overset{r}{X}_i = \rho h^r \frac{\partial^2 h^{r+1} v_{ir}}{\partial t^2},$$

$$\sum_{s=0}^{-1}(\cdots) \equiv 0, \quad i = \overline{1,3}, \quad r = 0,1,\cdots. \tag{3.27}$$

Multiplying (3.5) by h^r we obtain

$$(h^r H_{\alpha r})_{,\alpha} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{is} H_{is} + h^r H_{0r} + h^r \overset{r}{H} = \rho k h^r \frac{\partial^2 h^{r+1} \psi_r}{\partial t^2}. \tag{3.28}$$

Similarly, from (3.6) and (3.7) we have

$$(h^r D_{\alpha r})_{,\alpha} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{is} D_{is} + h^r \overset{r}{D} = h^r f_{er}, \quad r = 0,1,\cdots, \tag{3.29}$$

and

$$(h^r B_{\alpha r})_{,\alpha} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{is} B_{is} + h^r \overset{r}{B} = 0, \quad r = 0,1,\cdots, \tag{3.30}$$

Substituting (3.14) into (3.27); (3.16) and (3.18) into (3.28); (3.20) into (3.29); (3.22) into (3.30), respectively, we construct an equivalent infinite system with respect v_{ir}, ψ_r, $\tilde{\mathcal{X}}_r$, and $\tilde{\eta}_r$:

$$\frac{1}{2}\Big(E_{\alpha ikl}h^{2r+1}v_{kr,l}\Big)_{,\alpha} + \frac{1}{2}\Big(E_{\alpha ikl}h^{2r+1}v_{lr,k}\Big)_{,\alpha} + \frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ikl}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ikl}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(E^*_{\alpha ikl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ikl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha} + \Big(\tilde{b}_{\alpha i}h^{2r+1}\psi_r\Big)_{,\alpha}$$

$$+\Big(b_{\alpha i}^* h^{2r+1}\dot\psi_r\Big),_\alpha +\Big(d_{\alpha i\gamma} h^{2r+1}\psi_{r,\gamma}\Big),_\alpha +\Big(d_{\alpha ik}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1}\psi_s\Big),_\alpha$$

$$+\Big(d_{\alpha i\gamma}^* h^{2r+1}\dot\psi_{r,\gamma}\Big),_\alpha +\Big(d_{\alpha ik}^*\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1}\dot\psi_s\Big),_\alpha$$

$$+\Big(p_{\gamma\alpha i} h^{2r+1}\tilde\chi_{r,\gamma}\Big),_\alpha +\Big(p_{k\alpha i}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1}\tilde\chi_s\Big),_\alpha$$

$$+\Big(p_{\gamma\alpha i}^* h^{2r+1}\dot{\tilde\chi}_{r,\gamma}\Big),_\alpha +\Big(p_{k\alpha i}^*\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1}\dot{\tilde\chi}_s\Big),_\alpha$$

$$+\Big(q_{\gamma\alpha i} h^{2r+1}\tilde\eta_{r,\gamma}\Big),_\alpha +\Big(q_{k\alpha i}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1}\tilde\eta_s\Big),_\alpha$$

$$+\Big(q_{\gamma\alpha i}^* h^{2r+1}\dot{\tilde\eta}_{r,\gamma}\Big),_\alpha +\Big(q_{k\alpha i}^*\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1}\dot{\tilde\eta}_s\Big),_\alpha$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{js}\Big[\frac{1}{2}E_{jikl} h^{r+s+1}\Big(v_{ks,l}+v_{ls,k}\Big)+\frac{1}{2}E_{jikl}\sum_{s'=s+1}^{\infty} h^{r+s'+1}\Big(\overset{s}{b}_{ks'} v_{ls'}+\overset{s}{b}_{ls'} v_{ks'}\Big)$$

$$+\frac{1}{2}E_{jikl}^* h^{r+s+1}\Big(\dot v_{ks,l}+\dot v_{ls,k}\Big)+\frac{1}{2}E_{jikl}^*\sum_{s'=s+1}^{\infty} h^{r+s'+1}\Big(\overset{s}{b}_{ks'}\dot v_{ls'}+\overset{s}{b}_{ls'}\dot v_{ks'}\Big)$$

$$+\tilde b_{ij} h^{r+s+1}\psi_s + b_{ij}^* h^{r+s+1}\dot\psi_s + d_{ji\gamma} h^{r+s+1}\psi_{s,\gamma} + d_{jik}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'} h^{r+s'+1}\psi_{s'}$$

$$+d_{ji\gamma}^* h^{r+s+1}\dot\psi_{s,\gamma} + d_{jik}^*\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'} h^{r+s'+1}\dot\psi_{s'} + p_{\gamma ji} h^{r+s+1}\tilde\chi_{s,\gamma}$$

$$+p_{kji}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'} h^{r+s'+1}\tilde\chi_{s'} + p_{\gamma ji}^* h^{r+s+1}\dot{\tilde\chi}_{s,\gamma} + p_{kji}^*\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'} h^{r+s'+1}\dot{\tilde\chi}_{s'}$$

$$+q_{\gamma ji} h^{r+s+1}\tilde\eta_{s,\gamma} + q_{kji}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'} h^{r+s'+1}\tilde\eta_{s'} + q_{\gamma ji}^* h^{r+s+1}\dot{\tilde\eta}_{s,\gamma}$$

$$+q_{kji}^*\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'} h^{r+s'+1}\dot{\tilde\eta}_{s'}\Big] + h^r \overset{r}{X}_i = \rho h^r\frac{\partial^2 h^{r+1} v_{ir}}{\partial t^2},\quad i=\overline{1,3},\ \ r=0,1,\cdots,$$

$$\frac{1}{2}\Big(d_{kl\alpha} h^{2r+1} v_{kr,l}\Big),_\alpha +\frac{1}{2}\Big(d_{kl\alpha} h^{2r+1} v_{lr,k}\Big),_\alpha +\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks} h^{r+s+1} v_{ls}\Big),_\alpha$$

$$+\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls} h^{r+s+1} v_{ks}\Big),_\alpha +\frac{1}{2}\Big(d_{kl\alpha}^* h^{2r+1}\dot v_{kr,l}\Big),_\alpha +\frac{1}{2}\Big(d_{kl\alpha}^* h^{2r+1}\dot v_{lr,k}\Big),_\alpha$$

$$+\frac{1}{2}\Big(d^*_{kl\alpha}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(d^*_{kl\alpha}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}$$

$$+\Big(d_\alpha h^{2r+1}\psi_r\Big)_{,\alpha}+\Big(d^*_\alpha h^{2r+1}\dot{\psi}_r\Big)_{,\alpha}+\Big(\tilde{\alpha}_{\alpha k}h^{2r+1}\psi_{r,k}\Big)_{,,\alpha}$$

$$+\Big(\tilde{\alpha}_{\alpha k}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\psi_s\Big)_{,\alpha}+\Big(\alpha^*_{\alpha k}h^{2r+1}\dot{\psi}_{r,k}\Big)_{,\alpha}+\Big(\alpha^*_{\alpha k}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\dot{\psi}_s\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}d_{kli}h^{r+s+1}\Big(v_{ks,l}+v_{ls,k}\Big)+\frac{1}{2}d_{kli}\sum_{s'=s+1}^{\infty}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'}\Big)$$

$$+\frac{1}{2}d^*_{kli}h^{r+s+1}\Big(\dot{v}_{ks,l}+\dot{v}_{ls,k}\Big)+\frac{1}{2}d^*_{kli}\sum_{s'=s+1}^{\infty}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'}\Big)$$

$$+d_i h^{r+s+1}\psi_s+d^*_i h^{r+s+1}\dot{\psi}_s+\tilde{\alpha}_{ik}\Big(h^{r+s+1}\psi_{s,k}+\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'}h^{r+s'+1}\psi_{s'}\Big)$$

$$+\alpha^*_{ik}\Big(h^{r+s+1}\dot{\psi}_{s,k}+\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\psi}_{s'}\Big)\Big]$$

$$-\tilde{b}_{ij}\Big[\frac{1}{2}h^{2r+1}\Big(v_{ir,j}+v_{jr,i}\Big)+\frac{1}{2}\sum_{s=r+1}^{\infty}\overset{r}{b}_{is}h^{r+s+1}v_{js}+\frac{1}{2}\sum_{s=r+1}^{\infty}\overset{r}{b}_{js}h^{r+s+1}v_{is}\Big]$$

$$-\tilde{\xi}h^{2r+1}\psi_r-d_i\Big(h^{2r+1}\psi_{r,i}+\sum_{s=r+1}^{\infty}\overset{r}{b}_{is}h^{r+s+1}\psi_s\Big)$$

$$-b^*_{ij}\Big[\frac{1}{2}h^{2r+1}\Big(\dot{v}_{ir,j}+\dot{v}_{jr,i}\Big)+\frac{1}{2}\sum_{s=r+1}^{\infty}\overset{r}{b}_{is}h^{r+s+1}\dot{v}_{js}+\frac{1}{2}\sum_{s=r+1}^{\infty}\overset{r}{b}_{js}h^{r+s+1}\dot{v}_{is}\Big]$$

$$-\xi^*h^{2r+1}\dot{\psi}_r-d^*_i\Big(h^{2r+1}\dot{\psi}_{r,i}+\sum_{s=r+1}^{\infty}\overset{r}{b}_{is}h^{r+s+1}\dot{\psi}_s\Big)+h^r\overset{r}{H}=\rho k h^r\frac{\partial^2 h^{r+1}\psi_r}{\partial t^2},$$

$$r=0,1,\cdots,$$

$$\frac{1}{2}\Big(p_{\alpha kl}h^{2r+1}v_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}h^{2r+1}v_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha kl}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha kl}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(p^*_{\alpha kl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha kl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}$$

$$-\Big(\varsigma_{\alpha\gamma}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha}-\Big(\varsigma_{\alpha k}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_s\Big)_{,\alpha}$$

$$-\left(\tilde{a}_{\alpha\gamma}h^{2r+1}\tilde{\eta}_{r,\gamma}\right)_{,\alpha} - \left(\tilde{a}_{\alpha k}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_s\right)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}h^{r+s+1}p_{ikl}(v_{ks,l}+v_{ls,k}) + \frac{1}{2}p_{ikl}\sum_{s'=s+1}^{\infty}h^{r+s'+1}(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'})$$

$$+\frac{1}{2}p_{ikl}^{*}h^{r+s+1}(\dot{v}_{ks,l}+\dot{v}_{ls,k}) + \frac{1}{2}p_{ikl}^{*}\sum_{s'=s+1}^{\infty}h^{r+s'+1}(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'})$$

$$-\varsigma_{i\gamma}h^{r+s+1}\tilde{\chi}_{s,\gamma} - \varsigma_{ik}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}$$

$$-\tilde{a}_{i\gamma}h^{r+s+1}\tilde{\eta}_{s,\gamma} - \tilde{a}_{ik}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}\Big] + h^{r}\overset{r}{D} = h^{r}f_{er}, \quad r = 0, 1, \cdots ,$$

$$\frac{1}{2}\left(q_{\alpha kl}h^{2r+1}v_{kr,l}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}h^{2r+1}v_{lr,k}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\right)_{,\alpha}$$

$$+\frac{1}{2}\left(q_{\alpha kl}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}^{*}h^{2r+1}\dot{v}_{kr,l}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}^{*}h^{2r+1}\dot{v}_{lr,k}\right)_{,\alpha}$$

$$+\frac{1}{2}\left(q_{\alpha kl}^{*}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\right)_{,\alpha} + \frac{1}{2}\left(q_{\alpha kl}^{*}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\right)_{,\alpha}$$

$$-\left(\tilde{a}_{\alpha\gamma}h^{2r+1}\tilde{\chi}_{r,\gamma}\right)_{,\alpha} - \left(\tilde{a}_{\alpha k}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_s\right)_{,\alpha}$$

$$-\left(\xi_{\alpha\gamma}h^{2r+1}\tilde{\eta}_{r,\gamma}\right)_{,\alpha} - \left(\xi_{\alpha k}\sum_{s=r+1}^{\infty}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_s\right)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}q_{ikl}h^{r+s+1}(v_{ks,l}+v_{ls,k}) + \frac{1}{2}q_{ikl}\sum_{s'=s+1}^{\infty}h^{r+s'+1}(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'})$$

$$+\frac{1}{2}q_{ikl}^{*}h^{r+s+1}(\dot{v}_{ks,l}+\dot{v}_{ls,k}) + \frac{1}{2}q_{ikl}^{*}\sum_{s'=s+1}^{\infty}h^{r+s'+1}(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'})$$

$$-\tilde{a}_{i\gamma}h^{r+s+1}\tilde{\chi}_{s,\gamma} - \tilde{a}_{ik}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}$$

$$-\xi_{i\gamma}h^{r+s+1}\tilde{\eta}_{s,\gamma} - \xi_{ik}\sum_{s'=s+1}^{\infty}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}\Big] + h^{r}\overset{r}{B} = 0, \quad r = 0, 1, \cdots .$$

Assuming

$$v_{ir} = 0, \ \ i = \overline{1,3}; \ \ \psi_r = 0, \ \tilde{\chi}_r = 0, \ \tilde{\eta}_r = 0 \ \text{ for } \ r > N, \qquad (3.31)$$

34

we obtain governing system consisting of $6N+6$ equations of the Nth approximation with respect to (see Remark 3.1)

$$\overset{N}{v}_{ir}, \quad i = \overline{1,3}; \quad \overset{N}{\psi}_r, \quad \overset{N}{\tilde{\chi}}_r, \quad \overset{N}{\tilde{\eta}}_r, \quad r = \overline{0,N},$$

provided we consider the first $N+1$ equations for $r = \overline{0,N}$ for

$$\text{each} \quad i = 1,2,3, \quad \psi_r, \quad \tilde{\chi}_r, \quad \tilde{\eta}_r.$$

Namely (below superscript N on unknown functions is omitted),

$$\frac{1}{2}\Big(E_{\alpha ik\delta}h^{2r+1}v_{kr,\delta}\Big)_{,\alpha} + \frac{1}{2}\Big(E_{\alpha i\gamma l}h^{2r+1}v_{lr,\gamma}\Big)_{,\alpha} + \frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ik\delta}h^{2r+1}\dot{v}_{kr,\delta}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha i\gamma l}h^{2r+1}\dot{v}_{lr,\gamma}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(E^*_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha} + \frac{1}{2}\Big(E^*_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha} + \Big(\tilde{b}_{\alpha i}h^{2r+1}\psi_r\Big)_{,\alpha}$$

$$+\Big(b^*_{\alpha i}h^{2r+1}\dot{\psi}_r\Big)_{,\alpha} + \Big(d_{\alpha i\gamma}h^{2r+1}\psi_{r,\gamma}\Big)_{,\alpha} + \Big(d_{\alpha ik}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\psi_s\Big)_{,\alpha}$$

$$+\Big(d^*_{\alpha i\gamma}h^{2r+1}\dot{\psi}_{r,\gamma}\Big)_{,\alpha} + \Big(d^*_{\alpha ik}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\psi}_s\Big)_{,\alpha}$$

$$+\Big(p_{\gamma\alpha i}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha} + \Big(p_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_s\Big)_{,\alpha}$$

$$+\Big(p^*_{\gamma\alpha i}h^{2r+1}\dot{\tilde{\chi}}_{r,\gamma}\Big)_{,\alpha} + \Big(p^*_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\tilde{\chi}}_s\Big)_{,\alpha}$$

$$+\Big(q_{\gamma\alpha i}h^{2r+1}\tilde{\eta}_{r,\gamma}\Big)_{,\alpha} + \Big(q_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_s\Big)_{,\alpha}$$

$$+\Big(q^*_{\gamma\alpha i}h^{2r+1}\dot{\tilde{\eta}}_{r,\gamma}\Big)_{,\alpha} + \Big(q^*_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\tilde{\eta}}_s\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{js}\Big[\frac{1}{2}E_{jikl}h^{r+s+1}\Big(v_{ks,l}+v_{ls,k}\Big) + \frac{1}{2}E_{jikl}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'}\Big)$$

$$+\frac{1}{2}E^*_{jikl}h^{r+s+1}\Big(\dot{v}_{ks,l}+\dot{v}_{ls,k}\Big) + \frac{1}{2}E^*_{jikl}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'}\Big)$$

$$+\tilde{b}_{ij}h^{r+s+1}\psi_s + b^*_{ij}h^{r+s+1}\dot{\psi}_s + d_{ji\gamma}h^{r+s+1}\psi_{s,\gamma} + d_{jik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\psi_{s'}$$

$$+d^*_{ji\gamma}h^{r+s+1}\dot{\psi}_{s,\gamma} + d^*_{jik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\psi}_{s'} + p_{\gamma ji}h^{r+s+1}\tilde{\chi}_{s,\gamma}$$

$$+p_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'} + p^*_{\gamma ji}h^{r+s+1}\dot{\tilde{\chi}}_{s,\gamma} + p^*_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\tilde{\chi}}_{s'}$$

$$+q_{\gamma ji}h^{r+s+1}\tilde{\eta}_{s,\gamma} + q_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'} + q^*_{\gamma ji}h^{r+s+1}\dot{\tilde{\eta}}_{s,\gamma}$$

$$+q^*_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\tilde{\eta}}_{s'}\Big] + h^r\overset{r}{X}_i = \rho h^r\frac{\partial^2 h^{r+1}v_{ir}}{\partial t^2}, \qquad (3.32)$$

$$i=\overline{1,3}, \quad r=\overline{0,N},$$

$$\frac{1}{2}\Big(d_{kl\alpha}h^{2r+1}v_{kr,l}\Big)_{,\alpha} + \frac{1}{2}\Big(d_{kl\alpha}h^{2r+1}v_{lr,k}\Big)_{,\alpha} + \frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha} + \frac{1}{2}\Big(d^*_{kl\alpha}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha} + \frac{1}{2}\Big(d^*_{kl\alpha}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(d^*_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha} + \frac{1}{2}\Big(d^*_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}$$

$$+\Big(d_\alpha h^{2r+1}\psi_r\Big)_{,\alpha} + \Big(d^*_\alpha h^{2r+1}\dot{\psi}_r\Big)_{,\alpha} + \Big(\tilde{\alpha}_{\alpha k}h^{2r+1}\psi_{r,k}\Big)_{,,\alpha}$$

$$+\Big(\tilde{\alpha}_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\psi_s\Big)_{,\alpha} + \Big(\alpha^*_{\alpha k}h^{2r+1}\dot{\psi}_{r,k}\Big)_{,\alpha} + \Big(\alpha^*_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\psi}_s\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}d_{kli}h^{r+s+1}\Big(v_{ks,l}+v_{ls,k}\Big) + \frac{1}{2}d_{kli}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'}\Big)$$

$$+\frac{1}{2}d^*_{kli}h^{r+s+1}\Big(\dot{v}_{ks,l}+\dot{v}_{ls,k}\Big) + \frac{1}{2}d^*_{kli}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'}\Big)$$

$$+d_i h^{r+s+1}\psi_s + d^*_i h^{r+s+1}\dot{\psi}_s + \tilde{\alpha}_{ik}\Big(h^{r+s+1}\psi_{s,k} + \sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\psi_{s'}\Big)$$

$$+\alpha^*_{ik}\Big(h^{r+s+1}\dot{\psi}_{s,k} + \sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\psi}_{s'}\Big)\Big]$$

$$-\tilde{b}_{ij}\Big[\frac{1}{2}h^{2r+1}\Big(v_{ir,j}+v_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}v_{js} + \frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{js}h^{r+s+1}v_{is}\Big]$$

$$-\tilde{\xi}h^{2r+1}\psi_r - d_i\Big(h^{2r+1}\psi_{r,i} + \sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}\psi_s\Big)$$

$$-b_{ij}^*\Big[\frac{1}{2}h^{2r+1}\big(\dot{v}_{ir,j} + \dot{v}_{jr,i}\big) + \frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}\dot{v}_{js} + \frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{js}h^{r+s+1}\dot{v}_{is}\Big]$$

$$-\xi^*h^{2r+1}\dot{\psi}_r - d_i^*\Big(h^{2r+1}\dot{\psi}_{r,i} + \sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}\dot{\psi}_s\Big)$$

$$+h^r\overset{r}{H} = \rho k h^r\frac{\partial^2 h^{r+1}\psi_r}{\partial t^2}, \quad r = \overline{0,N}, \tag{3.33}$$

$$\frac{1}{2}\Big(p_{\alpha kl}h^{2r+1}v_{kr,l}\Big)_{,\alpha} + \frac{1}{2}\Big(p_{\alpha kl}h^{2r+1}v_{lr,k}\Big)_{,\alpha} + \frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha} + \frac{1}{2}\Big(p_{\alpha kl}^*h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha} + \frac{1}{2}\Big(p_{\alpha kl}^*h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(p_{\alpha kl}^*\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha} + \frac{1}{2}\Big(p_{\alpha kl}^*\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}$$

$$-\Big(\varsigma_{\alpha\gamma}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha} - \Big(\varsigma_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_s\Big)_{,\alpha}$$

$$-\Big(\tilde{a}_{\alpha\gamma}h^{2r+1}\tilde{\eta}_{r,\gamma}\Big)_{,\alpha} - \Big(\tilde{a}_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_s\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}h^{r+s+1}p_{ikl}(v_{ks,l} + v_{ls,k}) + \frac{1}{2}p_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}v_{ls'} + \overset{s}{b}_{ls'}v_{ks'})$$

$$+\frac{1}{2}p_{ikl}^*h^{r+s+1}(\dot{v}_{ks,l} + \dot{v}_{ls,k}) + \frac{1}{2}p_{ikl}^*\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}\dot{v}_{ls'} + \overset{s}{b}_{ls'}\dot{v}_{ks'})$$

$$-\varsigma_{i\gamma}h^{r+s+1}\tilde{\chi}_{s,\gamma} - \varsigma_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}$$

$$+\tilde{a}_{i\gamma}h^{r+s+1}\tilde{\eta}_{s,\gamma} + \tilde{a}_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}\Big] + h^r\overset{r}{D} = h^r f_{er}, \tag{3.34}$$

$$r = \overline{0,N}.$$

$$
\frac{1}{2}\Big(q_{\alpha kl}h^{2r+1}v_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}h^{2r+1}v_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}
$$

$$
+\frac{1}{2}\Big(q_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(q^{*}_{\alpha kl}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(q^{*}_{\alpha kl}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}
$$

$$
+\frac{1}{2}\Big(q^{*}_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(q^{*}_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}
$$

$$
-\Big(\tilde{a}_{\alpha\gamma}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha}-\Big(\tilde{a}_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_{s}\Big)_{,\alpha}
$$

$$
-\Big(\xi_{\alpha\gamma}h^{2r+1}\tilde{\eta}_{r,\gamma}\Big)_{,\alpha}-\Big(\xi_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_{s}\Big)_{,\alpha}
$$

$$
+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}q_{ikl}h^{r+s+1}(v_{ks,l}+v_{ls,k})+\frac{1}{2}q_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'})
$$

$$
+\frac{1}{2}q^{*}_{ikl}h^{r+s+1}(\dot{v}_{ks,l}+\dot{v}_{ls,k})+\frac{1}{2}q^{*}_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'})
$$

$$
-\tilde{a}_{i\gamma}h^{r+s+1}\tilde{\chi}_{s,\gamma}-\tilde{a}_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}
$$

$$
-\xi_{i\gamma}h^{r+s+1}\tilde{\eta}_{s,\gamma}-\xi_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}\Big]+h^{r}\overset{r}{B}=0, \qquad (3.35)
$$

$$
r=\overline{0,N}.
$$

In the Nth approximation (hierarchical model):

$$
(u_i,\varphi,\chi,\eta)(x_1,x_2,x_3,t)
$$

$$
\cong\sum_{r=0}^{N}N\Big(r+\frac{1}{2}\Big)h^{r}\Big(\overset{N}{v}_{ir},\overset{N}{\psi}_{r},\overset{N}{\tilde{\chi}}_{r},\overset{N}{\tilde{\eta}}_{r}\Big)(x_1,x_2,x_3,t)P_r(ax_3-b),
$$

where $\Big(\overset{N}{v}_{ir},\overset{N}{\psi}_{r},\overset{N}{\tilde{\chi}}_{r},\overset{N}{\tilde{\eta}}_{r}\Big)$ is a solution of the above system (3.32)-(3.35);

$$X_{ijr} = \frac{1}{2}E_{ijkl}h^{r+1}\Big(v_{kr,l} + v_{lr,k}\Big) + \frac{1}{2}E_{ijkl}\sum_{s=r+1}^{N}h^{s+1}\Big(\overset{r}{b}_{ks}v_{ls} + \overset{r}{b}_{ls}v_{ks}\Big)$$

$$+\frac{1}{2}E^*_{ijkl}h^{r+1}\Big(\dot{v}_{kr,l} + \dot{v}_{lr,k}\Big) + \frac{1}{2}E^*_{ijkl}\sum_{s=r+1}^{N}h^{s+1}\Big(\overset{r}{b}_{ks}\dot{v}_{ls} + \overset{r}{b}_{ls}\dot{v}_{ks}\Big)$$

$$+\tilde{b}_{ij}h^{r+1}\psi_r + b^*_{ij}h^{r+1}\dot{\psi}_r + d_{ij\gamma}h^{r+1}\psi_{r,\gamma} + d_{ijk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\psi_s + d^*_{ij\gamma}h^{r+1}\dot{\psi}_{r,\gamma}$$

$$+d^*_{ijk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\dot{\psi}_s + p_{\gamma ij}h^{r+1}\widetilde{\chi}_{r,\gamma} + p_{kij}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\chi}_s$$

$$+p^*_{\gamma ij}h^{r+1}\dot{\widetilde{\chi}}_{r,\gamma} + p^*_{kij}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\dot{\widetilde{\chi}}_s + q_{\gamma ij}h^{r+1}\widetilde{\eta}_{r,\gamma} + q_{kij}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\eta}_s$$

$$+q^*_{\gamma ij}h^{r+1}\dot{\widetilde{\eta}}_{r,\gamma} + q^*_{kij}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\dot{\widetilde{\eta}}_s, \quad i,j = \overline{1,3}, \ \ r = \overline{0,N};$$

$$H_{jr} = \frac{1}{2}d_{klj}h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}d_{klj}\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}v_{hs} + \overset{r}{b}_{ls}v_{ks})$$

$$+\frac{1}{2}d^*_{klj}h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}d^*_{klj}\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{hs} + \overset{r}{b}_{ls}\dot{v}_{ks})$$

$$+d_jh^{r+1}\psi_r + d^*_jh^{r+1}\dot{\psi}_r$$

$$+\tilde{\alpha}_{ji}\Big(h^{r+1}\psi_{r,i} + \sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{is}\psi_s\Big) + \alpha^*_{ji}\Big(h^{r+1}\dot{\psi}_{r,i} + \sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{is}\dot{\psi}_s\Big),$$

$$j = \overline{1,3}, \ r = \overline{0,N},$$

$$H_{0r} = -\tilde{b}_{ij}\Big[\frac{1}{2}h^{r+1}\Big(v_{ir,j} + v_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{N}h^{s+1}\Big(\overset{r}{b}_{is}v_{js} + \overset{r}{b}_{js}v_{is}\Big)\Big]$$

$$-\tilde{\xi}h^{r+1}\psi_r - d_i\Big(h^{r+1}\psi_{r,i} + \sum_{s=r+1}^{N}\overset{r}{b}_{is}\psi_s\Big)$$

$$-b^*_{ij}\Big[\frac{1}{2}h^{r+1}\Big(\dot{v}_{ir,j} + \dot{v}_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{N}h^{s+1}\Big(\overset{r}{b}_{is}\dot{v}_{js} + \overset{r}{b}_{js}\dot{v}_{is}\Big)\Big]$$

$$-\xi^* h^{r+1}\dot{\psi}_r - d_i^*\Big(h^{r+1}\dot{\psi}_{r,i} + \sum_{s=r+1}^{N} \overset{r}{b_{is}}\dot{\psi}_s\Big), \quad r = \overline{0, N};$$

$$D_{jr} = \frac{1}{2}p_{jkl}h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}p_{jkl}\sum_{s=r+1}^{N} h^{s+1}(\overset{r}{b_{ks}}v_{ls} + \overset{r}{b_{ls}}v_{ks})$$

$$+\frac{1}{2}p_{jkl}^* h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}p_{jkl}^*\sum_{s=r+1}^{N} h^{s+1}(\overset{r}{b_{ks}}\dot{v}_{ls} + \overset{r}{b_{ls}}\dot{v}_{ks})$$

$$-\varsigma_{j\gamma}h^{r+1}\widetilde{\chi}_{r,\gamma} - \varsigma_{jk}\sum_{s=r+1}^{N} h^{s+1}\overset{r}{b_{ks}}\widetilde{\chi}_s$$

$$-\tilde{a}_{j\gamma}h^{r+1}\widetilde{\eta}_{r,\gamma} - \tilde{a}_{jk}\sum_{s=r+1}^{N} h^{s+1}\overset{r}{b_{ks}}\widetilde{\eta}_s,$$

$$j = \overline{1, 3}, \quad r = \overline{0, N};$$

$$B_{jr} = \frac{1}{2}q_{jkl}h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}q_{jkl}\sum_{s=r+1}^{N} h^{s+1}(\overset{r}{b_{ks}}v_{ls} + \overset{r}{b_{ls}}v_{ks})$$

$$+\frac{1}{2}q_{jkl}^* h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}q_{jkl}^*\sum_{s=r+1}^{N} h^{s+1}(\overset{r}{b_{ks}}\dot{v}_{ls} + \overset{r}{b_{ls}}\dot{v}_{ks})$$

$$-\tilde{a}_{j\gamma}h^{r+1}\widetilde{\chi}_{r,\gamma} - \tilde{a}_{jk}\sum_{s=r+1}^{N} h^{s+1}\overset{r}{b_{ks}}\widetilde{\chi}_s$$

$$-\xi_{j\gamma}h^{r+1}\widetilde{\eta}_{r,\gamma} - \xi_{jk}\sum_{s=r+1}^{N} h^{s+1}\overset{r}{b_{ks}}\widetilde{\eta}_s \quad j = \overline{1, 3}, \ r = \overline{0, N};$$

$$e_{ijr} = \frac{1}{2}h^{r+1}\Big(v_{ir,j} + v_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{N} h^{s+1}\Big(\overset{r}{b_{is}}v_{js} + \overset{r}{b_{js}}v_{is}\Big),$$

$$i, j = \overline{1, 3}, \ r = \overline{0, N};$$

$$X_{njr} = X_{ijr}n_i$$

$$= \Big\{\frac{1}{2}E_{ijkl}h^{r+1}\Big(v_{kr,l} + v_{lr,k}\Big) + \frac{1}{2}E_{ijkl}\sum_{s=r+1}^{N} h^{s+1}\Big(\overset{r}{b_{ks}}v_{ls} + \overset{r}{b_{ls}}v_{ks}\Big)$$

$$+\frac{1}{2}E_{ijkl}^* h^{r+1}\Big(\dot{v}_{kr,l} + \dot{v}_{lr,k}\Big) + \frac{1}{2}E_{ijkl}^*\sum_{s=r+1}^{N} h^{s+1}\Big(\overset{r}{b_{ks}}\dot{v}_{ls} + \overset{r}{b_{ls}}\dot{v}_{ks}\Big)$$

40

$$+\tilde{b}_{ij}h^{r+1}\psi_r + b_{ij}^*h^{r+1}\dot{\psi}_r + d_{ij\gamma}h^{r+1}\psi_{r,\gamma} + d_{ijk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\psi_s + d_{ij\gamma}^*h^{r+1}\dot{\psi}_{r,\gamma}$$

$$+d_{ijk}^*\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\dot{\psi}_s + p_{\gamma ij}h^{r+1}\widetilde{\chi}_{r,\gamma} + p_{kij}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\chi}_s$$

$$+p_{\gamma ij}^*h^{r+1}\dot{\widetilde{\chi}}_{r,\gamma} + p_{kij}^*\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\dot{\widetilde{\chi}}_s + q_{\gamma ij}h^{r+1}\widetilde{\eta}_{r,\gamma} + p_{kij}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\eta}_s$$

$$+q_{\gamma ij}^*h^{r+1}\dot{\widetilde{\eta}}_{r,\gamma} + q_{kij}^*\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\dot{\widetilde{\eta}}_s\Bigg\}n_i, \quad j=\overline{1,3}, \quad r=\overline{0,N};$$

$$H_{nr} := H_{jr}n_j = \Bigg\{\frac{1}{2}d_{klj}h^{r+1}(v_{kr,l}+v_{lr,k}) + \frac{1}{2}d_{klj}\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}v_{hs}+\overset{r}{b}_{ls}v_{ks})$$

$$+\frac{1}{2}d_{klj}^*h^{r+1}(\dot{v}_{kr,l}+\dot{v}_{lr,k}) + \frac{1}{2}d_{klj}^*\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{hs}+\overset{r}{b}_{ls}\dot{v}_{ks})$$

$$+d_jh^{r+1}\psi_r + d_j^*h^{r+1}\dot{\psi}_r$$

$$+\tilde{\alpha}_{ji}\Big(h^{r+1}\psi_{r,i}+\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{is}\psi_s\Big) + \alpha_{ji}^*\Big(h^{r+1}\dot{\psi}_{r,i}+\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{is}\dot{\psi}_s\Big)\Bigg\}n_j,$$
$$r=\overline{0,N};$$

$$D_{nr} := D_{jr}n_j = \Bigg\{\frac{1}{2}p_{jkl}h^{r+1}(v_{kr,l}+v_{lr,k}) + \frac{1}{2}p_{jkl}\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}v_{ls}+\overset{r}{b}_{ls}v_{ks})$$

$$+\frac{1}{2}p_{jkl}^*h^{r+1}(\dot{v}_{kr,l}+\dot{v}_{lr,k}) + \frac{1}{2}p_{jkl}^*\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{ls}+\overset{r}{b}_{ls}\dot{v}_{ks})$$

$$-\varsigma_{j\gamma}h^{r+1}\widetilde{\chi}_{r,\gamma} - \varsigma_{jk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\chi}_s$$

$$-\tilde{a}_{j\gamma}h^{r+1}\widetilde{\eta}_{r,\gamma} - \tilde{a}_{jk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\eta}_s\Bigg\}n_j, \quad r=\overline{0,N};$$

$$B_{nr} := B_{jr}n_j = \left\{ \frac{1}{2}q_{jkl}h^{r+1}(v_{kr,l} + v_{lr,k}) + \frac{1}{2}q_{jkl}\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}v_{ls} + \overset{r}{b}_{ls}v_{ks}) \right.$$

$$+\frac{1}{2}q^*_{jkl}h^{r+1}(\dot{v}_{kr,l} + \dot{v}_{lr,k}) + \frac{1}{2}q^*_{jkl}\sum_{s=r+1}^{N}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{ls} + \overset{r}{b}_{ls}\dot{v}_{ks})$$

$$-\tilde{a}_{j\gamma}h^{r+1}\widetilde{\mathcal{X}}_{r,\gamma} - \tilde{a}_{jk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\mathcal{X}}_s$$

$$\left. -\xi_{j\gamma}h^{r+1}\widetilde{\eta}_{r,\gamma} - \xi_{jk}\sum_{s=r+1}^{N}h^{s+1}\overset{r}{b}_{ks}\widetilde{\eta}_s \right\}n_j, \quad r = \overline{0,N}.$$

From (3.23)-(3.26) and from the ICs following them we easily obtain BCs and ICs in terms of v_{ir}, ψ_r, $\bar{\mathcal{X}}_r$, $\bar{\eta}_r$, provided prismatic shells are not cusped ones (concerning BCs for cusped prismatic shells see Remark 3.2).

4 $N = 0$ Approximation

For the sake of simplicity we rewrite the equations (3.27), (3.28), (3.29), (3.30); (3.14), (3.16), (3.18), (3.20), (3.22), (3.9) for the $N = 0$ approximation and derive the governing equations of the $N = 0$ approximation. We first have

$$X_{\alpha j0,\alpha} + \overset{0}{X}_j = \rho\frac{\partial^2 h v_{j0}}{\partial t^2}, \quad j = \overline{1,3}, \tag{4.1}$$

$$H_{\alpha 0,\alpha} + H_{00} + \overset{0}{H} = \rho k\frac{\partial^2 h\psi_0}{\partial t^2}, \tag{4.2}$$

$$D_{\alpha 0,\alpha} + \overset{0}{D} = f_{e0}, \tag{4.3}$$

$$B_{\alpha 0,\alpha} + \overset{0}{B} = 0, \tag{4.4}$$

$$v_{j0} = \frac{u_{j0}}{h}, \quad \psi_0 = \frac{\varphi_0}{h};$$

$$X_{ij0} = \frac{1}{2}E_{ijkl}h\left(v_{k0,l} + v_{l0,k}\right) + \frac{1}{2}E^*_{ijkl}h\left(\dot{v}_{k0,l} + \dot{v}_{l0,k}\right)$$

$$+\tilde{b}_{ij}h\psi_0 + b^*_{ij}h\dot{\psi}_0 + d_{ij\gamma}h\psi_{0,\gamma} + d^*_{ij\gamma}h\dot{\psi}_{0,\gamma} + p_{\gamma ij}h\widetilde{\mathcal{X}}_{r,\gamma}$$

$$+p^*_{\gamma ij}h\widetilde{\dot{\mathcal{X}}}_{0,\gamma} + q_{\gamma ij}h\widetilde{\eta}_{0,\gamma} + q^*_{\gamma ij}h\widetilde{\dot{\eta}}_{0,\gamma}, \quad i,j = \overline{1,3}, \tag{4.5}$$

$$\tilde{\mathcal{X}}_0 := \frac{\mathcal{X}_0}{h}, \quad \tilde{\eta}_0 := \frac{\eta_0}{h};$$

$$H_{j0} = \frac{1}{2}d_{klj}h(v_{k0,l} + v_{l0,k}) + \frac{1}{2}d^*_{klj}h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) + d_j h\psi_0 + d^*_j h\dot{\psi}_0$$

$$+\tilde{\alpha}_{ji}h\psi_{0,i} + \alpha^*_{ji}h\dot{\psi}_{0,i}, \quad j = \overline{1,3}, \tag{4.6}$$

$$H_{00} = -\frac{1}{2}\tilde{b}_{ij}h\left(v_{i0,j} + v_{j0,i}\right) - \tilde{\xi}h\psi_0 - d_i h\psi_{0,i} - \frac{1}{2}b^*_{ij}h\left(\dot{v}_{i0,j} + \dot{v}_{j0,i}\right)$$

$$-\xi^* h\dot{\psi}_0 - d^*_i h\dot{\psi}_{0,i}; \tag{4.7}$$

$$D_{j0} = \frac{1}{2}p_{jkl}h(v_{k0,l} + v_{l0,k}) + \frac{1}{2}p^*_{jkl}h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) + \varsigma_{j\gamma}h\widetilde{\chi}_{0,\gamma}$$

$$+\tilde{a}_{j\gamma}h\widetilde{\eta}_{0,\gamma}, \quad j = \overline{1,3}; \tag{4.8}$$

$$B_{j0} = \frac{1}{2}q_{jkl}h(v_{k0,l} + v_{l0,k}) + \frac{1}{2}q^*_{jkl}h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) + \tilde{a}_{j\gamma}h\widetilde{\chi}_{0,\gamma}$$

$$+\xi_{j\gamma}h\widetilde{\eta}_{0,\gamma}, \quad j = \overline{1,3}; \tag{4.9}$$

$$e_{ij0} = \frac{1}{2}h\left(v_{i0,j} + v_{j0,i}\right), \quad i,j = \overline{1,3}, \tag{4.10}$$

respectively.

Now substituting (4.5)-(4.9) into (4.1)-(4.4), respectively, for the $N = 0$ approximation we obtain the following governing system of equations with respect to v_{i0}, ψ_0, $\tilde{\chi}_0$, $\tilde{\eta}_0$ in the following form:

$$\frac{1}{2}\left(E_{\alpha ik\delta}hv_{k0,\delta}\right),_\alpha + \frac{1}{2}\left(E_{\alpha i\gamma l}hv_{l0,\gamma}\right),_\alpha + \frac{1}{2}\left(E^*_{\alpha ik\delta}h\dot{v}_{k0,\delta}\right),_\alpha + \frac{1}{2}\left(E^*_{\alpha i\gamma l}h\dot{v}_{l0,\gamma}\right),_\alpha$$

$$+\left(\tilde{b}_{\alpha i}h\psi_0\right),_\alpha + \left(b^*_{\alpha i}h\dot{\psi}_0\right),_\alpha + \left(d_{\alpha i\gamma}h\psi_{0,\gamma}\right),_\alpha + \left(d^*_{\alpha i\gamma}h\dot{\psi}_{0,\gamma}\right),_\alpha + \left(p_{\gamma\alpha i}h\tilde{\chi}_{0,\gamma}\right),_\alpha$$

$$+\left(p^*_{\gamma\alpha i}h\dot{\tilde{\chi}}_{0,\gamma}\right),_\alpha + \left(q_{\gamma\alpha i}h\tilde{\eta}_{0,\gamma}\right),_\alpha + \left(q^*_{\gamma\alpha i}h\dot{\tilde{\eta}}_{0,\gamma}\right),_\alpha + \overset{0}{X}_i = \rho\frac{\partial^2 hv_{i0}}{\partial t^2},$$

$$i = \overline{1,3},$$

i.e.

$$\frac{1}{2}\left(E_{\alpha ik\delta}hv_{k0,\delta}\right),_\alpha + \frac{1}{2}\left(E_{\alpha i\gamma l}hv_{l0,\gamma}\right),_\alpha + \frac{1}{2}\left(E^*_{\alpha ik\delta}h\dot{v}_{k0,\delta}\right),_\alpha + \frac{1}{2}\left(E^*_{\alpha i\gamma l}h\dot{v}_{l0,\gamma}\right),_\alpha$$

$$+\tilde{b}_{\alpha i}h\psi_{0,\alpha} + \left(\tilde{b}_{\alpha i}h\right),_\alpha \psi_0 + b^*_{\alpha i}h\dot{\psi}_{0,\alpha} + \left(b^*_{\alpha i}h\right),_\alpha \dot{\psi}_0$$

$$+\left(d_{\alpha i\gamma}h\psi_{0,\gamma}\right),_\alpha + \left(d^*_{\alpha i\gamma}h\dot{\psi}_{0,\gamma}\right),_\alpha + \left(p_{\gamma\alpha i}h\tilde{\chi}_{0,\gamma}\right),_\alpha$$

$$+\left(p^*_{\gamma\alpha i}h\dot{\tilde{\chi}}_{0,\gamma}\right),_\alpha + \left(q_{\gamma\alpha i}h\tilde{\eta}_{0,\gamma}\right),_\alpha + \left(q^*_{\gamma\alpha i}h\dot{\tilde{\eta}}_{0,\gamma}\right),_\alpha + \overset{0}{X}_i = \rho\frac{\partial^2 hv_{i0}}{\partial t^2}, \tag{4.11}$$

$$i = \overline{1,3},$$

$$\frac{1}{2}\Big(d_{k\delta\alpha}hv_{k0,\delta}\Big)_{,\alpha} +\frac{1}{2}\Big(d_{\gamma l\alpha}hv_{l0,\gamma}\Big)_{,\alpha} +\frac{1}{2}\Big(d^*_{k\delta\alpha}h\dot{v}_{k0,\delta}\Big)_{,\alpha} +\frac{1}{2}\Big(d^*_{\gamma l\alpha}h\dot{v}_{l0,\gamma}\Big)_{,\alpha}$$

$$+\Big(d_\alpha h\psi_0\Big)_{,\alpha} +\Big(d^*_\alpha h\dot{\psi}_0\Big)_{,\alpha} +\Big(\tilde{\alpha}_{\alpha\delta}h\psi_{0,\delta}\Big)_{,\alpha} +\Big(\alpha^*_{\alpha\delta}h\dot{\psi}_{0,\delta}\Big)_{,\alpha}$$

$$-\frac{1}{2}\tilde{b}_{i\beta}hv_{i0,\beta} -\frac{1}{2}\tilde{b}_{\alpha j}hv_{j0,\alpha} -\tilde{\xi}h\psi_0 -d_\alpha h\psi_{0,\alpha}$$

$$-\frac{1}{2}b^*_{i\beta}h\dot{v}_{i0,\beta} -\frac{1}{2}b^*_{\alpha j}h\dot{v}_{j0,\alpha} -\xi^*h\dot{\psi}_0 -d^*_\alpha h\dot{\psi}_{0,\alpha} +\overset{0}{H} = \rho k\frac{\partial^2 h\psi_0}{\partial t^2},$$

i.e., because of $\tilde{b}_{ij} = \tilde{b}_{ji}$, $b^*_{ij} = b^*_{ji}$,

$$\frac{1}{2}\Big(d_{k\delta\alpha}hv_{k0,\delta}\Big)_{,\alpha} +\frac{1}{2}\Big(d_{\gamma l\alpha}hv_{l0,\gamma}\Big)_{,\alpha} +\frac{1}{2}\Big(d^*_{k\delta\alpha}h\dot{v}_{k0,\delta}\Big)_{,\alpha} +\frac{1}{2}\Big(d^*_{\gamma l\alpha}h\dot{v}_{l0,\gamma}\Big)_{,\alpha}$$

$$+\Big(d_\alpha h\Big)_{,\alpha}\psi_0 +\Big(d^*_\alpha h\Big)_{,\alpha}\dot{\psi}_0 +\Big(\tilde{\alpha}_{\alpha\delta}h\psi_{0,\delta}\Big)_{,\alpha} +\Big(\alpha^*_{\alpha\delta}h\dot{\psi}_{0,\delta}\Big)_{,\alpha} -\tilde{b}_{i\alpha}hv_{i0,\alpha}$$

$$-\tilde{\xi}h\psi_0 -b^*_{i\alpha}h\dot{v}_{i0,\alpha} -\xi^*h\dot{\psi}_0 +\overset{0}{H} = \rho k\frac{\partial^2 h\psi_0}{\partial t^2}, \tag{4.12}$$

$$\frac{1}{2}\Big(p_{\alpha k\delta}hv_{k0,\delta}\Big)_{,\alpha} +\frac{1}{2}\Big(p_{\alpha\gamma l}hv_{l0,\gamma}\Big)_{,\alpha} +\frac{1}{2}\Big(p^*_{\alpha k\delta}h\dot{v}_{k0,\delta}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(p^*_{\alpha\gamma l}h\dot{v}_{l0,\gamma}\Big)_{,\alpha} +\Big(\varsigma_{\alpha\gamma}h\tilde{\chi}_{0,\gamma}\Big)_{,\alpha} +\Big(\tilde{a}_{\alpha\gamma}h\tilde{\eta}_{0,\gamma}\Big)_{,\alpha} +\overset{0}{D} = f_{e0}, \tag{4.13}$$

$$\frac{1}{2}\Big(q_{\alpha k\delta}hv_{k0,\delta}\Big)_{,\alpha} +\frac{1}{2}\Big(q_{\alpha\gamma l}hv_{l0,\gamma}\Big)_{,\alpha} +\frac{1}{2}\Big(q^*_{\alpha k\delta}h\dot{v}_{k0,\delta}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(q^*_{\alpha\gamma l}h\dot{v}_{l0,\gamma}\Big)_{,\alpha} +\Big(\tilde{a}_{\alpha\gamma}h\tilde{\chi}_{0,\gamma}\Big)_{,\alpha} +\Big(\xi_{\alpha\gamma}h\tilde{\eta}_{0,\gamma}\Big)_{,\alpha} +\overset{0}{B} = 0. \tag{4.14}$$

For such equations and systems with order degeneracy see Section 8 of the present work and also [11]-[18].

Remark 4.1. System (4.11)-(4.14) we may also obtain from system (3.32)-(3.35) assuming there (3.31).

Evidently,

$$X_{nj0} = X_{ij0}n_i = \Big\{\frac{1}{2}E_{ijkl}h\Big(v_{k0,l} + v_{l0,k}\Big) + \frac{1}{2}E^*_{ijkl}h\Big(\dot{v}_{k0,l} + \dot{v}_{l0,k}\Big)$$

$$+\tilde{b}_{ij}h\psi_0 + b^*_{ij}h\dot{\psi}_0 + d_{ij\gamma}h\psi_{0,\gamma} + d^*_{ij\gamma}h\dot{\psi}_{0,\gamma} + p_{\gamma ij}h\widetilde{\chi}_{0,\gamma}$$

$$+p^*_{\gamma ij}h\dot{\widetilde{\chi}}_{0,\gamma} + q_{\gamma ij}h\widetilde{\eta}_{0,\gamma} + q^*_{\gamma ij}h\dot{\widetilde{\eta}}_{0,\gamma}\Big\} n_i, \quad j = \overline{1,3},$$

$$H_{n0} = H_{j0}n_j = \Big\{\frac{1}{2}d_{klj}h(v_{k0,l} + v_{l0,k}) + \frac{1}{2}d^*_{klj}h(\dot{v}_{k0,l} + \dot{v}_{l0,k})$$

$$+d_jh\psi_0 + d^*_jh\dot{\psi}_0 + \tilde{\alpha}_{ji}h\psi_{0,i} + \alpha^*_{ji}h\dot{\psi}_{0,i}\Big\} n_j,$$

$$D_{n0} = D_{j0}n_j = \left\{ \frac{1}{2}p_{jkl}h(v_{k0,l} + v_{l0,k}) + \frac{1}{2}p_{jkl}^*h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) \right.$$

$$\left. +\varsigma_{j\gamma}h\widetilde{\chi}_{0,\gamma} + \tilde{a}_{j\gamma}h\widetilde{\eta}_{0,\gamma} \right\} n_j,$$

$$B_{n0} = B_{j0}n_j = \left\{ \frac{1}{2}q_{jkl}h(v_{k0,l} + v_{l0,k}) + \frac{1}{2}q_{jkl}^*h(\dot{v}_{k0,l} + \dot{v}_{l0,k}) \right.$$

$$\left. +\tilde{a}_{j\gamma}h\widetilde{\chi}_{0,\gamma} + \xi_{j\gamma}h\widetilde{\eta}_{0,\gamma} \right\} n_j.$$

Now we apply the results of Section 8. To this end we rewrite the system (4.11)-(4.14) for the static case in the following matrix form

$$Lu := (A^{\alpha\beta}u_{,\alpha})_{,\beta} + E^{\alpha}u_{,\alpha} + Cu = F, \quad x \in \omega, \tag{4.15}$$

where

$$x := (x_1, x_2), u = (u_1, ..., u_6)^\top \equiv (v_{10}, v_{20}, v_{30}, \psi_0, \tilde{\chi}_0, \tilde{\eta}_0)^\top, \tag{4.16}$$

$$F := (F_1, ..., F_6)^\top \equiv (-\overset{0}{X}_1, -\overset{0}{X}_2, -\overset{0}{X}_3, -\overset{0}{H}, f_{e0} - \overset{0}{D}, -\overset{0}{B})^\top, \tag{4.17}$$

$$A^{\alpha\beta} := \|a_{kl}^{\alpha\beta}\|, \quad E^{\alpha} := \|e_{kl}^{\alpha}\| \in C^1(\bar{\omega});$$

$$C := \|c_{kl}\| \in C(\bar{\omega}), \ \alpha, \beta = 1, 2, \ k, l = \overline{1, 6}, \tag{4.18}$$

$$A^{\alpha\alpha} := h$$

$$\times \left\| \begin{matrix}
\frac{1}{2}(E_{\underline{\alpha}11\alpha} + E_{\underline{\alpha}1\alpha1}) & \frac{1}{2}(E_{\underline{\alpha}12\alpha} + E_{\underline{\alpha}1\alpha2}) & \frac{1}{2}(E_{\underline{\alpha}13\alpha} + E_{\underline{\alpha}1\alpha3}) & d_{\underline{\alpha}1\alpha} & p_{\alpha\underline{\alpha}1} & q_{\alpha\underline{\alpha}1} \\[4pt]
\frac{1}{2}(E_{\underline{\alpha}21\alpha} + E_{\underline{\alpha}2\alpha1}) & \frac{1}{2}(E_{\underline{\alpha}22\alpha} + E_{\underline{\alpha}2\alpha2}) & \frac{1}{2}(E_{\underline{\alpha}23\alpha} + E_{\underline{\alpha}2\alpha3}) & d_{\underline{\alpha}2\alpha} & p_{\alpha\underline{\alpha}2} & q_{\alpha\underline{\alpha}2} \\[4pt]
\frac{1}{2}(E_{\underline{\alpha}31\alpha} + E_{\underline{\alpha}3\alpha1}) & \frac{1}{2}(E_{\underline{\alpha}32\alpha} + E_{\underline{\alpha}3\alpha2}) & \frac{1}{2}(E_{\underline{\alpha}33\alpha} + E_{\underline{\alpha}3\alpha3}) & d_{\underline{\alpha}3\alpha} & p_{\alpha\underline{\alpha}3} & q_{\alpha\underline{\alpha}3} \\[4pt]
\frac{1}{2}(d_{1\alpha\underline{\alpha}} + d_{\alpha1\underline{\alpha}}) & \frac{1}{2}(d_{2\alpha\underline{\alpha}} + d_{\alpha2\underline{\alpha}}) & \frac{1}{2}(d_{3\alpha\underline{\alpha}} + d_{\alpha3\underline{\alpha}}) & \tilde{\alpha}_{\underline{\alpha}\alpha} & 0 & 0 \\[4pt]
\frac{1}{2}(p_{\underline{\alpha}1\alpha} + p_{\alpha\underline{\alpha}1}) & \frac{1}{2}(p_{\underline{\alpha}2\alpha} + p_{\alpha\underline{\alpha}2}) & \frac{1}{2}(p_{\underline{\alpha}3\alpha} + p_{\alpha\underline{\alpha}3}) & 0 & -\varsigma_{\underline{\alpha}\alpha} & -\tilde{a}_{\underline{\alpha}\alpha} \\[4pt]
\frac{1}{2}(q_{\underline{\alpha}1\alpha} + q_{\alpha\underline{\alpha}1}) & \frac{1}{2}(q_{\underline{\alpha}2\alpha} + q_{\alpha\underline{\alpha}2}) & \frac{1}{2}(q_{\underline{\alpha}3\alpha} + q_{\alpha\underline{\alpha}3}) & 0 & -\tilde{a}_{\underline{\alpha}\alpha} & -\xi_{\underline{\alpha}\alpha}
\end{matrix} \right\|,$$

$$\alpha = 1, 2, \tag{4.19}$$

$$A^{12} := h$$

$$\times \left\| \begin{matrix} E_{2111} & \frac{1}{2}(E_{2121} + E_{2112}) & \frac{1}{2}(E_{2131} + E_{2113}) & d_{211} & p_{121} & q_{121} \\[4pt] E_{2211} & \frac{1}{2}(E_{2221} + E_{2212}) & \frac{1}{2}(E_{2231} + E_{2213}) & d_{221} & p_{122} & q_{122} \\[4pt] E_{2311} & \frac{1}{2}(E_{2321} + E_{2312}) & \frac{1}{2}(E_{2331} + E_{2313}) & d_{231} & p_{123} & q_{123} \\[4pt] d_{112} & \frac{1}{2}(d_{212} + d_{122}) & \frac{1}{2}(d_{312} + d_{132}) & \tilde{\alpha}_{21} & 0 & 0 \\[4pt] p_{211} & \frac{1}{2}(p_{221} + p_{212}) & \frac{1}{2}(p_{231} + p_{213}) & 0 & -\varsigma_{21} & -\tilde{a}_{21} \\[4pt] q_{211} & \frac{1}{2}(q_{221} + q_{212}) & \frac{1}{2}(q_{231} + q_{213}) & 0 & -\tilde{a}_{21} & -\xi_{21} \end{matrix} \right\| , \quad (4.20)$$

$$A^{21} := h$$

$$\times \left\| \begin{matrix} \frac{1}{2}(E_{1112} + E_{1121}) & E_{1122} & \frac{1}{2}(E_{1132} + E_{1123}) & d_{112} & p_{211} & q_{211} \\[4pt] \frac{1}{2}(E_{1212} + E_{1221}) & E_{1222} & \frac{1}{2}(E_{1232} + E_{1223}) & d_{122} & p_{212} & q_{212} \\[4pt] \frac{1}{2}(E_{1312} + E_{1321}) & E_{1322} & \frac{1}{2}(E_{1332} + E_{1323}) & d_{132} & p_{213} & q_{213} \\[4pt] \frac{1}{2}(d_{121} + d_{211}) & d_{221} & \frac{1}{2}(d_{321} + d_{231}) & \tilde{\alpha}_{12} & 0 & 0 \\[4pt] \frac{1}{2}(p_{112} + p_{121}) & p_{122} & \frac{1}{2}(p_{132} + p_{123}) & 0 & -\varsigma_{12} & -\tilde{a}_{12} \\[4pt] \frac{1}{2}(q_{112} + q_{121}) & q_{122} & \frac{1}{2}(q_{132} + q_{123}) & 0 & -\tilde{a}_{12} & -\xi_{12} \end{matrix} \right\| , \quad (4.21)$$

$$E^{\alpha} := h \left\| \begin{matrix} 0 & 0 & 0 & \tilde{b}_{\alpha 1} & 0 & 0 \\ 0 & 0 & 0 & \tilde{b}_{\alpha 2} & 0 & 0 \\ 0 & 0 & 0 & \tilde{b}_{\alpha 3} & 0 & 0 \\ -\tilde{b}_{1\alpha} & -\tilde{b}_{2\alpha} & -\tilde{b}_{3\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \right\| , \quad \alpha = 1, 2;$$

$$(4.22)$$

$$C := \left\| \begin{matrix} 0 & 0 & 0 & (\tilde{b}_{\alpha 1} h)_{,\alpha} & 0 & 0 \\ 0 & 0 & 0 & (\tilde{b}_{\alpha 2} h)_{,\alpha} & 0 & 0 \\ 0 & 0 & 0 & (\tilde{b}_{\alpha 3} h)_{,\alpha} & 0 & 0 \\ 0 & 0 & 0 & (d_{\alpha} h)_{,\alpha} - \tilde{\xi} h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \right\| . \quad (4.23)$$

Keeping in mind some symmetry properties of the constitutive coefficients, from (4.20), (4.21) it follows that

$$
A^{12} := h
\begin{Vmatrix}
E_{2111} & E_{2121} & E_{2131} & d_{211} & p_{121} & q_{121} \\
E_{2211} & E_{2221} & E_{2231} & d_{221} & p_{122} & q_{122} \\
E_{2311} & E_{2321} & E_{2331} & d_{231} & p_{123} & q_{123} \\
d_{112} & d_{212} & d_{312} & \tilde{\alpha}_{21} & 0 & 0 \\
p_{211} & p_{221} & p_{231} & 0 & -\varsigma_{21} & -\tilde{a}_{21} \\
q_{211} & q_{221} & q_{231} & 0 & -\tilde{a}_{21} & -\xi_{21}
\end{Vmatrix},
\tag{4.24}
$$

$$
A^{21} := h
\begin{Vmatrix}
E_{1112} & E_{1122} & E_{1132} & d_{112} & p_{211} & q_{211} \\
E_{1212} & E_{1222} & E_{1232} & d_{122} & p_{212} & q_{212} \\
E_{1312} & E_{1322} & E_{1332} & d_{132} & p_{213} & q_{213} \\
d_{121} & d_{221} & d_{321} & \tilde{\alpha}_{12} & 0 & 0 \\
p_{112} & p_{122} & p_{132} & 0 & -\varsigma_{12} & -\tilde{a}_{12} \\
q_{112} & q_{122} & q_{132} & 0 & -\tilde{a}_{12} & -\xi_{12}
\end{Vmatrix},
\tag{4.25}
$$

Let for any vector

$$
\overset{(\alpha)}{\xi} = \left(\overset{(\alpha)}{\xi}_1, ..., \overset{(\alpha)}{\xi}_6 \right), \quad \alpha = 1, 2,
$$

satisfying

$$
\sum_{\alpha=1}^{2} \sum_{k=1}^{6} \left(\overset{(\alpha)}{\xi}_k \right)^2 > 0,
$$

inequality

$$
\overset{(\alpha)}{\xi} A^{\alpha\beta} \overset{(\beta)}{\xi} > 0, \quad x \in \bar{\omega} \backslash \gamma^0
$$

be valid, where

$$
\gamma^0 := \{ x \in \gamma := \partial\omega : A^{\alpha\beta}(x) = 0, \ \alpha, \beta = 1, 2 \}.
$$

So, system (4.15) with (4.19)-(4.23) is strongly elliptic on $\bar{\omega} \backslash \gamma^0$ while on γ^0 the order of equations degenerates.

In the case under consideration all the elements of the matrices $A^{\alpha\beta}$ contain as factors $h(x)$. Therefore, $\gamma^0 \subseteq \partial\omega$ coincides with the set, where $h(x) = 0$, provided all the constitutive coefficients do not vanish on $\partial\omega$, otherwise they will participate in formation of γ^0.

Let a piecewise smooth curve γ consist of smooth non-intersecting curves $\bar{\gamma}^{(k)}$ which may have only endpoints in common:

$$\gamma = \overset{p}{\underset{k=1}{\cup}} \bar{\gamma}^{(k)}$$

we assume that the Gauss-Ostrogradsky formula is applicable to the domain ω.

At points of smoothness of γ^0 let us consider the matrix

$$\Phi := E^{\alpha} \nu_{\alpha},$$

where $\nu := (\nu_1, \nu_2)$ is the inward normal at the above boundary points. Let further

$$\gamma_0 := \{x \in \gamma^0 \ : \ \Phi(x) = 0\}, \quad \gamma_1 := \{x \in \gamma^0 \ : \ \Phi(x) > 0\},$$

$$\gamma_2 := \{x \in \gamma^0 \ : \ \Phi(x) < 0\}, \quad \gamma_3 : \gamma \setminus \gamma^0.$$

Under matrix inequalities we mean inequalities for corresponding quadratic forms on vectors with nonzero length.

Let $x_0 \in \gamma^0$ be a common point of the neighbour pairs of the curves among $\overline{\gamma}^{(1)}, ..., \overline{\gamma}^{(p)}$. If there exists a neighbourhood of x_0 on γ which $\subset \left(\gamma_2 \cup \gamma_3\right)$, then the above point will be added to $\gamma_2 \cup \gamma_3$, otherwise it will be added to $\gamma_0 \cup \gamma_1$.

Let further

$$\gamma^0 = \gamma_0 \cup \gamma_1 \cup \gamma_2,$$

and consider BC

$$u|_{\gamma_2 \cup \gamma_3} = 0, \tag{4.26}$$

Definition 4.1. Let C_L be the class of bounded vectors u such that

$$u \in C^2(\omega) \cap C(\omega \cup \gamma_2 \cup \gamma_3),$$

$$A^{\alpha\beta} u_{,\alpha}, \ A^{\alpha\beta} u \ E^{\alpha} u \in C^1(\bar{\bar{\omega}}),$$

$$(A^{\alpha\beta} u_{,\alpha})|_{\gamma^0} = 0, \quad Lu \quad \text{be bounded.}$$

Definition 4.2. A vector $u \in C_L$ satisfying system (4.15) and BC (4.26) will be called a regular solution of the BVP (4.15), (4.26).

Definition 4.3. A vector $u \in L_2(\omega)$ will be called a weak solution of the BVP (4.15), (4.26) if the vector $F \in L_2(\omega)$ and u satisfies

$$\int_{\omega} vF dx = \int_{\omega} L^* v \cdot u dx$$

for any $v \in C_{L^*}$ satisfying the condition

$$v|_{\gamma_1 \cup \gamma_3} = 0, \quad v = (v_1, ..., v_6)$$

(the space of such vectors v will be denoted by V), L^* is the adjoint operator to L:

$$L^*(v) := (v_{,\alpha} A^{\alpha\beta})_{,\beta} - (vE^\alpha)_{,\alpha} + vC.$$

Theorem 4.1. *Let* $A^{12} = A^{21}$, *let the matrices* E^α, $\alpha = 1, 2$ *be symmetric, and*

$$E^\alpha_{,\alpha} - 2C \geq 0, \quad in \ \omega, \tag{4.27}$$

then homogeneous BVP, corresponding to the BVP (4.15), (4.26) has only the trivial solution in C_L *if*

$$uA^{\alpha\beta}u_{,\beta} + \frac{1}{2}uE^\alpha u \in C^1(\bar{G}), \quad \gamma_1 \cup \gamma_2 \cup \gamma_3 \neq \varnothing.$$

Theorem 4.2. *Let* $A^{12} = A^{21}$, *let the matrices* E^α *be symmetric, and*

$$\xi(E^\alpha_{,\alpha} - 2C)\xi \geq c_0 \sum_{k=1}^{m} \xi_k^2, \quad x \in \omega, \ c_0 = const > 0, \ \xi := (\xi_1, ..., \xi_6) \tag{4.28}$$

then there exists a weak solution of the BVP (4.15), (4.26).

Comparing (4.24) and (4.25) we easily obtain additional restrictions on the constitutive coefficients for fulfillment of $A^{12} = A^{21}$. Namely,

$$E_{1212} = E_{2211}, \quad E_{1312} = E_{2311} \quad E_{1322} = E_{2321};$$

$$d_{121} = d_{112}, \quad d_{221} = d_{212} \quad d_{321} = d_{312};$$

$$p_{112} = p_{211}, \quad p_{122} = p_{221} \quad p_{132} = p_{231};$$

$$q_{112} = q_{211}, \quad q_{122} = q_{221} \quad q_{132} = q_{231}.$$

For symmetry of matrices E^α, $\alpha = 1, 2$, we have to assume $\tilde{b}_{ij} = 0$ (see (4.22)), which implies that $E^\alpha \equiv 0$, $\alpha = 1, 2$. Since $\tilde{\xi} > 0$, (4.27) holds because of $E^\alpha \equiv 0$, $\alpha = 1, 2$, and (4.23). Indeed,

$$\xi(-2C)\xi = 2\left[\tilde{\xi}h - (d_\alpha h)_{,\alpha}\right](\xi_4)^2 \geq 0 \quad on \ \bar{\omega}.$$

For the same reason

$$\Phi|_{\gamma^0} = E^\alpha \nu_\alpha|_{\gamma^0} = 0,$$

hence

$$\gamma_1 = \varnothing, \quad \gamma_2 = \varnothing, \quad \gamma^0 = \gamma_0, \quad \gamma = \gamma_0 \cup \gamma_3.$$

So, we have to do with the Keldysh type BVP.

Thus, according to Theorem 4.1, a regular solution of BVP (4.15), (4.26) (the last takes the form

$$u_3|_{\gamma_3} = 0)$$

is unique, provided it exists.

In the case under consideration

$$\xi(E^\alpha,_\alpha - 2C)\xi = -2\xi C\xi = 2\Big[\tilde{\xi}h - (d_\alpha h),_\alpha\Big](\xi_4)^2 > 0 \ \ on \ \ \omega \cup \gamma_3$$

and (4.28) will be fulfilled if

$$\min_{\bar{\omega}}\Big[\tilde{\xi}h - (d_\alpha h),_\alpha\Big] > \frac{c_0}{2}.$$

But $h|_{\gamma^0} = 0$ and $\tilde{\xi}$ should be unbounded in a neighborhood of γ^0. If the last reasoning has a physical sense, then (4.28) holds and, according to Theorem 4.2, a weak solution of problem (4.15), (4.26) exists. If this physical assumption is not reasonable, then we apply an approach mentioned in Remark 8.2 and choose the multiplier $\tilde{\psi}$ suitably for ensuring the fulfillment of condition (4.28).

5 Transversely Isotropic Solids

Let us now consider the transversely isotropic elastic piezoelectric material in the case when the poling axis coincides with one of the material symmetry axes [19]. A material behavior is said to be transversely isotropic if it is invariant with respect to an arbitrary rotation about a given axis. This material behavior is of special importance in the modelling of fibre-reinforced composite materials with a coordinate axis in the fibre direction and assumed isotropic in cross-sections orthogonal to fibre direction [20] (in our case to poling axis as well, since in the case under consideration they coincide). The transverse isotropic model is also suitable for biological applications because it adequately describes the elastic properties of bundled fibers aligned in one direction [21] (see also [22]).

It is well-known [19] that the electric field that develops in piezoelectrics can be assumed to be quasi-static because the velocity of the elastic waves is much smaller than the velocity of electromagnetic waves. Therefore, the magnetic field due to the elastic waves is negligible $\mathbf{B} \approx 0$. This fact implies that

$$\frac{\partial \mathbf{B}}{\partial t} \approx 0.$$

So one of Maxwell's equations of electrodynamics becomes

$$rot\,\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \approx 0$$

and, as it was already assumed,

$$\mathbf{E} = -grad\,\chi.$$

Consequently, considering transversely isotropic piezoelectric continuum, it will be based on the governing equations of elastodynamics in the care of small deformations and quasi-electrostatic fields. Note that piezoelectric materials show in most cases a crystal structure with a symmetry of hexagonal 6 mm class. In the case when the poling axis coincides with one of the material symmetry axes these materials become transversely isotropic. Restricting to the case of time-harmonic motion with frequency o, i.e., all the sought quantities, s.c. free members of governing equations, and boundary data are represented as the products of e^{iot} and of the same quantities (to avoid redundant indices and symbols we leave the same notation) depending only on the space variables, from the governing equations of dynamics (2.1), (2.3), (2.4), (2.5), (2.8), we get the following governing equations

$$X_{ij,j} + \rho o^2 u_i = -\Phi_i, \quad i = \overline{1,3}; \tag{5.1}$$

$$D_{j,j} = f_e; \tag{5.2}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = \overline{1,3};$$

$$E_i = -\mathcal{X}_{,i}, \quad i = \overline{1,3};$$

$$\begin{pmatrix} X_{11} \\ X_{22} \\ X_{33} \\ X_{23} \\ X_{31} \\ X_{12} \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = C \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{5.3}$$

where (see [19])

$$C := $$
$$\begin{pmatrix}
E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\
E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\
E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 & 0 & 0 & p_{333} \\
0 & 0 & 0 & E_{2323} & 0 & 0 & 0 & p_{113} & 0 \\
0 & 0 & 0 & 0 & E_{2323} & 0 & p_{113} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1122}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{113} & 0 & -\varsigma_{11} & 0 & 0 \\
0 & 0 & 0 & p_{113} & 0 & 0 & 0 & -\varsigma_{11} & 0 \\
p_{311} & p_{311} & p_{333} & 0 & 0 & 0 & 0 & 0 & -\varsigma_{33}
\end{pmatrix} \tag{5.4}$$

From (5.3), (5.4) we have

$$X_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} - p_{311}E_3,$$

$$X_{22} = E_{1122}e_{11} + E_{1111}e_{22} + E_{1133}e_{33} - p_{311}E_3,$$

$$X_{33} = E_{1133}e_{11} + E_{1133}e_{22} + E_{3333}e_{33} - p_{333}E_3,$$

$$X_{23} = 2E_{2323}e_{23} - p_{113}E_2, \quad X_{31} = 2E_{2323}e_{31} - p_{113}E_1,$$

$$X_{12} = (E_{1111} - E_{1122})e_{12},$$

$$D_1 = 2p_{113}e_{13} + \varsigma_{11}E_1, \quad D_2 = 2p_{113}e_{23} + \varsigma_{11}E_2,$$

$$D_3 = p_{311}e_{11} + p_{311}e_{22} + p_{333}e_{33} + \varsigma_{33}E_3,$$

i.e.,

$$X_{11} = E_{1111}u_{1,1} + E_{1122}u_{2,2} + E_{1133}u_{3,3} - p_{311}E_3,$$

$$X_{22} = E_{1122}u_{1,1} + E_{1111}u_{2,2} + E_{1133}u_{3,3} - p_{311}E_3,$$

$$X_{33} = E_{1133}u_{1,1} + E_{1133}u_{2,2} + E_{3333}u_{3,3} - p_{333}E_3,$$

$$X_{23} = E_{2323}(u_{2,3} + u_{3,2}) - p_{113}E_2, \quad X_{31} = E_{2323}(u_{3,1} + u_{1,3}) - p_{113}E_1, \qquad (5.5)$$

$$X_{12} = \frac{1}{2}(E_{1111} - E_{1122})(u_{1,2} + u_{2,1}),$$

$$D_1 = p_{113}(u_{3,1} + u_{1,3}) + \varsigma_{11}E_1, \quad D_2 = p_{113}(u_{2,3} + u_{3,2}) + \varsigma_{11}E_2,$$

$$D_3 = p_{311}u_{1,1} + p_{311}u_{2,2} + p_{333}u_{3,3} + \varsigma_{33}E_3.$$

Conditions of Anti-plane Piezoelectric State [19] have the form

1. $u_1 \equiv 0, \quad u_2 \equiv 0, \quad u_3 \not\equiv 0;$
2. $X_{13} \not\equiv 0, \quad X_{23} \not\equiv 0; \quad X_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad X_{33} \equiv 0;$
3. $e_{13} \not\equiv 0, \quad e_{23} \not\equiv 0; \quad e_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad e_{33} \equiv 0;$ \qquad (5.6)
4. $E_1 \not\equiv 0, \quad E_2 \not\equiv 0, \quad E_3 \equiv 0;$
5. $D_1 \not\equiv 0, \quad D_2 \not\equiv 0, \quad D_3 \equiv 0.$

Taking into account (5.6), from the first three relations of (5.5) we have

$$u_{3,3} \equiv 0, \quad u_3 = u_3(x_1, x_2);$$

the fourth and fifth relations of (5.5) give

$$X_{23} = E_{2323}u_{3,2} - p_{113}E_2, \tag{5.7}$$

$$X_{31} = E_{2323}u_{3,1} - p_{113}E_1, \tag{5.8}$$

respectively;
the sixth of (5.5) is identically fulfilled;
the seventh and eighth relations of (5.5) give

$$D_1 = p_{113}u_{3,1} + \varsigma_{11}E_1, \tag{5.9}$$

$$D_2 = p_{113}u_{3,2} + \varsigma_{11}E_2. \tag{5.10}$$

respectively;
the ninth of (5.5) is identically fulfilled.

From the first two of (5.1) it follows that

$$\Phi_\alpha \equiv 0, \quad \alpha = 1, 2, \tag{5.11}$$

the third of (5.1) will have the form

$$X_{31,1} + X_{32,2} + \rho o^2 u_3 = -\Phi_3; \tag{5.12}$$

while (5.2) will have the form

$$D_{1,1} + D_{2,2} = f_e. \tag{5.13}$$

Substituting (5.7) and (5.8) into (5.12), and (5.9) and (5.10) into (5.13), we get

$$(E_{2323}u_{3,1})_{,1} + (E_{2323}u_{3,2})_{,2} - (p_{113}E_1)_{,1} - (p_{113}E_2)_{,2} + \rho o^2 u_3 = -\Phi_3,$$

and

$$(p_{113}u_{3,1})_{,1} + (p_{113}u_{3,2})_{,2} + (\varsigma_{11}E_1)_{,1} + (\varsigma_{11}E_2)_{,2} = f_e,$$

respectively.

Taking into account

$$E_\alpha = -\chi_{,\alpha}, \quad \alpha = 1, 2,$$

we obtain the following governing equations in the anti-plane piezoelectric state

$$(E_{2323}u_{3,1})_{,1} + (E_{2323}u_{3,2})_{,2} + (p_{113}\chi_{,1})_{,1} + (p_{113}\chi_{,2})_{,2} + \rho o^2 u_3 = -\Phi_3,$$

$$(p_{113}u_{3,1})_{,1} + (p_{113}u_{3,2})_{,2} - (\varsigma_{11}\chi_{,1})_{,1} - (\varsigma_{11}\chi_{,2})_{,2} = f_e,$$

i.e.,

$$(E_{2323}u_{3,\alpha})_{,\alpha} + (p_{113}\chi_{,\alpha})_{,\alpha} + \rho o^2 u_3 = -\Phi_3, \tag{5.14}$$

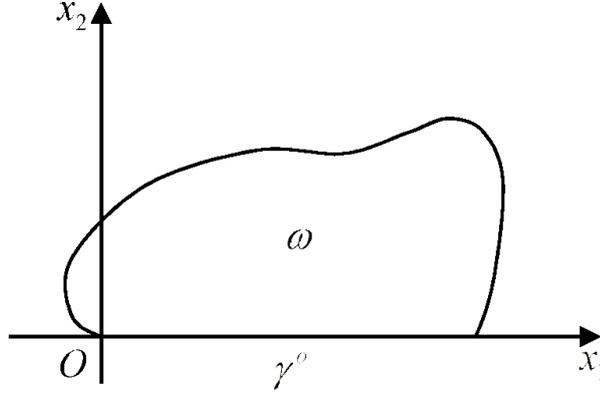$$(p_{113}u_{3,\alpha})_{,\alpha} - (\varsigma_{11}\chi_{,\alpha})_{,\alpha} = f_e. \tag{5.15}$$

Figure 5.1:

Let the plane domain of interest have the form given in Figure 5.1 and let

$$E_{2323} = E_0 x_2^{\kappa_1}, \quad E_0 = const > 0, \quad \kappa_1 = const \geq 0;$$

$$p_{113} = p_0 x_2^{\kappa_2}, \quad p_0 = const > 0, \quad \kappa_2 = const \geq 0;$$

$$\varsigma_{11} = \varsigma_0 x_2^{\kappa_3}, \quad \varsigma_0 = const > 0, \quad \kappa_3 = const \geq 0,$$

then (5.14) and (5.15) take the forms

$$E_0 (x_2^{\kappa_1} u_{3,\alpha})_{,\alpha} + p_0 (x_2^{\kappa_2} \chi_{,\alpha})_{,\alpha} + \rho o^2 u_3 = -\Phi_3, \tag{5.16}$$

and

$$p_0 (x_2^{\kappa_2} u_{3,\alpha})_{,\alpha} - \varsigma_0 (x_2^{\kappa_3} \chi_{,\alpha})_{,\alpha} = f_e, \tag{5.17}$$

respectively.

We will consider the following two cases. To this end first we state the following

Theorem 5.1 (Jaiani, see [12]). *If the coefficients a_α, $\alpha = 1, 2$, and c of the equation*

$$x_2^{\kappa_\alpha} u_{,\alpha\alpha} + a_\alpha(x_1, x_2) u_{,\alpha} + c(x_1, x_2) u = 0, \quad c \leq 0, \quad \kappa_\alpha = const \geq 0, \quad \alpha = 1, 2,$$

are analytic in $\overline{\omega}$, then
(i) if either $\kappa_2 < 1$, or $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) < x_2^{\kappa_2 - 1} \tag{5.18}$$

in $\overline{\omega}_\delta$ for some $\delta = const > 0$, where

$$\omega_\delta := \{(x_1, x_2) \in \omega \ : \ 0 < x_2 < \delta\},$$

the Dirichlet problem (Problem D, $u \in C^2(\omega) \cap C(\overline{\omega})$) is well-posed;

(ii) if $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) \geq x_2^{\kappa_2 - 1} \tag{5.19}$$

in ω_δ and $a_1(x_1, x_2) = O(x_2^{\kappa_1})$, $x_2 \to 0_+$ (O is the Landau symbol), the Keldysh problem (Problem E, bounded $u \in C^2(\omega) \cap C(\bar{\omega} \setminus \gamma^0)$) is well-posed.

Case 1. $\kappa_i = \kappa = const \geq 0$, $i = \overline{1, 3}$.

After some actions, from (5.16) and (5.17) we get

$$(\varsigma_0 E_0 + p_0^2)(x_2^\kappa u_{3,\alpha})_{,\alpha} + \varsigma_0 \rho o^2 u_3 = -\varsigma_0 \Phi_3 + p_0 f_e, \tag{5.20}$$

and

$$(p_0^2 + \varsigma_0 E_0)(x_2^\kappa \chi_{,\alpha})_{,\alpha} + p_0 \rho o^2 u_3 = -p_0 \Phi_3 - E_0 f_e. \tag{5.21}$$

(5.20) and (5.21) we rewrite as

$$x_2 u_{3,\alpha\alpha} + \kappa u_{3,2} + \varsigma_0 (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho o^2 u_3$$

$$= (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-\varsigma_0 \Phi_3 + p_0 f_e), \tag{5.22}$$

and

$$x_2 \chi_{,\alpha\alpha} + \kappa \chi_{,2} + p_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho o^2 u_3$$

$$= (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \tag{5.23}$$

respectively.

In the static case $o = 0$ and from (5.22), (5.23) we obtain separate equations

$$x_2 u_{3,\alpha\alpha} + \kappa u_{3,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-\varsigma_0 \Phi_3 + p_0 f_e) \tag{5.24}$$

$$x_2 \chi_{,\alpha\alpha} + \kappa \chi_{,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \tag{5.25}$$

with respect to u_3 and χ, correspondingly.

We will consider

Problem D. Find solutions $u_3, \chi \in C^2(\omega) \cap C(\bar{\omega})$ of the system (5.24), (5.25) by their values prescribed on $\partial \omega$

and

Problem E. Find bounded solutions $u_3, \chi \in C^2(\omega) \cap C(\omega \cup (\partial \omega \setminus \gamma^0))$ of the system (5.24), (5.25) by their values prescribed only on the arc $\partial \omega \setminus \overline{\gamma^0}$.

Now we are about to apply Theorem 5.1 but this theorem concerns the homogeneous equation whereas equations (5.24) and (5.25) are nonhomogeneous ones. If we find their particular solutions which are continuous on $\bar{\omega}$, then in usual way we reduce the problems under consideration to BVPs for the homogeneous equations corresponding to equations (5.24), (5.25) with boundary data changed according to boundary values of the particular solutions. In case of Problem E it is sufficient to find a bounded particular solution continuous on $\bar{\omega} \setminus \overline{\gamma^0}$. If Φ_3 and f_e depend only on x_2, then we easily find such particular solutions, depending only on x_2. For another way of finding of desired particular solutions we refer the reader to [[23], pp. 75-78]. Further applying to nonhomogeneous equations Theorem 5.1 we will always have in mind this observation. According to Theorem 5.1 it follows

Theorem 5.2. *The values of u_3 and χ should be prescribed on the entire boundary (Problem D) for $\kappa < 1$, while on the part of the boundary, where $x_2 = 0$, should be freed at all of boundary conditions (Problem E) for $\kappa \geq 1$. Both problems are uniquely solvable in the classical sense.*

Proof. Indeed, for $\kappa < 1$ and $\kappa \geq 1$, correspondingly, (5.18) and (5.19) are fulfilled, which proves the theorem. $\qquad\square$

We now consider system (5.22), (5.23), assuming that the right-hand sides belong to $L_2(\omega)$ and

$$x_2^{1-\kappa}\rho \in C(\bar{\omega}).$$

Equations (5.22) and (5.23) we rewrite as follows

$$(x_2 u_{3,\alpha})_{,\alpha} + (\kappa - 1)u_{3,2} + \varsigma_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\rho o^2 u_3$$

$$= (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-\varsigma_0 \Phi_3 + p_0 f_e), \tag{5.26}$$

and

$$(x_2 \chi_{,\alpha})_{,\alpha} + (\kappa - 1)\chi_{,2} = -p_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\rho o^2 u_3$$

$$+ (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \tag{5.27}$$

Intending to apply results of Section 9, we check for the first equation (5.26) of the system which contains only the unknown u_3, the condition (9.15). In our case it looks like

$$\frac{1}{2}E^\alpha_{,\alpha} - C = \frac{1}{2}(\kappa - 1) - \varsigma_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\rho o^2 > c_0 > 0, \text{ on } \bar{\omega}.$$

The last condition will be satisfied for

$$o^2 < \left[\frac{1}{2}(\kappa - 1) - c_0\right](E_0 + \varsigma_0^{-1} p_0^2)\left[\max_{(x_1,x_2)\in\bar{\omega}} (x_2^{1-\kappa}\rho)\right]^{-1} \tag{5.28}$$

provided

$$\kappa > 1 + 2c_0, \tag{5.29}$$

where c_0 may be arbitrary small.

Therefore there exists an $\mathcal{H}$-weak solution u_3 of Problem E for equation (5.26). Substituting the found u_3 into equation (5.27) we obtain the equation only with respect to χ. We now apply Theorem 5.1 to equation (5.27) and conclude that there exist unique classical solutions for Problem D if $\kappa < 1$ and Problem E if $\kappa \geq 1$. Evidently, classical solutions are $\mathcal{H}$-weak solutions as well

So, we have proved the following

Theorem 5.3. *An $\mathcal{H}$-weak solution (u_3, χ) of Problem E for system (5.26), (5.27) exists under conditions (5.28), (5.29). The last actually means*

$$\kappa > 1$$

because of arbitrarily smallness of the constant c_0.

Remark 5.1. If $\bar{\omega}$ is a stripe $\{-\infty < x_1 < +\infty,\ 0 \le x_2 \le L = const\}$ and all the quantities depend only on x_2 (it means that we consider cylindrical strain) then in the static case $(o = 0)$ from (5.20) and (5.21) we obtain

$$(x_2^{\kappa} u_{3,2})_{,2} = (\varsigma_0 E_0 + p_0^2)^{-1}(-\varsigma_0 \Phi_3 + p_0 f_e)$$

and

$$(x_2^{\kappa} \chi_{,2})_{,2} = (\varsigma_0 E_0 + p_0^2)^{-1}(-p_0 \Phi_3 - E_0 f_e),$$

respectively. Their general solutions have the forms

$$u_3(x_2) = (\varsigma_0 E_0 + p_0^2)^{-1} \int_L^{x_2} \frac{d\tau}{\tau^{\kappa}} \int_L^{\xi} [-\varsigma_0 \Phi_3(t) + p_0 f_e(t)]dt + c_2^1$$

$$+c_1^1 \begin{cases} (1-\kappa)^{-1}(x_2^{1-\kappa} - L^{1-\kappa}) & \text{for} \quad \kappa \ne 1, \\ \ln x_2 - \ln L & \text{for} \quad \kappa = 1 \end{cases}$$

and

$$\chi(x_2) = (\varsigma_0 E_0 + p_0^2)^{-1} \int_L^{x_2} \frac{d\tau}{\tau^{\kappa}} \int_L^{\xi} [-p_0 \Phi_3(t) - E_0 f_e(t)]dt + c_2^2$$

$$+c_1^2 \begin{cases} (1-\kappa)^{-1}(x_2^{1-\kappa} - L^{1-\kappa}) & \text{for} \quad \kappa \ne 1, \\ \ln x_2 - \ln L & \text{for} \quad \kappa = 1. \end{cases}$$

In the case under consideration BCs look like

$$u_3(0) = c_0^1,\ \chi(0) = c_0^2;\ \ u_3(L) = c_L^1,\ \chi(L) = c_L^2 \text{ (Problem D)};$$

$$u_3(x_2) = O(1),\ \chi(x_2) = O(1),\ x_2 \to 0+;\ \ u_3(L) = c_L^1,\ \chi(L) = c_L^2 \text{ (Problem E)}.$$

From these BCs we easily calculate constants

$$c_{\beta}^{\alpha},\ \ \alpha, \beta = 1, 2,\ \text{ for } \ \kappa < 1\ \text{ (Problem D)}$$

and

$$c_2^{\alpha},\ \ \alpha = 1, 2,\ \text{ for } \ \kappa \ge 1\ \text{ (Problem E)},$$

in the last case $c_1^{\alpha} = 0$, $\alpha = 1, 2$, (otherwise solutions will be unbounded) and some restrictions on $\Phi_3(x_2)$, $f_e(x_2)$ are required as well.

Case 2. $\kappa_2 = \kappa_3 = \kappa = const \ge 0$.
After some actions, from (5.16) and (5.17) we get

$$((p_0^2 x_2^{\kappa} + \varsigma_0 E_0 x_2^{\kappa_1})u_{3,\alpha})_{,\alpha} + \varsigma_0 \rho o^2 u_3 = -\varsigma_0 \Phi_3 + p_0 f_e, \tag{5.30}$$

$$\varsigma_0 (x_2^{\kappa} \chi_{,\alpha})_{,\alpha} = p_0(x_2^{\kappa} u_{3,\alpha})_{,\alpha} - f_e. \tag{5.31}$$

So, for $\kappa_1 = 0$ and any $\kappa \geq 0$, equation (5.30) is not a degenerate one, while equation (5.31) is a degenerate one. If $o = 0$, i.e., we deal with the static case and from (5.30), (5.31) we arrive at the system

$$((p_0^2 x_2^\kappa + \varsigma_0 E_0)u_{3,\alpha})_{,\alpha} = -\varsigma_0 \Phi_3 + p_0 f_e, \tag{5.32}$$

$$\varsigma_0 (x_2^\kappa \chi_{,\alpha})_{,\alpha} = p_0 (x_2^\kappa u_{3,\alpha})_{,\alpha} - f_e. \tag{5.33}$$

As (5.32) is not a degenerate equation, the values of u_3 should be prescribed on the entire boundary (Problem D), while, according to Theorem 5.1, the values of χ should be prescribed on the entire boundary (Problem D) for $0 \leq \kappa < 1$ and the part where $x_2 = 0$ should be freed of BCs (Problem E) for $\kappa \geq 1$. It will be clear if we rewrite (5.33) in the following form

$$x_2 \chi_{,\alpha\alpha} + \kappa \chi_{,2} = \varsigma_0^{-1}[p_0 (x_2^\kappa u_{3,\alpha})_{,\alpha} - f_e]x_2^{1-\kappa}.$$

Indeed, for $\kappa < 1$ and $\kappa \geq 1$, correspondingly, (5.18) and (5.19) are realized. So we have proved the following

Theorem 5.4. *Problem D for equation (5.32) for all $\kappa \geq 0$ is uniquely solvable in the classical sense. Problem D for $0 \leq \kappa < 1$ and Problem E for $\kappa \geq 1$ for equation (5.33) are uniquele solvable in the classical sense. In other words Problem D for system (5.32), (5.33) has a unique classical solution, while Problem E has a unique classical solution for $\kappa \geq 1$.*

Remark 5.2. Similarly to Case 1 we solve BVPs in the explicit form in the case of cylindrical strain (see Remark 5.1).

Now we come back to the time-harmonic motion and follow Section 9.

Let $\kappa_1 > \kappa$, then we rewrite (5.30) as follows

$$(p_0^2 x_2 + \varsigma_0 E_0 x_2^{\kappa_1 - \kappa + 1})u_{3,\alpha\alpha} + (\kappa p_0^2 + \kappa_1 \varsigma_0 E_0 x_2^{\kappa_1 - \kappa})u_{3,2}$$

$$+\varsigma_0 o^2 \rho x_2^{1-\kappa} u_3 = x_2^{1-\kappa}(p_0 f_e - \varsigma_0 \Phi_3). \tag{5.34}$$

Now, we write equation (5.34) in the following form

$$((p_0^2 x_2 + \varsigma_0 E_0 x_2^{\kappa_1 - \kappa + 1})u_{3,\alpha})_{,\alpha} + (\kappa - 1)(p_0^2 + \varsigma_0 E_0 x_2^{\kappa_1 - \kappa})u_{3,2}$$

$$+\varsigma_0 o^2 \rho x_2^{1-\kappa} u_3 = x_2^{1-\kappa}(p_0 f_e - \varsigma_0 \Phi_3). \tag{5.35}$$

Conditions (9.2) lead to the restrictions

$$\kappa_1 \geq \kappa + 1,$$

$$x_2^{1-\kappa} \rho \in C(\bar{\omega}). \tag{5.36}$$

Equation (5.35) has an order degeneracy by $x_2 = 0$, i.e., on γ^0. According to (9.5) and (9.6)

$$\Phi|_{x_2=0} = E^1 \nu_i|_{x_2=0} = E^2 \nu_2|_{x_2=0} = E^2|_{x_2=0}$$

$$= [(\kappa - 1)p_0^2 - \varsigma_0 E_0(1 - \kappa)x_2^{\kappa_1 - \kappa}]|_{x_2=0} = (\kappa - 1)p_0^2$$

and consequently

$$\gamma^0 = \begin{cases} \gamma_1 & \text{for } \kappa > 1, \\ \gamma_2 & \text{for } \kappa < 1, \\ \gamma_0 & \text{for } \kappa = 1, \end{cases}$$

which means that we have to do with the Dirichlet problem for $\kappa < 1$ and with the Keldysh problem for $\kappa \geq 1$, since $\gamma_0 \cup \gamma_1$ should be freed of BC and at $\gamma_0 \cup \gamma_1$ BC should be replaced by boundedness of u_3 in a neighborhood of $\gamma_0 \cup \gamma_1$.

Condition (9.15) looks like

$$-\frac{1}{2}\varsigma_0 E_0(1 - \kappa)(\kappa_1 - \kappa)x_2^{\kappa_1 - \kappa - 1} - \varsigma_0 o^2 \rho x_2^{1-\kappa} > c_0 > 0 \ \text{ in } \ \overline{\omega},$$

which is not valid since the limit of the left hand side, because of (5.36), tends to

$$\begin{cases} -\infty & \text{for } \kappa > 1; \\ \varsigma_0 o^2 \rho < 0 & \text{for } \kappa = 1, \kappa_1 \geq 2; \\ 0 & \text{for } \kappa < 1, \ \kappa_1 > \kappa + 1; \\ -\frac{1}{2}\varsigma_0 E_0(1 - \kappa) < 0; & \text{for } \kappa < 1, \ \kappa_1 > \kappa + 1, \end{cases}$$

as $x_2 \to 0+$.

To avoid this obstacle we have to apply an approach described in Remark 8.2. For an example of application of the above-mentioned approach we refer the reader to [13], where cusped elastic prismatic shells are considered.

Similarly, when $\kappa > \kappa_1 + 1$, we first rewrite (5.30) as

$$(p_0^2 x_2^{\kappa - \kappa_1 + 1} + \varsigma_0 E_0 x_2)u_{3,\alpha\alpha} + (\kappa p_0^2 x_2^{\kappa - \kappa_1} + \kappa_1 \varsigma_0 E_0)u_{3,2}$$

$$+\varsigma_0 \rho o^2 x_2^{1-\kappa_1} u_3 = x_2^{1-\kappa_1}(p_0 f_e - \varsigma_0 \Phi_3), \tag{5.37}$$

then as follows

$$\left((p_0^2 x_2^{\kappa - \kappa_1 + 1} + \varsigma_0 E_0 x_2)u_{3,\alpha} \right)_{,\alpha} + (\kappa_1 - 1)(p_0^2 x_2^{\kappa - \kappa_1} + \varsigma_0 E_0)u_{3,2}$$

$$+\varsigma_0 \rho o^2 x_2^{1-\kappa_1} u_3 = x_2^{1-\kappa_1}(p_0 f_e - \varsigma_0 \Phi_3) \tag{5.38}$$

and carry out the corresponding reasonings.

After substituting the found solutions of Problem E for equation (5.34), ((5.37)) into (5.31) we arrive at the already investigated problem for equation (5.31) with respect to X with the result that for X Problem E is well-posed for $\kappa > 1$ ($\kappa > 2$).

6 Antiplane Deformation of Piezoelectrics in N=0 Approximation

In the $N = 0$ Approximation conditions of the antiplane state look like

1. $v_{10} \equiv 0, \quad v_{20} \equiv 0, \quad v_{30} \not\equiv 0;$

2. $X_{130} \not\equiv 0, \quad X_{230} \not\equiv 0, \quad X_{\alpha\beta0} \equiv 0, \quad \alpha, \beta = 1, 2; \quad X_{330} \equiv 0;$

3. $e_{130} \not\equiv 0, \quad e_{230} \not\equiv 0, \quad e_{\alpha\beta0} \equiv 0, \quad \alpha, \beta = 1, 2; \quad e_{330} \equiv 0;$

4. $E_{10} \not\equiv 0, \quad E_{20} \not\equiv 0, \quad E_{30} \equiv 0;$

5. $D_{10} \not\equiv 0, \quad D_{20} \not\equiv 0, \quad D_{30} \equiv 0.$

Then from (4.11)-(4.14) we have

$$\frac{1}{2}(E_{\alpha\beta3\delta}hv_{30,\delta}),_\alpha + \frac{1}{2}(E_{\alpha\beta\gamma3}hv_{30,\gamma}),_\alpha + (p_{\gamma\alpha\beta}h\tilde{\chi}_{0,\gamma}),_{\gamma\alpha} + \overset{0}{X}_\beta = 0, \quad \beta = 1, 2,$$

$$\frac{1}{2}(E_{\alpha33\delta}hv_{30,\delta}),_\alpha + \frac{1}{2}(E_{\alpha3\gamma3}hv_{30,\gamma}),_\alpha + (p_{\gamma\alpha3}h\tilde{\chi}_{0,\gamma}),_\alpha + \overset{0}{X}_3 = \rho h\ddot{v}_{30}, \qquad (6.1)$$

$$-(\varsigma_{\alpha\gamma}h\tilde{\chi}_{0,\gamma}),_\alpha + \frac{1}{2}(p_{\alpha3\delta}hv_{30,\delta}),_\alpha + \frac{1}{2}(p_{\alpha\gamma3}hv_{30,\gamma}),_\alpha + \overset{0}{D} = f_{e0}.$$

If we consider transversely isotropic piezoelectric materials, then

$$E_{2323} = E_{1313} \not\equiv 0;$$

$$E_{2222} = E_{1111} \not\equiv 0, \quad E_{1122} \not\equiv 0, \quad E_{2233} = E_{1133} \not\equiv 0, \quad E_{3333} \not\equiv 0; \qquad (6.2)$$

$$p_{223} = p_{113} \not\equiv 0;$$

$$\varsigma_{22} = \varsigma_{11} \not\equiv 0, \quad \varsigma_{33} \not\equiv 0.$$

Other elastic, piezoelectric, and dielectric permittivity constants are identically zero with regard to reciprocal symmetries.

Therefore, from (6.1), by virtue of (6.2), we obtain

$$\overset{0}{X}_\beta = 0, \quad \beta = 1, 2;$$

$$\frac{1}{2}(E_{1331}hv_{30,1}),_1 + \frac{1}{2}(E_{2332}hv_{30,2}),_2 + \frac{1}{2}(E_{1313}hv_{30,1}),_1 + \frac{1}{2}(E_{2323}hv_{30,2}),_2$$

$$+(p_{113}h\tilde{\chi}_{0,1}),_1 + (p_{223}h\tilde{\chi}_{0,2}),_2 + \overset{0}{X}_3 = \rho h\ddot{v}_{30}$$

and

$$-(\varsigma_{11}h\tilde{\chi}_{0,1}),_1 - (\varsigma_{22}h\tilde{\chi}_{0,2}),_2 + \frac{1}{2}(p_{131}hv_{30,1}),_1 + \frac{1}{2}(p_{232}hv_{30,2}),_2$$

$$+\frac{1}{2}(p_{113}hv_{30,1}),_1 + \frac{1}{2}(p_{223}hv_{30,2}),_2 + \overset{0}{D} = f_{e0}.$$

Whence, taking into account

$$E_{1331} = E_{3131} = E_{2332} = E_{2323},$$

$$p_{223} = p_{113}, \quad \varsigma_{11} = \varsigma_{22},$$

we get

$$(E_{2323}hv_{30,\alpha})_{,\alpha} + (p_{113}h\widetilde{\chi}_{0,\alpha})_{,\alpha} + \overset{0}{X}_3 = \rho h \ddot{v}_{30}. \tag{6.3}$$

and

$$(p_{113}hv_{30,\alpha})_{,\alpha} - (\varsigma_{11}h\widetilde{\chi}_{0,\alpha})_{,\alpha} + \overset{0}{D} = f_{e0}, \tag{6.4}$$

respectively.

Now assuming

$$E_{2323}h = E_0 x_2^{\kappa_1}, \quad E_0 = const > 0, \quad \kappa_1 = const \geq 0;$$

$$p_{113}h = p_0 x_2^{\kappa_2}, \quad p_0 = const > 0, \quad \kappa_2 = const \geq 0; \tag{6.5}$$

$$\varsigma_{11}h = \varsigma_0 x_2^{\kappa_3}, \quad \varsigma_0 = const > 0, \quad \kappa_3 = const \geq 0,$$

in the case of time-harmonic vibration from (6.3), (6.4) we arrive at equations (5.16), (5.17); and (similarly to Section 5) at systems (5.20), (5.21); (5.22), (5.23); (5.26), (5.27); (5.31), (5.30); (5.31), for $\kappa_1 > \kappa$ (5.34) i.e. (5.35); and (5.31), for $\kappa > \kappa_1$ (5.37) i.e. (5.38); of the antiplane state in the three-dimensional formulation, where u_3, χ, Φ_3, and f_e should be replaced by v_{30}, $\widetilde{\chi}_0$, $\overset{0}{X}_3$, and $f_{e0} - \overset{0}{D}$ (the factor h should be put in before the zero moment of a weighted unknown v_{30}), respectively. Namely, we obtain the following systems (6.6), (6.7); (6.8), (6.9); (6.10), (6.11); (6.12), (6.13); (6.14), (6.15); (6.14), for $\kappa_1 > \kappa$ (6.16) i.e. (6.17); and (6.14), for $\kappa > \kappa_1$ (6.18) i.e. (6.19):

$$E_0(x_2^{\kappa_1}v_{30,\alpha})_{,\alpha} + p_0(x_2^{\kappa_2}\widetilde{\chi}_{0,\alpha})_{,\alpha} + \rho o^2 h v_{30} = -\overset{0}{X}_3, \tag{6.6}$$

$$p_0(x_2^{\kappa_2}v_{30,\alpha})_{,\alpha} - \varsigma_0(x_2^{\kappa_3}\widetilde{\chi}_{0,\alpha})_{,\alpha} = f_{e0} - \overset{0}{D}; \tag{6.7}$$

$$(\varsigma_0 E_0 + p_0^2)(x_2^{\kappa}v_{30,\alpha})_{,\alpha} + \varsigma_0 \rho h o^2 v_{30} = -\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}), \tag{6.8}$$

$$(p_0^2 + \varsigma_0 E_0)(x_2^{\kappa}\widetilde{\chi}_{0,\alpha})_{,\alpha} + p_0 \rho h o^2 v_{30} = -p_0 \overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D}); \tag{6.9}$$

$$x_2 v_{30,\alpha\alpha} + \kappa v_{30,2} + \varsigma_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho o^2 h v_{30}$$

$$= (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \left[-\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}) \right], \tag{6.10}$$

$$x_2\tilde{\chi}_{0,\alpha\alpha} + \kappa\tilde{\chi}_{0,2} + p_0(\varsigma_0 E_0 + p_0^2)^{-1}x_2^{1-\kappa}\rho o^2 h v_{30}$$

$$= (\varsigma_0 E_0 + p_0^2)^{-1}x_2^{1-\kappa}\Big[-p_0\overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D})\Big]; \tag{6.11}$$

$$(x_2 v_{30,\alpha})_{,\alpha} + (\kappa - 1)v_{30,2} + \varsigma_0(\varsigma_0 E_0 + p_0^2)^{-1}x_2^{1-\kappa}\rho o^2 h v_{30}$$

$$= (\varsigma_0 E_0 + p_0^2)^{-1}x_2^{1-\kappa}\Big[-\varsigma_0\overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D})\Big], \tag{6.12}$$

$$(x_2\tilde{\chi}_{0,\alpha})_{,\alpha} + (\kappa - 1)\tilde{\chi}_{0,2} = -p_0(\varsigma_0 E_0 + p_0^2)^{-1}x_2^{1-\kappa}\rho o^2 h v_{30}$$

$$+(\varsigma_0 E_0 + p_0^2)^{-1}x_2^{1-\kappa}\Big[-p_0\overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D})\Big]; \tag{6.13}$$

$$\varsigma_0(x_2^\kappa\widetilde{\chi}_{0,\alpha})_{,\alpha} = p_0(x_2^\kappa v_{30,\alpha})_{,\alpha} - (f_{e0} - \overset{0}{D}), \tag{6.14}$$

$$((p_0^2 x_2^\kappa + \varsigma_0 E_0 x_2^{\kappa_1})v_{30,\alpha})_{,\alpha} + \varsigma_0\rho h o^2 v_{30} = -\varsigma_0\overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}) \tag{6.15}$$

i.e. (for $\kappa_1 \geq \kappa$)

$$(p_0^2 x_2 + \varsigma_0 E_0 x_2^{\kappa_1-\kappa+1})v_{30,\alpha\alpha} + (\kappa p_0^2 + \kappa_1\varsigma_0 E_0 x_2^{\kappa_1-\kappa})v_{30,2}$$

$$+\varsigma_0\rho o^2 x_2^{1-\kappa}h v_{30} = x_2^{1-\kappa}\Big[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0\overset{0}{X}_3\Big] \tag{6.16}$$

i.e.

$$((p_0^2 x_2 + \varsigma_0 E_0 x_2^{\kappa_1-\kappa+1})v_{30,\alpha})_{,\alpha} + (\kappa - 1)(p_0^2 + \varsigma_0 E_0 x_2^{\kappa_1-\kappa})v_{30,2}$$

$$+\varsigma_0\rho o^2 x_2^{1-\kappa}h v_{30} = x_2^{1-\kappa}[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0\overset{0}{X}_3] \tag{6.17}$$

and (for $\kappa > \kappa_1$)

$$(p_0^2 x_2^{\kappa-\kappa_1+1} + \varsigma_0 E_0 x_2)v_{30,\alpha\alpha} + (\kappa p_0^2 x_2^{\kappa-\kappa_1} + \kappa_1\varsigma_0 E_0)v_{30,2}$$

$$+\varsigma_0\rho o^2 x_2^{1-\kappa_1}h v_{30} = x_2^{1-\kappa_1}\Big[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0\overset{0}{X}_3\Big] \tag{6.18}$$

i.e.

$$((p_0^2 x_2^{\kappa-\kappa_1+1} + \varsigma_0 E_0 x_2) v_{30,\alpha})_{,\alpha} + (\kappa_1 - 1)(p_0^2 x_2^{\kappa-\kappa_1} + \varsigma_0 E_0) v_{30,2}$$

$$+ \varsigma_0 \rho o^2 x_2^{1-\kappa_1} h v_{30} = x_2^{1-\kappa_1}\left[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0 \overset{0}{X}_3 \right]. \tag{6.19}$$

In the static case ($o = 0$) from (6.6), (6.7); (6.8), (6.9); (6.10), (6.11); (6.12), (6.13); (6.14), (6.15), (6.16), (6.17) and (6.18), (6.19) we have

$$E_0(x_2^{\kappa_1} v_{30,\alpha})_{,\alpha} + p_0(x_2^{\kappa_2} \tilde{\chi}_{0,\alpha})_{,\alpha} = -\overset{0}{X}_3, \tag{6.20}$$

$$p_0(x_2^{\kappa_2} v_{30,\alpha})_{,\alpha} - \varsigma_0(x_2^{\kappa_3} \tilde{\chi}_{0,\alpha})_{,\alpha} = f_{e0} - \overset{0}{D}; \tag{6.21}$$

$$(\varsigma_0 E_0 + p_0^2)(x_2^{\kappa} v_{30,\alpha})_{,\alpha} = -\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}), \tag{6.22}$$

$$(p_0^2 + \varsigma_0 E_0)(x_2^{\kappa} \widetilde{\chi}_{0,\alpha})_{,\alpha} = -p_0 \overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D}); \tag{6.23}$$

$$x_2 v_{30,\alpha\alpha} + \kappa v_{30,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\left[-\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}) \right], \tag{6.24}$$

$$x_2 \tilde{\chi}_{0,\alpha\alpha} + \kappa \tilde{\chi}_{0,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\left[-p_0 \overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D}) \right]; \tag{6.25}$$

$$(x_2 v_{30,\alpha})_{,\alpha} + (\kappa - 1) v_{30,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\left[-\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}) \right], \quad (6.26)$$

$$(x_2 \tilde{\chi}_{0,\alpha})_{,\alpha} + (\kappa - 1) \tilde{\chi}_{0,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}\left[-p_0 \overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D}) \right]; \quad (6.27)$$

$$\varsigma_0(x_2^{\kappa} \widetilde{\chi}_{0,\alpha})_{,\alpha} = p_0(x_2^{\kappa} v_{30,\alpha})_{,\alpha} - (f_{e0} - \overset{0}{D}), \tag{6.28}$$

$$((p_0^2 x_2^{\kappa} + \varsigma_0 E_0 x_2^{\kappa_1}) v_{30,\alpha})_{,\alpha} = -\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}) \tag{6.29}$$

i.e. (for $\kappa_1 > \kappa$)

$$(p_0^2 x_2 + \varsigma_0 E_0 x_2^{\kappa_1-\kappa+1}) v_{30,\alpha\alpha} + (\kappa p_0^2 + \kappa_1 \varsigma_0 E_0 x_2^{\kappa_1-\kappa}) v_{30,2}$$

$$= x_2^{1-\kappa}\left[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0 \overset{0}{X}_3 \right], \tag{6.30}$$

i.e.

63

$$((p_0^2 x_2 + \varsigma_0 E_0 x_2^{\kappa_1 - \kappa + 1}) v_{30,\alpha})_{,\alpha} + (\kappa - 1)(p_0^2 + \varsigma_0 E_0 x_2^{\kappa_1 - \kappa}) v_{30,2}$$

$$= x_2^{1-\kappa} \left[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0 \overset{0}{X}_3 \right] \tag{6.31}$$

and (for $\kappa > \kappa_1$)

$$(p_0^2 x_2^{\kappa - \kappa_1 + 1} + \varsigma_0 E_0 x_2) v_{30,\alpha\alpha} + (\kappa p_0^2 x_2^{\kappa - \kappa_1} + \kappa_1 \varsigma_0 E_0) v_{30,2}$$

$$= x_2^{1-\kappa_1} \left[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0 \overset{0}{X}_3 \right], \tag{6.32}$$

i.e.

$$((p_0^2 x_2^{\kappa - \kappa_1 + 1} + \varsigma_0 E_0 x_2) v_{30,\alpha})_{,\alpha} + (\kappa_1 - 1)(p_0^2 x_2^{\kappa - \kappa_1} + \varsigma_0 E_0) v_{30,2}$$

$$= x_2^{1-\kappa_1} \left[p_0(f_{e0} - \overset{0}{D}) - \varsigma_0 \overset{0}{X}_3 \right], \tag{6.33}$$

respectively.

Let
$$E_{2323} = \widetilde{E}_0 x_2^{\kappa_1^1}, \qquad h = h_0 x_2^{\kappa_1^2}, \qquad \widetilde{E}_0, h_0 = const > 0;$$
$$p_{113} = \widetilde{p}_0 x_2^{\kappa_2^2}, \qquad \widetilde{p}_0 = const > 0; \ \varsigma_{11} = \widetilde{\varsigma}_0 x_2^{\kappa_3^1}, \qquad \widetilde{\varsigma}_0 = const > 0;$$
$$\kappa_1^2, \kappa_j^1 = const > 0, \qquad j = \overline{1,3}.$$
$$\kappa_1 = const = \kappa_1^1 + \kappa_1^2 \geq 0; \quad \kappa_2 = const = \kappa_2^1 + \kappa_1^2 \geq 0; \quad \kappa_3 = const = \kappa_3^1 + \kappa_1^2 \geq 0.$$

then
$$E_{2323} h = E_0 x_2^{\kappa_1}, \ where \ E_0 = \widetilde{E}_0 h_0 = const > 0;$$
$$p_{113} h = p_0 x_2^{\kappa_2}, \ where \ p_0 = \widetilde{p}_0 h_0 = const > 0; \tag{6.34}$$
$$\varsigma_{11} h = \varsigma_0 x_2^{\kappa_3}, \ where \ \varsigma_0 = \widetilde{\varsigma}_0 h_0 = const > 0.$$

Taking into accounts (6.5), (6.34), from (6.1) it is evident that the following theorem is true.

Theorem 6.1. *If s.c. elastic, piezoelectric, dielectric constants (coefficients) are independent of space points i.e. $\kappa_i^1 = 0$, $i = \overline{1,3}$, then we have to do with cusped prismatic shells for $\kappa_i = \kappa_1^2 > 0$, $i = \overline{1,3}$. In this case peculiarities, arising by setting BCs and those arising by prismatic shells of constant thickness i.e. $\kappa_1^2 = 0$ when the elastic, piezoelectric, and dielectric coefficients are changing according to (6.5) i.e. $\kappa_i = \kappa_i^1 > 0$, $i = \overline{1,3}$, coincide. The stress-strain states coincide as well.*

We now are in a position to reformulate results of Section 5 for the matter under consideration in Section 6.

Consequently, we may draw the following conclusions.

Case 1. $\kappa_i = \kappa = const \geq 0$, $i = \overline{1,3}$, i.e. $\kappa = \kappa^1 + \kappa_1^2$, where $\kappa^1 := \kappa_i^1$, $i = \overline{1,3}$.

Theorem 6.2. *In the static case ($o = 0$) for system (6.24), (6.25) according to Theorem 5.1 it follows that the values of v_{30} and $\tilde{\chi}_0$ should be prescribed on the entire boundary (Problem D) for $\kappa < 1$, while on the part of the boundary, where $x_2 = 0$, should be freed of BCs (Problem E) for $\kappa \geq 1$. Both problems are uniquely solvable in the classical sense.*

Theorem 6.3. *An $\mathcal{H}$-weak solution $(v_{30} \, \tilde{\chi}_0)$ of Problem E for system (6.12), (6.13) exists under the following conditions:*

$$x_2^{1-\kappa}\rho h = h_0 x_2^{1-\kappa^1}\rho \in C(\bar{\omega}),$$

$$o^2 < \left[\frac{1}{2}(\kappa - 1) - c_0\right](E_0 + \varsigma_0^{-1}p_0^2)\left[\max_{(x_1,x_2)\in\bar{\omega}}(h_0 x_2^{1-\kappa}\rho)\right]^{-1},$$

$$\kappa > 1 + 2c_0.$$

Remark 6.1. Similarly to Remark 5.1 we construct in the explicit form solutions of BVPs for system (6.22), (6.23) in the case of the cylindrical strain.

   Case 2.) $\kappa_2 = \kappa_3 = \kappa = const \geq 0$, i.e., $\kappa_2^1 = \kappa_3^1$.

Theorem 6.4. *For $\kappa_1 = 0$ and any $\kappa \geq 0$, the values of v_{30} (see (6.29)) should be prescribed on the entire boundary (Problem D), while (see (6.28)), according to Theorem 5.1, the values of $\tilde{\chi}_0$ should be prescribed on the entire boundary (Problem D) for $0 \leq \kappa < 1$ and the part where $x_2 = 0$ should be freed of BCs (Problem E) for $\kappa \geq 1$.*

   If either $\kappa_1 \geq \kappa + 1$ or $\kappa \geq \kappa_1 + 1$, $\kappa_1 > 1$, we carry out reasonings for systems (6.31), (6.28) and (6.33), (6.28) similar to Section 5 for systems (5.35), (5.31) and (5.38), (5.31).

Remark 6.2. Just as in section 5 the problems D and E we solve in the explicit forms in the case of cylindrical strain (see Remark 5.1 and Remark 5.2). The governing equations obtained from (6.22), (6.23), (6.28), (6.29) have the following forms, correspondingly

$$(\varsigma_0 E_0 + p_0^2)(x_2^\kappa v_{30,2})_{,2} = -\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}),$$

$$(p_0^2 + \varsigma_0 E_0)(x_2^\kappa \tilde{\chi}_{0,2})_{,2} = -p_0 \overset{0}{X}_3 - E_0(f_{e0} - \overset{0}{D});$$

$$\varsigma_0(x_2^\kappa \tilde{\chi}_{0,2})_{,2} = p_0(x_2^\kappa v_{30,2})_{,2} - (f_{e0} - \overset{0}{D}),$$

$$((p_0^2 x_2^\kappa + \varsigma_0 E_0 x_2^{\kappa_1})v_{30,2})_{,2} = -\varsigma_0 \overset{0}{X}_3 + p_0(f_{e0} - \overset{0}{D}).$$

7 $N = 0$ Approximation for Porous Isotropic Elastic Prismatic Shells

In the case under consideration, assuming the constitutive coefficients λ and μ (the Lamé constants), $\tilde{\alpha}$, $\tilde{b}$, and $\tilde{\xi}$ to be constants from (4.11)-(4.14) we get the following governing system

$$\mu\Big[(hv_{\alpha 0,\beta}),_\alpha + (hv_{\beta 0,\alpha}),_\alpha\Big] + \lambda(hv_{\gamma 0,\gamma}),_\beta + \tilde{b}(h\psi_0),_\beta + \overset{0}{X}_\beta = \rho h\ddot{v}_{\beta 0}, \quad \beta = 1,2; \quad (7.1)$$

$$\mu(hv_{30,\alpha}),_\alpha + \overset{0}{X}_3 = \rho h\ddot{v}_{30}; \quad (7.2)$$

$$\tilde{\alpha}(h\psi_{0,\alpha}),_\alpha - \tilde{b}hv_{\gamma 0,\gamma} - \tilde{\xi}h\psi_0 + \overset{0}{H} = \rho h\ddot{\psi}_0 - \mathcal{F}_0. \quad (7.3)$$

BCs for the weighted displacements and the weighted volume fraction are non-classical in the case of cusped prismatic shells (see Figures 3.2, 3.3). Namely, we are not always able to prescribe them at cusped edges.

Let ω be a domain bounded by a sufficiently smooth arc $(\partial\omega \setminus \overline{\gamma^0})$ lying in the half-plane $x_2 > 0$ and a segment $\overline{\gamma^0}$ of the x_1-axis $(x_2 = 0)$.

If the thickness looks like

$$2h(x_1, x_2) = h_0 x_2^\kappa, \qquad h_0, \ \kappa = const > 0, \quad (7.4)$$

then we can prescribe the displacements and volume fraction at the cusped edge $\overline{\gamma}_0$ if $\kappa < 1$, while we cannot do it if $\kappa \geq 1$.

Let us show it for the particular case of deformation when

$$v_{\alpha 0} \equiv 0, \ \ \alpha = 1,2; \ \ v_{30} \not\equiv 0.$$

Then in the static case, taking into account (7.4), from (7.2), (7.3) we get

$$x_2 v_{30,\alpha\alpha} + \kappa v_{30,2} = 2(\mu h_0)^{-1} x_2^{1-\kappa} \overset{\circ}{X}_3, \quad (7.5)$$

$$x_2\psi_{0,\alpha\alpha} + \kappa\psi_{0,2} - \xi\alpha^{-1}x_2\psi_0 = -2(\tilde{\alpha}h_0)^{-1}x_2^{1-\kappa}\Big(\overset{\circ}{H} + \mathcal{F}_0\Big), \quad (7.6)$$

respectively.

Problem D $\Big($Dirichlet Problem: Find solutions

$$v_{30}, \psi_0 \in C^2(\omega) \cap C(\bar\omega)$$

of (7.5), (7.6) by their values prescribed on $\partial\omega\Big)$

and

Problem E $\Big($Keldysh Problem: Find bounded solutions

$$v_{30}, \psi_0 \in C^2(\omega) \cap C(\omega \cup (\partial\omega \setminus \overline{\gamma^0}))$$

of (7.5), (7.6) by their values prescribed only on the arc $\partial\omega \setminus \overline{\gamma^0}\Big)$

are uniquely solvable for equations (7.5), (7.6) by $\kappa_2 < 1$ and $\kappa_2 \geq 1$, correspondingly. It follows from Theorem 5.1. Indeed, from (5.18) and (5.19), it follows

$a_2(x_1, x_2) = \kappa < 1$ for Problem D and $a_2(x_1, x_2) = \kappa \geq 1$ for Problem E, respectively, since $\kappa_1 = \kappa_2 = 1$.

To the general system (7.1)-(7.3) in the static and dynamical (time-harmonic motion) cases we apply results obtained for the more general system (see Section 8).

8 Systems of Elliptic Equations of the Second Order with an Order Degeneracy

Let $G \subset R^n$, $n \geq 2$, be a bounded domain, $x := (x_1, ..., x_n)$, $\Gamma := \partial G$, and

$$Lu := (A^{ij}u_{,i})_{,j} + E^i u_{,i} + Cu = F, \tag{8.1}$$

i.e.,

$$L_k u := (a_{kl}^{ij} u_{l,i})_{,j} + e_{kl}^i u_{l,i} + c_{kl} u_l = F_k, \quad k = \overline{1, m},$$

where

$$u = (u_1, ..., u_m)^\top, \quad F := (F_1, ..., F_m)^\top,$$

$$A^{ij} := \|a_{kl}^{ij}\|, \quad E^i := \|e_{kl}^i\| \in C^1(\bar{G}); \tag{8.2}$$

$$C := \|c_{kl}\| \in C(\bar{G}), \quad i, j = \overline{1, n}, \quad k, l = \overline{1, m},$$

where we use the Einstein convention, and indices after a comma, as usually., mean differentiation with respect to the corresponding variables.

Let

$$\overset{(i)}{\xi} A^{ij} \overset{(j)}{\xi} > 0, \quad x \in \bar{G} \setminus \Gamma^0, \tag{8.3}$$

for all $\xi^{(i)} := \left(\xi_1^{(i)}, ..., \xi_m^{(i)}\right)$, $i = \overline{1, n}$, such that

$$\sum_{i=1}^{n} \sum_{k=1}^{m} \left(\xi_k^{(i)}\right)^2 > 0,$$

where

$$\Gamma^0 := \{x \in \Gamma : A^{ij}(x) = 0, \quad i, j = \overline{1, n}\}. \tag{8.4}$$

So, the system (8.1) is strongly elliptic on $\bar{G} \setminus \Gamma^0$ with the equations' order degeneracy on Γ^0.

Let

$$\Gamma = \overset{p}{\underset{k=1}{\cup}} \overline{\Gamma}^{(k)},$$

where $\overline{\Gamma}^{(k)}$, $k = \overline{1, p}$, are smooth hypersurfaces with possibly common boundary points, and let the Gauss-Ostrogradsky formula be applicable to the domain G.

At points of smoothness of Γ^0 let us consider matrix

$$\Phi := E^i \nu_i, \tag{8.5}$$

where $\nu := (\nu_1, ..., \nu_n)$ is the inward normal at the above boundary points. Let further

$$\Gamma_0 := \{x \in \Gamma^0 \, : \, \Phi(x) = 0\}, \ \ \Gamma_1 := \{x \in \Gamma^0 \, : \, \Phi(x) > 0\},$$

$$\Gamma_2 := \{x \in \Gamma^0 \, : \, \Phi(x) < 0\}, \ \ \ \Gamma_3 : \Gamma \setminus \Gamma^0. \tag{8.6}$$

Under matrix inequalities we mean inequalities for the corresponding quadratic forms on vectors with nonzero length.

Let $x_0 \in \Gamma^0$ be a common point of the hypersurfaces $\overline{\Gamma}^{(i_1)}, ..., \overline{\Gamma}^{(i_q)}$ for certain $i_1, ..., i_q \in \{1, ..., p\}$, $q = \overline{2, p}$, $i_k \neq i_q$, $k \neq l$. If there exists such a neighbourhood $\omega_{\underline{i}}$, $\underline{i} := (i_1, ..., i_q)$, of the point x_0, that

$$\left(\omega_{\underline{i}} \cap \left(\underset{l=1}{\overset{q}{\cup}} \Gamma^{(i_l)} \right) \right) \subset \left(\Gamma_2 \cup \Gamma_3 \right),$$

then the above point will be added to $\Gamma_2 \cup \Gamma_3$, otherwise it will be added to $\Gamma_0 \cup \Gamma_1$.

Let further $\Gamma^0 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ and

$$u|_{\Gamma_2 \cup \Gamma_3} = 0. \tag{8.7}$$

Definition 8.1. Let C_L be the class of bounded vectors u such that

$$u \in C^2(G) \cap C(G \cup \Gamma_2 \cup \Gamma_3),$$

$$A^{ij}u_{,i}, \ A^{ij}u, \ E^i u \in C^1(\bar{G}),$$

$$(A^{ij}u_{,i})|_{\Gamma^0} = 0, \ \ Lu \ \text{ be bounded on } \ G. \tag{8.8}$$

Definition 8.2. A vector $u \in C_L$ satisfying system (8.1) and BC (8.7) will be called a regular solution of the BVP (8.1), (8.7).

Definition 8.3. A vector $u \in L_2(\Gamma)$ will be called an $L_2(G)$-weak solution of the BVP (8.1), (8.7) if $F \in L_2(G)$ and

$$\int_G v \, F dx = \int_G L^* v \cdot u dx \tag{8.9}$$

is valid for any $v \in C_{L^*}$, satisfying the condition

$$v|_{\Gamma_1 \cup \Gamma_3} = 0, \tag{8.10}$$

(the space of such vectors v will be denoted by V, if for v and $u \in C_L$ with (8.7) equality (8.9) is valid), L^* is the adjoint operator to L:

$$L^* v := (v_{,i} A^{ij})_{,j} - (v E^i)_{,i} + v C,$$

i.e.,

$$L_l^* v := \left(v_{k,i} a_{kl}^{ij} \right)_{,j} - \left(v_k e_{kl}^i \right)_{,i} + v_k c_{kl}, \ \ l = \overline{1, m},$$

and C_{L^*} is defined for the operator L^* like C_L for the operator L.

It is easily seen that if there exists a regular solution of problem (8.1), (8.7), it will be also a week solution. Indeed, if $v \in C^2(\bar{G})$ and $u \in C_{L^*}$ then we have the following Green formula

$$\int_G v \, Ludx - \int_G L^* v \cdot udx = \int_\Gamma [(v_{,i} A^{ij} u - v A^{ij} u_{,i}) \nu_j - v\Phi u]d\Gamma, \quad A^{ij} = A^{ji}. \quad (8.11)$$

Presenting Γ as $\Gamma_3 \cup \Gamma^0$, then the first sum under the integral will be equal to zero on Γ_3 because of (8.7) and on Γ^0 because of (8.4). The second sum will be equal to zero on Γ^0, by virtue of (8.8), and on Γ_3, by virtue of (8.10). We now present Γ as $\Gamma_3 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_0$, then on $\Gamma_3 \cup \Gamma_1 \cup \Gamma_2$ the third summand becomes zero because of (8.7), (8.10) and on Γ_0 because of (8.6). Thus, the right hand side is zero and, since $Lu = F$, from (8.11) it follows (8.9).

Theorem 8.1. *Let $A^{ij} = A^{ji}$, let the matrices E^i, $i = \overline{1,n}$, be symmetric, and*

$$E^i{}_{,i} - 2C \geq 0, \quad x \in G, \quad (8.12)$$

then homogeneous BVP corresponding to the BVP (8.1), (8.7) has only the trivial solution in C_L if

$$uA^{ij} u_{,j} + \frac{1}{2} u E^i u \in C^1(\bar{G}), \quad \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \varnothing.$$

Proof. Integrating the equality (see [24])

$$uLu + u_{,i} A^{ij} u_{,j} + \frac{1}{2} u(E^i{}_{,i} - 2C)u = (uA^{,j} u_{,j} + \frac{1}{2} u E^i u)_{,i} \quad (8.13)$$

on G and using the Gauss-Ostrogradsky formula, by virtue of (8.6)-(8.8) and

$$Lu = 0,$$

we obtain

$$\int_G [u_{,i} A^{ij} u_{,j} + \frac{u}{2}(E^i_{,i} - 2C)u]dG + \frac{1}{2} \int_{\Gamma_1} u\Phi u d\Gamma_1 = 0.$$

Hence, in view of (8.3), (8.12), and $\Phi > 0$ on Γ_1, we arrive at

$$u_{,i} A^{ij} u_{,j} = 0, \quad x \in G.$$

But, according to (8.3), it is admissible only if

$$u_{,i} = 0, \quad i = \overline{1,n}, \quad x \in G.$$

Therefore, $u = const$ and if $\Gamma_2 \cup \Gamma_3 \neq \varnothing$, then by virtue of (8.7),

$$u \equiv 0. \quad (8.14)$$

If $\Gamma_2 \cup \Gamma_3 = \varnothing$ but $\Gamma_1 \neq \varnothing$, then (8.14) is valid since

$$u|_{\Gamma_1} = 0$$

because of
$$u\Phi u|_{\Gamma_1} = 0 \ \text{ and } \ \Phi|_{\Gamma_1} > 0.$$

If
$$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \varnothing,$$

but inequality (8.12) is strong, then it is easily seen that
$$u(E_{,i}^{\,i} - 2C)u = 0, \ \ x \in G.$$

Therefore, (8.14) is valid. □

Theorem 8.2. *Let $A^{ij} = A^{ji}$, let the matrices E^i, $i = \overline{1,n}$, be symmetric, and*

$$\xi(E^i{}_{,i} - 2C)\xi \geq c_0 \sum_{k=1}^{m} \xi_k^2, \ \ x \in G, \ c_0 = const > 0, \ \ \xi := (\xi_1, ..., \xi_m), \qquad (8.15)$$

then there exists a weak solution of BVP (8.1), (8.7).

Proof. According to the Abstract Existence Principle (G. Fichera, see Theorem 8.3 below and [15], and also [16]) it follows that for the existence of a weak solution the fulfilment of the following inequality is sufficient

$$\|v\|_2 \leq K\|L^*v\|_2, \qquad (8.16)$$

where

$$K = \text{const}, \quad \|v\|_2 := \left(\int\limits_G \sum_{k=1}^{m} v_k^2 dG \right)^{\frac{1}{2}}.$$

Indeed, let $\mathcal{V}$ be an abstract linear manifold of the real field; let B_α $\alpha = 1, 2$, be real Banach spaces; let $\mathcal{M}_\alpha$, $\alpha = 1, 2$, be linear homomorphisms of $\mathcal{V}$ into B_α; let F and u be vectors of conjugate spaces B_1^* and B_2^*, respectively. Let us consider for any $v \in \mathcal{V}$ the following functional equation

$$< F, \mathcal{M}_1(v) > = < u, \mathcal{M}_2(v) > \qquad (8.17)$$

where F is a given, and u is an unknown vector. We denote: by $\mathcal{V}_2$ a kernel of the homomorphism $\mathcal{M}_2$
$$\mathcal{V}_2 := \{v \in \mathcal{V} : \mathcal{M}_2(v) = 0\};$$

by $\mathcal{M}_1(\mathcal{V}_2)$ an image of $\mathcal{V}_2$ on B_1 by homomorphism $\mathcal{M}_1$; and by $\overline{\mathcal{M}_1(\mathcal{V}_2)}$ its closure.

Let us consider a factor space

$$Q := B_1/\overline{\mathcal{M}_1(\mathcal{V}_2)}.$$

Let M_1 be a homomorphism, which maps $v \in \mathcal{V}$ on a class of equivalence $[\mathcal{M}_1(v)]$ of the factor space Q. A given vector F should fulfil the following necessary condition

$$< F, \mathcal{M}_1(v_2) > = 0, \ \ v_2 \in \mathcal{V}_2. \qquad (8.18)$$

The following existence principle holds:

Theorem 8.3 (**Abstract Existence Principle (G. Fichera)**). *A solution u of functional equation (8.17) exists for any fixed F satisfying the condition (8.18) if and only if there exists a constant K such that for any $v \in \mathcal{V}$ the inequality*

$$\|M_1(v)\|_Q \leq K\|\mathcal{M}_2(v)\|_{B_2} \tag{8.19}$$

holds.

Let $\mathcal{A}$ be a closed subspace of the space B_2^*, consisting of the vectors u_0, solutions of the "homogeneous" problem

$$< u_0, \mathcal{M}_2(v) >= 0, \quad v \in \mathcal{V}.$$

We denote by $\mathcal{F}$ the Banach factor space $\mathcal{F} := B_2^*/\mathcal{A}$. For any $F \in B_1^*$, satisfying the compatibility conditions (8.18) (that is to say for any element of the adjoint space Q^*), there exists uniquely defined $U \in \mathcal{F}$ such that, if u is any element in the equivalence class U, then u is a solution of equation (8.17) with

$$\|U\|_{\mathcal{F}} \leq K\|F\|_{B_1^*}. \tag{8.20}$$

Inequality (8.20) is said to be a dual inequality of (8.19).

In our case under consideration instead of the functional equation (8.17) we have equation (8.9), $\mathcal{V} \equiv V$, $\mathcal{M}_1$ is an identical operator,

$$\mathcal{M}_2 \equiv L^*, \ B_1 \equiv B_2 \equiv B_1^* \equiv B_2^* \equiv L_2(G), \ Q \equiv L_2 \backslash \bar{V}_2,$$

where

$$V_2 := \{v \in V : L^*(v) = 0\},$$

$$\|M_1(v)\|_Q \equiv \|[\mathcal{M}_1(v)]\|_Q \equiv \|[v]\|_Q := \inf_{v_2 \in \bar{V}_2} \|v + v_2\|_2,$$

$$\mathcal{F} \equiv L_2/\mathcal{A}, \quad \mathcal{A} := \{u_0 \in L_2(\omega) : (u_0, L^*v) = 0\};$$

$$\|U\|_{\mathcal{F}} := \inf_{u_0 \in \bar{\mathcal{A}}} \|u + u_0\|_2,$$

and Fichera's Abstract Existence Principle takes the form:

Theorem 8.4. *A solution $u \in L_2(G)$ of the functional equation (8.9) exists for any $F \in L_2(G)$, satisfying condition*

$$\int_G Fv_2 dx = 0, \quad v_2 \in V_2, \tag{8.21}$$

if and only if there exists a constant K such that for any $v \in V$ the inequality

$$\|[v]\|_Q = \inf_{v_2 \in V_2} \|v + v_2\|_2 \leq K\|L^*(v)\|_2, \tag{8.22}$$

is valid; moreover

$$\|U\|_{\mathcal{F}} = \inf_{v_0 \in \mathcal{A}} \|u + u_0\|_2 \leq K\|F\|_2.$$

Let now (8.16) be valid, then from

$$L^*(v_2) = 0$$

it follows that $v_2 \equiv 0$, i.e., $V_2 \equiv \{0\}$. Therefore,

$$\inf_{v_2 \in V_2} \|v + v_2\|_2 = \|v\|_2,$$

and (8.22) holds, by virtue of (8.16). Hence, there exists a solution of the functional equation (8.9), i.e., a weak solution of problem (8.1), (8.7). It is clear, that in this case condition (8.21) is fulfilled for any $F \in L_2(G)$.

In order to establish equality (8.16) we write identity (8.13) for the operator L^*:

$$L^*v \cdot v + v_{,i} A^{ij} v_{,j} + \frac{v}{2}(E^i_{,i} - 2C)v = (v_{,j} A^{ij} v - \frac{v}{2} E^i v)_{,i}.$$

After integrating the last equality on G and using the Gauss-Ostrogradsky formula, by virtue of (8.4), (8.5), (8.10) we have

$$\int_Q L^*v \cdot v dx + \int_G [v_{,i} A^{ij} v_{,j} + \frac{v}{2}(E^i_{,i} - 2C)v] dG - \frac{1}{2} \int_{\Gamma_2} v\Phi v d\Gamma_2 = 0.$$

Therefore, in view of (8.3), (8.6), (8.15), we have

$$-\int_G L^*v \cdot v dx \geq \frac{1}{2} \int_G v(E^i_{,i} - 2C)v dG \geq \overset{*}{c} \int_G \sum_{k=1}^m v_k^2 dG = \overset{*}{c}\|v\|_2^2, \quad \overset{*}{c} = \mathrm{const.}$$

On the other hand, applying the Hölder enequality, we get

$$\int_G (-L^*v)v dx \leq \| -L^*v\|_2 \cdot \|v\|_2 = \|L^*v\|_2 \cdot \|v\|_2.$$

Hence,

$$\overset{*}{c}\|v\|_2^2 \leq \|L^*v\|_2 \cdot \|v\|_2,$$

i.e, (8.16) is valid. $\qquad\qquad\square$

Remark 8.1. The system with nonsmooth coefficients (they are infinite on a subset of Γ^0) we reduce to the present case by means of smoothing factors.

Remark 8.2. Let us introduce new unknown functions ω_i, $i = \overline{1,m}$, by relations

$$u_i = \tilde{\psi} w_i, \quad i = 1, m, \quad \tilde{\psi} \in C^2(\bar{G}), \quad \tilde{\psi} > 0 \text{ on } \bar{G}. \qquad (8.23)$$

Note, that the matrix Φ is invariant to change of unknown functions (8.23), in the sense of change of sign since for the system obtained

$$\hat{L}_w := (\hat{A}^{ij} w_{,i})_{,j} + \hat{E}^i w_{,i} + \hat{C}w = F, \quad w = (w_1 \cdots w_m), \qquad (8.24)$$

where

$$\hat{A}^{ij} = \tilde{\psi} A^{ij}, \quad \hat{E}^i = \tilde{\psi} E^i + \tilde{\psi}_{,j} A^{ij}, \quad \hat{C} = \tilde{\psi} C,$$

by virtue of (8.4), it looks like

$$\hat{\Phi} = \hat{E}^i \nu_i = (\tilde{\psi} E^i + \tilde{\psi}_{,j} A^{ij}) \nu_i = \tilde{\psi} E^i \nu_i = \tilde{\psi} \Phi.$$

If the function ψ is such that for the system (8.24) the condition (8.15) is valid, then under assumptions of Theorem 8.2 there exists a weak solution of the problem

$$\hat{L} w = F, \quad w|_{\Gamma_2 \cup \Gamma_3} = 0, \tag{8.25}$$

i.e.,

$$(v, F) = (\hat{L}^* v, w),$$

where $\hat{L}^*$ is the operator conjugate to the operator $\hat{L}$. But

$$(\hat{L}^* v, w) = (\hat{L}^* v, \tilde{\psi}^{-1} u) = (\tilde{\psi}^{-1} \hat{L}^* v, u) = (\hat{L}^* v, u),$$

since if $A^{ij} = A^{ji}$, then

$$\tilde{\psi}^{-1} \hat{L}^* v = \hat{L}^* v.$$

Thus,

$$(v, F) = (L^* v, u),$$

i.e, a weak solution of problem (8.1), (8.7) exists and according to (8.23), we get it, from the weak solution of problem (8.25).

9 The $\mathcal{H}$-weak Solution for a Single Equation

For the convenience of the reader we repeat the relevant material from [15] (compare with [16] Chapter I, §4) with proofs in a slightly changed form, thus making our exposition of the present work self-contained.

Let $G \subset R^n$ be a domain, $\Gamma := \partial \Gamma$, and

$$Lu := (A^{ij} u_{,i})_{,j} + E^i u_{,i} + Cu = F, \tag{9.1}$$

where real functions

$$A^{ij}, \ E^i \in C^1(\bar{G}); \ C \in C(\bar{G}), \ i, j = \overline{1,n}. \tag{9.2}$$

As usual we use the Einstein's summation convention and indices after a comma mean differentiation with respect to the corresponding variables.

Let

$$\xi_i A^{ij} \xi_j > 0, \ x \in \bar{G} \setminus \Gamma^0, \tag{9.3}$$

for any $\xi_i \in R^1$, $i = \overline{1,n}$, such that

$$\sum_{i=1}^n \xi_i^2 > 0,$$

where

$$\Gamma^0 := \{ x \in \Gamma \ : \ A^{ij}(x) = 0, \ i, j = \overline{1,n} \}. \tag{9.4}$$

So, equation (9.1) is elliptic on $\bar{G} \setminus \Gamma^0$ with order degeneracy on Γ^0.

Let
$$\Gamma = \bigcup_{k=1}^{p} \overline{\Gamma}^{(k)},$$

where $\overline{\Gamma}^{(k)}$, $k = \overline{1,p}$, are smooth hypersurfaces with possibly common boundary points, and to the domain G the Gauss-Ostrogradsky formula is applicable.

In points of smoothness of Γ^0 let us consider the function, which we shall call Fichera's function,
$$\Phi := E^i \nu_i, \tag{9.5}$$

where $\nu := (\nu_1, ..., \nu_n)$ is the inward normal at the above boundary points. Let further

$$\Gamma_0 := \{x \in \Gamma^0 \ : \ \Phi(x) = 0\}, \ \ \Gamma_1 := \{x \in \Gamma^0 \ : \ \Phi(x) > 0\},$$

$$\tag{9.6}$$

$$\Gamma_2 := \{x \in \Gamma^0 \ : \ \Phi(x) < 0\}, \ \ \ \Gamma_3 : \Gamma \setminus \Gamma^0.$$

Let $x_0 \in \Gamma^0$ be a common point of the hypersurfaces $\overline{\Gamma}^{(i_1)}, ..., \overline{\Gamma}^{(i_q)}$ for certain $i_1, ..., i_q \in \{1, ..., p\}$, $2 \leq q \leq p$, $i_k \neq i_q$, $k \neq l$. If there exists a neighbourhood $\omega_{\underline{i}}$, $\underline{i} := (i_1, ..., i_q)$, of the point x_0, such that

$$\left(\omega_{\underline{i}} \cap \left(\bigcup_{l=1}^{q} \Gamma^{(i_l)}\right)\right) \subset \left(\Gamma_2 \cup \Gamma_3\right),$$

then the above point will be added to $\Gamma_2 \cup \Gamma_3$, otherwise it will be added to $\Gamma_0 \cup \Gamma_1$.

Let further $\Gamma^0 = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$,

$$u|_{\Gamma_2 \cup \Gamma_3} = 0, \tag{9.7}$$

Definition 9.1. Let C_L be the class of bounded functionss u such that

$$u \in C^2(G) \cap C(G \cup \Gamma_2 \cup \Gamma_3),$$

$$A^{ij}u_{,i}, \ A^{ij}u, \ E^i u \in C^1(\bar{G}),$$

$$(A^{ij}u_{,i})|_{\Gamma^0} = 0, \ \ Lu \ \ \text{be bounded in} \ \ G. \tag{9.8}$$

Definition 9.2. A function $u \in C_L$ satisfying system (9.1) and BC (9.7) will be called a regular solution of the BVP (9.1), (9.7).

For a function $u \in C_L$ and a function $v \in C^1(\bar{G})$ the integral identity

$$\int_G v L u \, dx = - \int_G [A^{ij}v_{,j}\, u_{,i} + u E^i v_{,i} + (E_{,i} - C)uv] dx$$

$$- \int_{\Gamma_3} v A^{ij}u_{,i}\, n_j d\Gamma - \int_{\Gamma} uv\Phi d\Gamma, \tag{9.9}$$

where n is an inward normal, holds.

Let $\mathcal{W}$ be the class of functions belonging to $C^1(\bar{G})$ and vanishing on Γ_3 (when not empty). If $u \in C_L$ and vanishes a.e. on $\Gamma_2 \cup \Gamma_3$, then for any $v \in \mathcal{W}$ the identity

$$\int_G vLudx = -\int_G [A^{ij}v_{,j}\,u_{,i} + uE^i v_{,i} + (E_{,i} - C)uv]dx - \int_{\Gamma_1} uv\Phi d\Gamma. \qquad (9.10)$$

is satisfied (see (9.9)).

Let us introduce a scalar product in $\mathcal{W}$ in the following way

$$(u,v)_{\mathcal{H}} = \int_G (A^{ij}u_{,i}\,v_{,j} + uv)dx + \int_{\Gamma_1 \cup \Gamma_2} uv|\Phi|d\Gamma.$$

The space $\mathcal{H}$ will be the Hilbert space, obtained by functional completion from $\mathcal{W}$ with the introduced scalar product.

Let us consider for $u, v \in \mathcal{W}$ the bilinear form

$$B(u,v) = -\int_G [A^{ij}v_{,j}\,u_{,i} + uE^i v_{,i} + (E_{,i} - C)uv]dx - \int_{\Gamma_1} uv\Phi d\Gamma. \qquad (9.11)$$

It is easily seen that

$$|B(u,v)| \le K\Big(\int_G (|gradv|^2 + v^2)dx + \int_{\Gamma_1} |v|^2 d\Gamma\Big)^{\frac{1}{2}} \|u\|_{\mathcal{H}},$$

where K is a constant, depending on the coefficients of L. For any fixed $v \in \mathcal{W}$, $B(u,v)$ can be considered as a linear bounded functional of u, defined on $\mathcal{H}$.

Definition 9.3. A function $u \in \mathcal{H}$ will be called an $\mathcal{H}$-weak solution of the BVP (9.1), (9.7) if $F \in L_2(G)$ and

$$\int_G vFdx = B(u,v) \qquad (9.12)$$

is valid for any $v \in \mathcal{W}$.

Let $\mathcal{V}$ (see Section 8 and take $m = 1$, then we get corresponding to equation (9.1) results) coincide with the class of functions C_{L^*} satisfying a.e. the BC

$$v|_{\Gamma_1 \cup \Gamma_3} = 0, \qquad (9.13)$$

(the space of such vectors v will be denoted by V, if for v and $u \in C_L$ with (9.7) equality (9.12) is valid), L^* is the adjoint operator to L:

$$L^*v := (v_{,i}\,A^{ij})_{,j} - (vE^i)_{,i} + vC,$$

then any $\mathcal{H}$-weak solution is an L_2-weak solution.

According to the representation theorem of linear functionals in Hilbert spaces, we have for $u \in \mathcal{H}$, $v \in \mathcal{W}$:

$$B(u,v) = (u, T(v))_{\mathcal{H}},$$

where $T(v)$ is a linear transformation defined in $\mathcal{W}$ and with range in $\mathcal{H}$.

For $u \in \mathcal{W}$, and $v \in \mathcal{W}$, from (9.11) we have

$$B(u,v) = -\int_G [A^{ij} v_{,j}\, u_{,i} + \frac{1}{2} u E^i v_{,i} - \frac{1}{2} v E^i u_{,i} + \left(\frac{1}{2} E^i{}_{,i} - C\right) uv]\, dx$$
$$-\frac{1}{2}\int_{\Gamma_1} uv\Phi\, d\Gamma + \frac{1}{2}\int_{\Gamma_2} uv\Phi\, d\Gamma, \qquad (9.14)$$

since

$$\int_G u E^i v_{,i}\, dx = \frac{1}{2}\int_G u E^i v_{,i}\, dx + \frac{1}{2}\int_G u E^i v_{,i}\, dx$$

$$= \frac{1}{2}\int_G \left[u E^i v_{,i} + \frac{1}{2}\left(u E^i v\right)_{,i} - \frac{1}{2} u_{,i}\, E^i v - \frac{1}{2} u E^i v \right] dx$$

$$= \frac{1}{2}\int_G u E^i v_{,i}\, dx - \frac{1}{2}\int_{\Gamma_1 \cup \Gamma_2} u\Phi v\, d\Gamma - \frac{1}{2}\int_G u_{,i}\, E^i v\, dx - \frac{1}{2}\int_G u E^i{}_{,i}\, v\, dx.$$

Let us suppose that

$$\frac{1}{2} E^i{}_{,i} - C > c_0 = const > 0 \ \ in \ \ G \cup \Gamma. \qquad (9.15)$$

This condition is satisfied if we assume C negative and $|C|$ large enough. If (9.15) is satisfied we easily get from (9.14) for $v \in \mathcal{W}$:

$$|B(v,v)| = \int_G \left[A^{ij} v_{,j}\, v_{,i} + \left(\frac{1}{2} E^i{}_{,i} - C\right) v^2 \right] dx + \frac{1}{2}\int_{\Gamma_1} v^2 \Phi\, d\Gamma$$

$$+ \frac{1}{2}\int_{\Gamma_2} v^2 \Phi\, d\Gamma \geq \lambda_0 \|v\|_{\mathcal{H}}^2 \quad (\lambda_0 > 0).$$

Therefore,

$$\|v\|_{\mathcal{H}}^2 \leq \frac{1}{\lambda_0}|B(v,v)| \leq \frac{1}{\lambda_0}|(v, T(v))| \leq \frac{1}{\lambda_0}\|v\|_{\mathcal{H}}\|T(v)\|_{\mathcal{H}},$$

i.e.,

$$\left(\int_G v^2 dx\right)^{\frac{1}{2}} \leq \|v\|_{\mathcal{H}} \leq \frac{1}{\lambda_0}\|T(v)\|_{\mathcal{H}}. \qquad (9.16)$$

Hence, from Fichera's Abstract Existence Principle (see Theorem 8.3) we deduce:

Theorem 9.1. *If condition (9.15) is satisfied, for any $F \in L_2(G)$ an $\mathcal{H}$-weak solution of problem (9.1), (9.7) exists.*

The uniqueness of the $\mathcal{H}$-weak solution is connected with the continuity of the bilinear form $B(u,v)$ with respect to the pair (u,v). When this is the case then, since $B(u,v)$ can be extended by continuity in $\mathcal{H} \times \mathcal{H}$, from (9.16) uniqueness of the $\mathcal{H}$-weak solution follows easily.

Remark 9.1. $B(u,v)$ is continuous with respect to the pair (u,v) in the case when L is self-adjoint, i.e., when $E^i = 0$. This is easilly seen from (9.14). In this case, if C is negative in $G \cup \Gamma$, an $\mathcal{H}$-weak solution exists for any given $f \in L_2(G)$ and is unique.

10 Mathematical Moments

Let $f(x_1, x_2, x_3)$ be a given function in $\overline{\Omega}$ having integrable partial derivatives, let f_r be its r-th order moment defined as follows

$$f_r(x_1, x_2) := \int\limits_{\overset{(-)}{h}(x_1, x_2)}^{\overset{(+)}{h}(x_1, x_2)} f(x_1, x_2, x_3) P_r(ax_3 - b) dx_3,$$

where (see Section 3)

$$a(x_1, x_2) := \frac{1}{h(x_1, x_2)}, \quad b(x_1, x_2) := \frac{\widetilde{h}(x_1, x_2)}{h(x_1, x_2)},$$

$$2h(x_1, x_2) = \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) > 0,$$

$$2\widetilde{h}(x_1, x_2) = \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2) > 0,$$

and

$$P_r(\tau) = \frac{1}{2^r r!} \frac{d^r (\tau^2 - 1)^r}{d\tau^r}, \quad r = 0, 1, \cdots,$$

are the r-th order Legendre polynomials with the orhogonality property

$$\int\limits_{-1}^{+1} P_m(\tau) P_n(\tau) d\tau = \frac{2}{2m+1} \delta_{mn}.$$

From here, substituting

$$\tau = ax_3 - b = \frac{2}{\overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2)} x_3 - \frac{\overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2)}{\overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2)},$$

we have

$$\left(m + \frac{1}{2}\right) a \int\limits_{\overset{(-)}{h}(x_1, x_2)}^{\overset{(+)}{h}(x_1, x_2)} P_m(ax_3 - b) P_n(ax_3 - b) dx_3 = \delta_{mn}.$$

Using the well-known formulas of integration by parts (with respect to x_3) and differentiation with respect to a parameter of integrals depending on parameters (x_α), taking into account $P_r(1) = 1$, $P_r(-1) = (-1)^r$, we deduce

$$\int\limits_{\overset{(-)}{h}(x_1, x_2)}^{\overset{(+)}{h}(x_1, x_2)} P_r(ax_3 - b) f_{,3}\, dx_3 = -a \int\limits_{\overset{(-)}{h}(x_1, x_2)}^{\overset{(+)}{h}(x_1, x_2)} P_r'(ax_3 - b) f dx_3 + \overset{(+)}{f} - (-1)^r \overset{(-)}{f}, \quad (10.1)$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 = f_{r,\alpha} - \overset{(+)}{f}\,\overset{(+)}{h}_{,\alpha} + (-1)^r\overset{(-)}{f}\,\overset{(-)}{h}_{,\alpha}$$

$$-\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P'_r(ax_3-b)(a_{,\alpha}\,x_3-b_{,\alpha})f\,dx_3, \quad \alpha=1,2, \tag{10.2}$$

where superscript prime means differentiation with respect to the argument ax_3-b, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables, $\overset{(\pm)}{f}:=f[x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)]$. Applying the following relations from the theory of the Legendre polynomials (see e.g. [25], p. 299 or p. 338-339 of the second edition)

$$P'_r(\tau) = \sum_{s=0}^{r}(2s+1)\frac{1-(-1)^{r+s}}{2}P_s(\tau)^{\text{ii}},$$

$$\tau P'_r(\tau) = rP_r(\tau) + P'_{r-1}(\tau) = rP_r(\tau) + \sum_{s=0}^{r-1}(2s+1)\frac{1+(-1)^{r+s}}{2}P_s(\tau)^{\text{iii}} \tag{10.3}$$

and, in view of $\dfrac{a_{,\alpha}}{a}=(\ln a)'=-\dfrac{h_{,\alpha}}{h}$, $\dfrac{a_{,\alpha}}{a}b=\widetilde{h}a_{,\alpha}$, $b_{,\alpha}=(\widetilde{h}a)_{,\alpha}$, it is easily seen that

$$P'_r(ax_3-b)(a_{,\alpha}\,x_3-b_{,\alpha}) = \frac{a_{,\alpha}}{a}(ax_3-b)P'_r(ax_3-b) + (\frac{a_{,\alpha}}{a}b-b_{,\alpha})P'_r(ax_3-b)$$

$$= -h_{,\alpha}\,h^{-1}(ax_3-b)P'_r(ax_3-b) - \widetilde{h}_{,\alpha}\,h^{-1}P'_r(ax_3-b)$$

$$= -\overset{r}{a}_{\alpha r}P_r(ax_3-b) - \sum_{s=0}^{r-1}\overset{r}{a}_{\alpha s}P_s(ax_3-b)^{\text{iv}}, \tag{10.4}$$

$^{\text{ii}}$on the top of the symbol $\sum$ both $r-1$ and r are true since the last term equals zero.
$^{\text{iii}}$on the top of the symbol $\sum$ both $r-2$ and $r-1$ are true since the last term equals zero.
$^{\text{iv}}$since

$$\sum_{s=0}^{r-1}(2s+1)\left[\frac{h_{,\alpha}+(-1)^{r+s}h_{,\alpha}}{2h}+\frac{\widetilde{h}_{,\alpha}-(-1)^{r+s}\widetilde{h}_{,\alpha}}{2h}\right]P_s(ax_3-b)$$

$$=\sum_{s=0}^{r-1}\frac{(2s+1)}{2h}\left(\frac{\overset{(+)}{h}_{,\alpha}-\overset{(-)}{h}_{,\alpha}+\overset{(+)}{h}_{,\alpha}(-1)^{r+s}-\overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\right.$$

$$\left.+\frac{\overset{(+)}{h}_{,\alpha}+\overset{(-)}{h}_{,\alpha}-\overset{(+)}{h}_{,\alpha}(-1)^{r+s}-\overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\right)P_s(ax_3-b)$$

$$=\sum_{s=0}^{r-1}(2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}P_s(ax_3-b)$$

78

where

$$\overset{r}{a}_{\alpha\underline{r}} := r\frac{h_{,\alpha}}{h}, \quad \overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r.$$

Now, bearing in mind (10.4) and (10.3), from (10.1) and (10.2) we have

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3$$

$$= f_{r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{\alpha s}f_s - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha}, \quad \alpha = 1,2, \qquad (10.5)$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r}\overset{r}{a}_{3s}f_s + \overset{(+)}{f} - (-1)^r\overset{(-)}{f}, \qquad (10.6)$$

respectively. Here

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}.$$

Let

$$f(x_1,x_2,x_3) = \sum_{r=0}^{\infty} a\Big(r+\frac{1}{2}\Big)f_r(x_1,x_2)P_r(ax_3-b), \qquad (10.7)$$

then

$$\overset{(\pm)}{f} := f(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)) = \sum_{s=0}^{\infty} a\Big(s+\frac{1}{2}\Big)f_s(\pm1)^s$$

$$= \sum_{s=0}^{\infty} \frac{(\pm1)^s(2s+1)}{2h}f_s, \quad i = \overline{1,3}, \qquad (10.8)$$

whence

$$\overset{(+)}{f} - (-1)^r\overset{(-)}{f} = -\sum_{s=0}^{\infty}\overset{r}{a}_{3s}f_s, \quad i = \overline{1,3}, \qquad (10.9)$$

$$\overset{(+)}{f}\overset{(+)}{h}_{,\alpha} - (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha} = \sum_{s=0}^{\infty}\overset{r}{a}^{*}_{\alpha s}f_s, \quad i = \overline{1,3}, \quad \alpha = 1,2, \qquad (10.10)$$

where

$$\overset{r}{a}^{*}_{\alpha s} = \overset{r}{a}_{\alpha s}, \quad s \neq r, \quad \overset{r}{a}^{*}_{\alpha\underline{r}} = (2r+1)\frac{h_{,\alpha}}{h}.$$

Substituting (10.10) and (10.9) into (10.5) and (10.6), respectively, we get

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 \;=\; f_{r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{\alpha s}f_s - \sum_{s=0}^{\infty}\overset{r}{a}^{*}_{\alpha s}f_s$$

$$=\; f_{r,\alpha} + \sum_{s=r}^{\infty}\overset{r}{b}_{\alpha s}f_s, \qquad (10.11)$$

79

where

$$\overset{r}{b}_{js} := -\overset{r}{a}_{js}, \ s > r; \ \ \overset{r}{b}_{js} = 0, \ s < r;$$

$$\overset{r}{b}_{\alpha\underline{r}} := \overset{r}{a}_{\alpha\underline{r}} - \overset{r}{a}^{*}_{\alpha\underline{r}} = -(r+1)\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2h}, \ \ \overset{r}{b}_{3\underline{r}} = 0,$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b)f_{,3}\, dx_3 \ = \ \sum_{s=0}^{r}\overset{r}{a}_{3s}f_s - \sum_{s=0}^{\infty}\overset{r}{a}_{3s}f_s$$

$$= \ -\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}f_s, \qquad (10.12)$$

respectively.

If $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are known (prescribed), then from (10.5) and (10.6), correspondingly, we obtain

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b)f_{,\alpha}\, dx_3 = f_{r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{\alpha s}f_s$$

$$+ \overset{(+)}{f}\overset{(+)}{n}_\alpha\sqrt{1 + (\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2} + (-1)^r\overset{(-)}{f}\overset{(-)}{n}_\alpha\sqrt{1 + (\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2} \ (10.13)$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b)f_{,3}\, dx_3 = \sum_{s=0}^{r}\overset{r}{a}_{3s}f_s$$

$$+ \overset{(+)}{f}\overset{(+)}{n}_3\sqrt{1 + (\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2} + (-1)^r\overset{(-)}{f}\overset{(-)}{n}_3\sqrt{1 + (\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2}, \ (10.14)$$

since

$$\overset{(\pm)}{n}_\alpha = \frac{\mp\overset{(\pm)}{h}_{,\alpha}}{\sqrt{1 + (\overset{(\pm)}{h}_{,1})^2 + (\overset{(\pm)}{h}_{,2})^2}}, \ \ \overset{(\pm)}{n}_3 = \frac{\pm 1}{\sqrt{1 + (\overset{(\pm)}{h}_{,1})^2 + (\overset{(\pm)}{h}_{,2})^2}}.$$

11 Conclusions

1. Differential hierarchical models for piezoelectric nonhomogeneous viscoelastic Kelvin-Voigt prismatic shells with voids are constructed. The ways of investigation of boundary value problems and initial boundary value problems, including the case of cusped prismatic shells [4], are indicated and some preliminary results are presented.

2. It is shown that in the case of hierarchical models of cusped prismatic shells, depending on the character of vanishing of the thickness at the lateral boundary of the prismatic shell, for well-posedness of the boundary value and initial boundary value problems the setting of boundary conditions is nonclassical, in general. Namely, in the case of nonclassical setting of boundary conditions they should be either weighted ones or the cusped edge should be freed from boundary conditions. In other words, at cusped edges: in the case of piezoelectric viscoelastic materials the displacements, volume fraction, and electric potential cannot always be prescribed.

3. If either elastic, piezoelectric, and dielectric constitutive coefficients are independent of the space points while the thickness of the prismatic shell vanishes in some way at some part of the boundary of the prismatic shell or the thickness of the prismatic shell is constant while the elastic, piezoelectric, and dielectric constitutive coefficients vanish in the same way at the same part of the boundary of the prismatic shell, then peculiarities of setting the boundary conditions for the displacement in the first case and those arising for the volume fraction function and the electric potential in the second case coincide. The stress-strain states coincide as well.

4. Antiplane deformation of piezoelectric nonhomogeneous transversely isotropic materials in the three-dimensional formulation and in $N = 0$ approximation is analysed. Some boundary value problems are solved in explicit forms in concrete cases.

References

[1] I. Vekua. On a way of calculating of prismatic shells. *Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences*, 21:191-259, 1995 (Russian).

[2] I. Vekua. The theory of thin shallow shells of variable thickness. *Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences*, 30:5-103, 1965 (Russian).

[3] I.N. Vekua. *Shell Theory: General Methods of Construction.* Pitman Advanced Publishing Program, Boston-London, Melbourne, 1985.

[4] G. Jaiani. *Cusped Shell-Like Structures.* Springer, Heidelberg-Dordrecht-London-New York, 2011.

[5] D. Natroshvili. *Mathematical Problems of Thermo-Electro-Magneto-Elasticity.* Lecture Notes of TICMI, 12, Tbilisi University Press, 2011.

[6] G. Avalishvili, M. Avalishvili, Wofgang H. Müller. Investigation of three-dimensional boundary value problem for thermoelastic piezoelectric solids. *Bulletin of TICMI*, 21(2):65-79, 2018.

[7] D. Ieşan. On a theory of thermoviscoelastic materials with voids. *J. Elasticity*, 104(1):369-384, 2011.

[8] S.C. Cowin, J.W. Nunziato. *Linear elastic materials with voids.* J. Elasticity, 13, 125-147, 1983.

[9] M.M. Svanadze. Steady vibrations problem in the theory of viscoelasticity for Kelvin-Voigt materials with voids. *Proc. Appl. Math. Mech.*, 12:283-284, 2012.

[10] M.M. Svanadze. Potential method in the linear theory of viscoelastic materials with voids. *J. Elasticity*, 114(1):101-126, 2014.

[11] G. Jaiani. *Theory of Cusped Euler-Bernoulli Beams and Kirchhoff-Love Plates.* Lecture Notes of TICMI, 3, Tbilisi University Press, 2002.

[12] G. Jaiani. On a generalization of the Keldysh theorem. *Georgian Mathematical Journal*, 2(3):291-297, 1995.

[13] G. Jaiani. The first boundary value problem of cusped prismatic shell theory in zero approximation of Vekua theory. *Proceedings of I. Vekua Institute of Applied Mathematics*, 29:5-38, 1988 (in Russian, Georgian and English summaries).

[14] G. Jaiani. Initial and boundary value problems for singular differential equations and applications to the theory of cusped bars and plates. *Complex methods for partial differential equations (ISAAC Serials, 6), Eds.: H. Begehr, O. Celebi, W. Tutschke. Kluwer, Dordrecht*:113-149, 1999.

[15] G. Fichera. On a unified theory of boundary value problems for elliptic-parabolic equations of second order. *Boundary Problems in Differ. Equat., Madison, Univ. Wisconsin Press, edited by Langer*:97-120, 1960.

[16] O.A. Oleinik, E.B. Radkevich. *Second Order Equations with the Non-begative Charachteristic Form.* Matematicheski Analis 1969 (Itogi Nauki, VINITI Akad. Nauk SSSR), Moscow, 1971 (Russian).

[17] J. Dufner. Randwertprobleme fuer systeme linearer partieller Differentialgleichungen zweiter Ordnung vom elliptisch-parabolischen Typ. *Atti Acc. Naz. Lincei, Memorie, Sc. fis., mat. e nat. S.* 8(11):17-60, 1972.

[18] J. Dufner. On the solvability of boundary value problems for elliptic-parabolic systems of second order. *Atti Acc. Naz.Lincei, Rend.Sc.fis.mat.e nat.*, 57:147-155, 1974.

[19] P.S. Dineva, D. Gross, R. Müller, T. Rangelov. Dynamic fracture of piezoelectric materials solution of time-harmonic problems via BIEM. *Solid Mechanics and Its Applications*, 212, 2014.

[20] H. Altenbach, J. Altenbach, W. Kissing. *Mechanics of Composite Structural Elements.* Springer-Verlag, Heidelberg 2004.

[21] R. Sinkus, et. al. Imaging Anisotropic and Viscous Properties of Brest Tissue by Magnetic Resonance-Elastography, Magnetic Resonance in Medicine. *Wiley Interscience*, 53:372-387, 2005.

[22] Ph. L. Gould, *Introduction to Linear Elasticity.* 3rd Edition, Springer, New-York, Heidelberg, Dordrecht, London, 2013.

[23] G. Jaiani, *Solution of Some Problems for a Degenerate Elliptic Equation of Higher Order and their Applications to Prismatic Shells.* Tbilisi University Press, Tbilisi, 1982, 178 p. (in Russian; Georgian and English summaries).

[24] A. Bitsadze. *Some Classes of Partial Differential Equations.* Gordon and Breach Science Publishers, 504 p., 1988.

[25] G. Jaiani. *Mathematical Models of Mechanics of Continua.* Tbilisi University Press, Tbilisi, 338 p., 2004 (The second edition, 526 p., 2018) (Georgian).

Previous Issues

1. **W.-L. Schulze**. *Pseudo-differential Calculus and Applications to Non-smooth Configurations.* Volume 1, 2000,

2. **S. Camiz.** *Exploratory 2- and 3-way Data Analisys and Applications.* Volume 2, 2001, 44 p.

3. **G. Jaiani**. *Theory of Cusped Euler-Bernoulli Beams and Kirchhoff-Love Plates*. Volume 3, 2002, 129 p.

4. **G. Jaiani, S. Kharibegashvili, D. Natroshvili, W.L. Wendland**. *Hierarchical Models for Cusped Plates and Beams.* Volume 4, 2003, 121 p.

5. **A. Bernardini, G. Bretti, P.E. Ricci.** *Laguerre-type Exponentials, Multidimensional Special Polynomials and Applications.* Volume 5, 2004, 28 p.

6. **Ts. Gabeskiria, G. Jaiani, J. Antidze, G. Datashvili.** *English-Georgian-Russian-German-French-Italian Glossary of Mathematical Terms.* Volume 6, 2005, 228 p.

7. **J. Rogava, M. Tsiklauri**. *High Order Accuracy Decomposition Schemes for Evolution Problem.* Volume 7, 2006, 164 p.

8. Volume 8, 2007 **was dedicated to the Centenary of Ilia Vekua.**

   It contains the following articles:

   - **R.P. Gilbert, G.V. Jaiani.** *Ilia Vekua's Centenary*
   - **H. Begehr, T. Vaitekhovich.** *Complex Partial Differential Equations in a Manner of I.N. Vekua.*pp. 15-26
   - **B.-W. Schulze.** *Operators on Configurations with Singular Geometry.* pp.27-42
   - **V. Kokilashvili, V. Paatashvili.** *On the Riemann-Hilbert Problem in Weighted Classes of Cauchy Type Integrals with Density from $L^{P(\cdot)}(\Gamma)$.* pp. 43-52
   - **Tavkhelidze.** *Classification of a Wide Set of Geometric Figures.* pp. 53-61

9. **N. Chinchaladze**. *On Some Nonclassical Problems for Differential Equations and Their Applications to the Theory of Cusped Prismatic Shells*. Volume 9, 2008,

10. **D. Caratelli, B. Germano, J. Gielis, M.X. He, P. Natalini, P.E. Ricci**. *Fourier Solution of the Dirichlet Problem for the Laplace and Helmholtz Equations in Starlike Domains*. Volume 10, 2009, 64 p.

11. **A. Cialdea.** *The L^P–Dissipativity of Partial Defferential Operators.* Volume 11, 2010, 94 p.

12. **D. Natroshvili.** *Mathematical Problems of Thermo-Electro-Magneto-Elasticity.* Volume 12, 2011, 128 p.

13. **G. Akhalaia, G. Giorgadze, V. Jikia, N. Kaldani, G. Makatsaria, N. Manjavidze.** *Elliptic Systems on Riemann Surface.* Volume 13, 2012, 155 p.

14. **I. Vekua.** *On Metaharmonic Functions.* Volume 14, 2013, 62 p.

15. **Wolfgang H. Müller & Paul Lofink.** *The Movement of the Earth: Modelling the Flattening Parameter.* Volume 15, 2014, 42 p.

16. **W.H. Müller, E.N. Vilchevskaya, & A.B. Freidin.** *Structural Changes in Micro-Materials: Phenomenology, Theory, Applications, and Simulations.* Volume 16, 2015, 74 p.

17. Volume 17, 2016 contains the following articles**:**

   - **Alexander Meskhi.** *Multilinear Integral Operators in Weighted Function Spaces,* pp. 5-18

   - **Alice Fialowski.** *The Moduli Space and Versal Deformations Of Algebraic Structures,* pp. 19-34

   - **Reinhold Kienzler.** *Material Conservation and Balance Laws in Linear Elasticity with Applications,* pp. 35-65

18. **V.Yu. Belashov, E.S. Belashova, O.A. Kharshiladze.** *Nonlinear Wave Structures of the Soliton and Vortex Types in Complex Continuous Media: Theory, Simulation, Applications.* Volume 18, 2017, 90 p.

Information for the Authors