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On the Simple Layer Potential Ansatz for Steady Elastic Oscillations

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Abstract. We consider the Dirichlet problem for steady elastic oscillations. The main result concerns the solvability of the boundary integral system of equations of the first kind arising when we impose the Dirichlet boundary condition to a simple layer potential. Such a result is here obtained by using the theories of differential forms and reducible operators.

Keywords: Steady elastic oscillations; potential theory; integral representations.

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1. Introduction

Potential methods for the basic boundary value problems related to steady elastic oscillations have been developed in [8]. In particular, the representability of the solution of the Dirichlet problem has been obtained by means of a double layer potential. If we look for the solution of this problem in the form of a simple layer potential, we obtain an integral system of the first kind on the boundary.

The same problem for Lamé system was previously considered in [3]. There an existence theorem for the relevant integral system of the first kind on the boundary was obtained following a method given in [1] for the Laplace equation. This method hinges on the theory of reducible operators and on the theory of differential forms, it does not use the theory of pseudodifferential operators and could be considered as an extension to higher dimensions of Muskhelishvili's method (see [2]). Later, this approach was extended to different BVPs for several partial differential equations and systems in simply and multiple connected domains (see [5] and the references therein).

The aim of the present paper is to show how to extend this method to the Dirichlet problem for steady elastic oscillations.

The paper is structured as follows. Section 2 is devoted to some notations and definitions, whereas Section 3 deals with auxiliary results in potential theory. In Section 4 we construct a reducing operator that we use in the study of the integral system of the first kind arising when we impose the Dirichlet boundary condition to a simple layer potential. In Section 5 we find a solution of the Dirichlet problem in terms of a simple layer potential.

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2. Definitions

Throughout this paper, Ω is a bounded domain (open connected set) of $\mathbb{R}^3$ such that its boundary is a Lyapunov hypersurface Σ (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $\lambda \in (0,1]$), and such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected; $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outwards unit normal vector at the point $x = (x_1, x_2, x_3) \in \Sigma$. The symbol $|\cdot|$ denotes the Euclidean norm for elements of $\mathbb{R}^3$.

Given the set of constants λ, μ, ρ satisfying the conditions

$$\mu, \rho > 0, \quad 3\lambda + 2\mu > 0,$$

the homogeneous system of elastostatic oscillations has the form

$$\mu\Delta u + (\lambda+\mu)\nabla \mathrm{div}\ u + \rho\omega^2 u = 0 \tag{1}$$

where $u : \mathbb{R}^3 \to \mathbb{C}^3$ is the displacement vector and $\omega \in \mathbb{R}$ is the oscillation frequency [8, p. 48]. It is convenient to write the basic equation (1) in a matrix form. To this end we consider the (3×3) matrix differential operator

$$A(\partial_x, \omega) = \Big(A_{jk}(\partial_x, \omega)\Big)_{j,k=1,2,3}$$

whose entries are

$$A_{jk}(\partial_x, \omega) = \delta_{jk}(\mu\Delta + \rho\omega^2) + (\lambda+\mu)\frac{\partial^2}{\partial x_j \partial x_k}, \quad j,k = 1,2,3,$$

δ_{jk} being the delta Kronecker symbol. Equation (1) becomes

$$A(\partial_x, \omega)u = 0. \tag{2}$$

When $\omega = 0$ we simply write $A(\partial_x)$.

The (3×3) matrix differential operator

$$T(\partial_x, n) = \Big(T_{jk}(\partial_x, n)\Big)_{j,k=1,2,3}$$

is introduced, where

$$T_{jk}(\partial_x, n(x)) = \lambda n_j(x)\frac{\partial}{\partial x_k} + \mu n_k(x)\frac{\partial}{\partial x_j} + \mu\delta_{jk}\frac{\partial}{\partial n(x)}.$$

T is known as the stress operator (see [8, p.57]). The matrix of the fundamental solutions of the homogeneous oscillations system (2) has the form

$$\Gamma(x, \omega) = \Big(\Gamma_{kj}(x, \omega)\Big)_{j,k=1,2,3}$$

where

$$\Gamma_{kj}(x,\omega) = \sum_{l=1}^{2} \Big(\delta_{kj}\alpha_l + \beta_l \frac{\partial^2}{\partial x_k \partial x_j}\Big) \frac{e^{ik_l|x|}}{|x|},$$

i is the imaginary unity, the non-negative constants k_1 and k_2 are determined by

$$k_1^2 = \rho\omega^2(\lambda + 2\mu)^{-1}, \qquad k_2^2 = \rho\omega^2\mu^{-1},$$

and

$$\alpha_l = \delta_{2l}(2\pi\mu)^{-1}, \qquad \beta_l = (-1)^l(2\pi\rho\omega^2)^{-1}.$$

$\Gamma(x,\omega)$ is called Kupradze's matrix (see [8, p. 85]). Each column and each row of this matrix satisfy (2) for $x \neq 0$.

Let

$$\widetilde{\Gamma}(x,\omega) = \Gamma(x,\omega) - \Gamma(x), \tag{3}$$

$\Gamma(x)$ being Somigliana's matrix (see [8, p. 84]).

We recall the following estimates (see [8, pp. 87-88]):

$$|\Gamma_{kj}(x,\omega)| \leq \frac{c(\lambda,\mu)}{|x|}, \qquad k,j = 1,2,3, \ \forall x \neq 0,$$

$$|\widetilde{\Gamma}_{kj}(x,\omega)| \leq |\omega|\widetilde{c}(\lambda,\mu), \qquad k,j = 1,2,3,$$

and

$$\Big|\frac{\partial \widetilde{\Gamma}_{kj}(x,\omega)}{\partial x_l}\Big| \leq \omega^2\overline{c}(\lambda,\mu), \qquad k,j,l = 1,2,3 \tag{4}$$

where $c(\lambda,\mu)$, $\widetilde{c}(\lambda,\mu)$ and $\overline{c}(\lambda,\mu)$ are positive constants, depending on λ and μ only.

The symbol $C^h(\Omega)$ ($h \in \mathbb{N}$) stands for the space of all complex-valued continuous functions whose derivatives are continuously differentiable up to the order h in Ω. Moreover, the Hölder space $C^{h,\beta}(\Omega)$ consists of all functions, defined in Ω, having continuous derivatives up to order $h \in \mathbb{N}$ and such that the partial derivatives of order h are Hölder continuous with exponent $\beta \in (0,1]$.

If u is a h-form in Ω, the symbol du denotes the differential of u, while $*u$ denotes the dual Hodge form.

From now on we consider $p \in (1,+\infty)$. By $L^p(\Sigma)$ we denote the space of p-integrable complex-valued functions defined on Σ. By $L^p_h(\Sigma)$ we mean the space of the differential forms of degree $h \geq 1$ whose components belong to $L^p(\Sigma)$.

The Sobolev space $W^{1,p}(\Sigma)$ can be defined as the space of functions in $L^p(\Sigma)$ such that their weak differential belongs to $L^p_1(\Sigma)$. If $u \in [W^{1,p}(\Sigma)]^3$, by du we denote the vector (du_1, du_2, du_3).

Finally, we write $\underset{\Sigma}{*} w = w_0$ if w is an 2-form on Σ and $w = w_0 d\sigma$.

In what follows, we shall distinguish by apices + and − the limit obtained by approaching the boundary Σ from Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$, respectively, that is

$$u^+(x) = \lim_{\Omega \ni y \to x} u(y) \quad \text{and} \quad u^-(x) = \lim_{\mathbb{R}^3 \setminus \overline{\Omega} \ni y \to x} u(y).$$

If B_1 and B_2 are two Banach spaces and $S : B_1 \to B_2$ is a continuous linear operator, we say that S can be reduced on the left if there exists a continuous linear operator $R : B_2 \to B_1$ such that $RS = I + T$, where I stands for the identity operator on B_1 and $T : B_1 \to B_1$ is compact. Analogously, one can define an operator S reducible on the right. If S is a reducible operator, its range is closed and then the equation $S\alpha = \beta$ has a solution if and only if $\langle \gamma, \beta \rangle = 0$, for any $\gamma \in B_2^*$ such that $S^*\gamma = 0$, S^* being the adjoint of S (see, e.g., [7] or [9]).

3. Auxiliary results

We need some results about the BVPs

$$\begin{cases} v \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 & \\ A(\partial_x, \omega) v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Sigma \end{cases} \tag{5}$$

and

$$\begin{cases} w \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 & \\ A(\partial_x, \omega) w = 0 & \text{in } \Omega \\ T(\partial_x, n) w = 0 & \text{on } \Sigma. \end{cases} \tag{6}$$

Denote by $\mathcal{V}_0$ and $\mathcal{W}_0$ the spaces of solutions of (5) and (6), respectively. First, observe that $\mathcal{V}_0 = \{0\}$ ($\mathcal{W}_0 = \{0\}$) whenever ω^2 is not a Dirichlet (traction) eigenvalue of (5) ((6)). Moreover, let us define

$$V = \left\{ T(\partial_x, n) v \Big|_{\Sigma} \ : \ v \in \mathcal{V}_0 \right\}$$

and

$$W = \{ w|_{\Sigma} \ : \ w \in \mathcal{W}_0 \}$$

We are interested in the kernels of the boundary integral operators $\mp I + K$ and $\pm I + K^*$, where

$$K\varphi(x) = \int_{\Sigma} [T(\partial_y, n(y)) \Gamma(x - y, \omega)]' \varphi(y) \, d\sigma_y \,,$$

the prime denoting the transpose of a matrix, and

$$K^*\psi(x) = \int_\Sigma [T(\partial_x, n(x))\Gamma(x-y,\omega)]\psi(y)\, d\sigma_y\,.$$

In view of [8, Theorems 2.2 and 2.3, p.413–415], we have

$$\mathcal{N}(I+K) = W \quad \text{ and } \quad \mathcal{N}(I-K^*) = V. \tag{7}$$

From (4), (3) and [8, p. 236 and p. 355] it follows

$$\dim \mathcal{N}(I+K) = \dim \mathcal{N}(I+K^*) = m_T \in \mathbb{N} \tag{8}$$

and

$$\dim \mathcal{N}(I-K) = \dim \mathcal{N}(I-K^*) = m_D \in \mathbb{N}\,.$$

If ω^2 is not an interior traction (Dirichlet) eigenvalue, then $m_T = 0$ ($m_D = 0$).

Lemma 3.1: *Suppose $m_T \neq 0$. Let $\{\phi^1, \dots, \phi^{m_T}\}$ is a basis of $\mathcal{N}(I+K^*)$ and define*

$$w^j(x) = \int_\Sigma \Gamma(x-y,\omega)\phi^j(y)\, d\sigma_y, \qquad x \in \mathbb{R}^3 \setminus \Sigma$$

$j = 1, \dots, m_T$. Then

$$\phi^j(x) = -\frac{1}{2}[T(\partial_x, n(x))w^j(x)]^- \qquad \text{on } \Sigma,$$

$j = 1, \dots, m_T$, and the vector functions

$$\psi^j(x) = -[\overline{w}^j(x)]^- \qquad x \in \Sigma,$$

$j = 1, \dots, m_T$, form a basis for $\mathcal{N}(I+K)$.

Proof: Note that $[T(\partial_x, n)w^j]^+ = 0$ on Σ because $\phi^j + K^*\phi^j = 0$ ($j = 1, \dots, m_T$). By applying the jump relation (see [8, p. 416])

$$[T(\partial_x, n(x))w^j(x)]^+ - [T(\partial_x, n(x))w^j(x)]^- = 2\phi^j(x), \qquad x \in \Sigma,$$

we get

$$\phi^j(x) = -\frac{1}{2}[T(\partial_x, n(x))w^j(x)]^-, \qquad x \in \Sigma.$$

Moreover, $[\overline{w}^j]^+ = [\overline{w}^j]^- \in W = \mathcal{N}(I+K)$ (see (7)).

If we assume that α_j ($j = 1, \dots, m_T$) solve

$$\sum_{j=1}^{m_T} \alpha_j \int_\Sigma \psi^j \, \phi^l \, d\sigma = 0, \quad l = 1, \dots, m_T,$$

and if we set

$$w = \sum_{j=1}^{m_T} \overline{\alpha}_j w^j,$$

we have that

$$\int_\Sigma [\overline{w}]^- [T(\partial_x, n)w]^- \, d\sigma = 0.$$

Then, arguing as in the proof of [8, Theorem 2.13, p. 132], we get

$$\lim_{R\to+\infty} \int_{|x|=R} |w_p|^2 \, d\sigma = \lim_{R\to+\infty} \int_{|x|=R} |w_s|^2 \, d\sigma = 0, \tag{9}$$

where $w = w_p + w_s$,

$$\text{curl } w_p = 0, \quad (\Delta + k_1^2) w_p = 0\,,$$

$$\text{div } w_s = 0, \quad (\Delta + k_2^2) w_s = 0$$

in Ω (see [8, Theorem 2.5, p. 123]). In view of Lemma 2.14 in [8, p. 134], (9) along with radiation conditions lead to $w = 0$ in $\mathbb{R}^3 \setminus \Omega$. Hence $[T(\partial_x, n)w]^- = 0$ on Σ and this implies that

$$\sum_{j=1}^{m_T} \alpha_j \phi^j = 0,$$

that is $\alpha_j = 0$ $(j = 1, \dots, m_T)$ due to the fact that ϕ^j are linearly independent. Since the determinant of the matrix

$$\int_\Sigma \psi^j \, \phi^l \, d\sigma = \frac{1}{2} \int_\Sigma [\overline{w}^j]^- [T(\partial_x, n)w^l]^- \, d\sigma \quad j, l = 1, \dots, m_T$$

does not vanish, $\psi^1, \dots, \psi^{m_T}$ are linearly independent and, in view of (8), form a basis of $\mathcal{N}(I + K)$. □

In an analogue to Lemma 3.1, we have the following result.

Lemma 3.2: *Suppose $m_D \neq 0$. Let $\{\eta^1, \dots, \eta^{m_D}\}$ be a basis of $\mathcal{N}(I - K)$ and* *define*

$$v^j(x) = \int_\Sigma [T(\partial_y, n(y))\Gamma(x-y,\omega)]'\eta^j(y)\, d\sigma_y, \qquad x \in \mathbb{R}^3 \setminus \Sigma$$

$j = 1, \ldots, m_D$. *Then*

$$\eta^j = \frac{1}{2}[v^j]^- \qquad on\ \Sigma,$$

$j = 1, \ldots, m_D$, *and the vector functions*

$$[\chi^j]^- = [T(\partial_x, n)\overline{v}^j]^- \qquad on\ \Sigma,$$

$j = 1, \ldots, m_D$, *form a basis of* $\mathcal{N}(I - K^*)$.

4. Reduction of a certain integral equation

Given $f \in [W^{1,p}(\Sigma)]^3$ $(1 < p < \infty)$ such that

$$\int_\Sigma f\, T(\partial_x, n)v\, d\sigma = 0, \qquad \forall\, v \in \mathcal{V}_0, \tag{10}$$

we want to determine a solution of the Dirichlet problem

$$\begin{cases} A(\partial_x, \omega)u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma \end{cases} \tag{11}$$

in the form of a simple layer potential

$$u(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi(y)\, d\sigma_y, \quad x \in \Omega \tag{12}$$

with density $\varphi \in [L^p(\Sigma)]^3$. Observe that conditions (10) are necessary for the solvability of the problem (11) because of Green's formulas.

By imposing the boundary condition to (12), an integral system of equations of the first kind

$$\int_\Sigma \Gamma(x-y,\omega)\varphi(y)\, d\sigma_y = f(x) \tag{13}$$

arises on Σ. Following [1], we take the differential d of both sides of system (13) and the singular integral system

$$\int_\Sigma d_x[\Gamma(x-y,\omega)]\varphi(y)\, d\sigma_y = df(x) \tag{14}$$

comes out. Note that in (14) the unknown is a vector function $\varphi \in [L^p(\Sigma)]^3$, while the data is a vector whose components are differential forms of degree 1 belonging to $L^p_1(\Sigma)$.

We are going to show that the operator on the left-hand side of (14), acting from $[L^p(\Sigma)]^3$ into $[L_1^p(\Sigma)]^3$, can be reduced on the left.

First, we recall the next result proved in [3] (see also [4] for higher dimensions).

Lemma 4.1: *The singular integral operator* $R : [L^p(\Sigma)]^3 \longrightarrow [L_1^p(\Sigma)]^3$

$$R_j\varphi(x) = \int_\Sigma d_x[\Gamma_{jk}(x-y)]\varphi_k(y)\,d\sigma_y \tag{15}$$

($j = 1,2,3$) can be reduced on the left. A reducing operator of (15) *is the integral operator* $R^{'} : [L_1^p(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$ *defined as*

$$\begin{aligned} R_i^{'}[\psi](x) =& \frac{(\lambda+\mu)(\lambda+2\mu)}{(\lambda+3\mu)}\mathcal{K}_{jj}[\psi](x)\,n_i(x) + \mu\mathcal{K}_{ij}[\psi](x)\,n_j(x) + \\ &+ \mu\frac{(\lambda+\mu)}{(\lambda+3\mu)}\mathcal{K}_{ji}[\psi](x)\,n_j(x) \end{aligned} \tag{16}$$

($i = 1,2,3$), where

$$\mathcal{K}_{js}[\psi](x) = *\int_\Sigma d_x[s_1(x-y)]\wedge\psi_j(y)\wedge dx^s - \delta_{ihp}^{123}\int_\Sigma \frac{\partial}{\partial x_s}[K_{ij}(x-y)]\wedge\psi_h(y)\wedge dy^p,$$

$$s_1(x-y) = -\frac{1}{4\pi|y-x|}\sum_{j=1}^{3} dx^j dy^j$$

and

$$K_{ij}(x-y) = \frac{1}{4\pi}\frac{(y_i-x_i)(y_j-x_j)}{|y-x|^3}.$$

Theorem 4.2: *Let* $S : [L^p(\Sigma)]^3 \to [L_1^p(\Sigma)]^3$ *be the singular integral operator*

$$S\varphi(x) = \int_\Sigma d_x[\Gamma(x-y,\omega)]\varphi(y)\,d\sigma_y, \qquad x\in\Sigma.$$

Then S can be reduced on the left by the singular integral operator (16).

Proof: In view of (3)

$$\begin{aligned} S\varphi(x) &= \int_\Sigma d_x[\Gamma(x-y)]\varphi(y)\,d\sigma_y + \int_\Sigma d_x[\widetilde{\Gamma}(x-y,\omega)]\varphi(y)\,d\sigma_y \\ &= R\varphi(x) + \Lambda\varphi(x), \qquad x\in\Sigma. \end{aligned}$$

The operator (16) reduces S because

$$R'S = R'R + R'\Lambda$$

is a Fredholm operator. In fact R' reduces R (Lemma 4.1) and $R'\Lambda$ is compact (see (4)). $\square$

5. Representation theorem

In this section we obtain the representability of a solution of the Dirichlet problem (11) by means of a simple layer potential. To this end we define the space in which we look for such a solution.

Definition 5.1: We say that the vector function u belongs to $\mathcal{S}^p$ if and only if there exists $\varphi \in [L^p(\Sigma)]^3$ such that u can be represented by the simple layer potential (12).

Theorem 5.2: *Let $f \in [C^{1,\lambda}(\Sigma)]^3$. There exists a solution of the Dirichlet problem*

$$\begin{cases} u \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 & \\ A(\partial_x, \omega)u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma \end{cases} \tag{17}$$

if and only if conditions (10) *are satisfied. Moreover, any solution can be represented as*

$$u(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi(y)\, d\sigma_y, \quad x \in \Omega\,,$$

with $\varphi \in [C^{0,\lambda}(\Sigma)]^3$.

Proof: In view of [8, pp. 426-428] there exists a solution of (17) if and only if conditions (10) are satisfied, and any solution can be represented as a double layer potential

$$u(x) = \int_\Sigma [T(\partial_y, n(y))\Gamma(x-y,\omega)]'\varphi(y)\, d\sigma_y, \qquad x \in \Omega$$

with $\varphi \in [C^{0,\lambda}(\Sigma)]^3$.

Now set

$$z(x) = \rho\omega^2 \int_\Omega \Gamma(x-y)u(y)\, dy, \qquad x \in \Omega.$$

It is well known that $z \in [C^{2,\lambda}(\overline{\Omega})]^3$. Moreover, z satisfies $A(\partial_x)z = -\rho\omega^2 u$ in Ω, and then $A(\partial_x)u = A(\partial_x)z$ in Ω.
The vector function $v = u - z$ satisfies the Dirichlet problem

$$\begin{cases} v \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 & \\ A(\partial_x)v = 0 & \text{in } \Omega \\ v = f - z & \text{on } \Sigma. \end{cases}$$

The unique solution v can be represented as a simple layer potential

$$v(x) = \int_\Sigma \Gamma(x-y)\psi(y)\, d\sigma_y, \quad x \in \Omega$$

with density $\psi \in [C^\lambda(\Sigma)]^3$ (see [3]). Then $v \in [C^{1,\lambda}(\overline{\Omega})]^3$ and $u = v + z \in [C^{1,\lambda}(\overline{\Omega})]^3$ too.

Consider the BVP

$$\begin{cases} w \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 \\ A(\partial_x,\omega)w = 0 & \text{in } \Omega \\ T(\partial_x,n)w = T(\partial_x,n)u & \text{on } \Sigma\,. \end{cases} \tag{18}$$

According to [8, pp. 428-431] a solution of (18) exists and it can be represented by a simple layer potential with density in $[C^{0,\lambda}(\Sigma)]^3$. The vector function $g = u - w$ satisfies the BVP

$$\begin{cases} g \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 \\ A(\partial_x,\omega)g = 0 & \text{in } \Omega \\ T(\partial_x,n)g = 0 & \text{on } \Sigma. \end{cases}$$

If ω^2 is not a traction eigenvalue, $g = 0$, and hence $u = w$, i.e. u is a simple layer potential with density in $[C^{0,\lambda}(\Sigma)]^3$.
If ω^2 is a traction eigenvalue, we define

$$u^j(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi^j(y)\, d\sigma_y, \qquad j = 1,\dots,m_T$$

$\{\varphi^1,\dots,\varphi^{m_T}\}$ being a basis of the kernel of the operator $I + K$.

Observe that $\varphi^j \in [C^{0,\lambda}(\Sigma)]^3$ for each $j = 1,\dots,m_T$. In view of Lemma 3.1

$$g(x) = \sum_{j=1}^{m_T} c_j u^j(x) = \sum_{j=1}^{m_T} c_j \int_\Sigma \Gamma(x-y,\omega)\varphi^j(y)\, d\sigma_y, \qquad x \in \Sigma.$$

Moreover

$$g(x) = \sum_{j=1}^{m_T} c_j \int_\Sigma \Gamma(x-y,\omega)\varphi^j(y)\, d\sigma_y, \qquad x \in \Omega$$

because $\widetilde{g}(x) = g(x) - \sum_{j=1}^{m_T} c_j \int_\Sigma \Gamma(x-y,\omega)\varphi^j(y)\, d\sigma_y$ satisfies $A(\partial_x,\omega)\widetilde{g} = 0$ in Ω and $\widetilde{g} = T(\partial_x,n(x))\widetilde{g} = 0$ on Σ. Thus g is a simple layer potential with density in $[C^{0,\lambda}(\Sigma)]^3$, and then the same is for u. □

We are now in a position to prove an existence theorem for the singular integral equation $S\varphi = w$, i.e.

$$\int_\Sigma d_x[\Gamma(x-y,\omega)]\varphi(y)\,d\sigma_y = w(x) \quad \text{a.e. on } \Sigma, \tag{19}$$

where $w \in [L^p_1(\Sigma)]^3$ is given and $\varphi \in [L^p(\Sigma)]^3$ is looked for.

To this end, we recall the following lemma.

Lemma 5.3: *[6, Lemma 1] Let $g_1, g_2, \dots, g_m \in L^p(\Sigma)$ $(1 < p < \infty)$ be linearly independent and $h \in (0,1]$. If $\psi \in L^p_1(\Sigma)$ is such that*

$$\int_\Sigma du \wedge \psi = 0, \quad \forall\, u \in C^{1,h}(\Sigma) : \int_\Sigma u g_j\, d\sigma = 0,\ j = 1, \dots, m$$

then $\psi \in W^{1,p}_1(\Sigma)$ and $d\psi = \sum_{j=1}^m c_j g_j\, d\sigma$.

Theorem 5.4: *Given $w \in [L^p_1(\Sigma)]^3$, there exists $\varphi \in [L^p(\Sigma)]^3$ solution of the singular integral equation* (19) *if and only if*

$$\int_\Sigma \gamma_j \wedge w_j = 0, \qquad j = 1, 2, 3$$

for every $\gamma \in [W^{1,q}_1(\Sigma)]^3$, $(q = p/(p-1))$ such that $d\gamma = T(\partial_x, n)v d\sigma$, with $v \in \mathcal{V}_0$.

Proof: Consider the adjoint S^* of S, i.e. the operator $S^* : [L^q_1(\Sigma)]^3 \longrightarrow [L^q(\Sigma)]^3$ whose components are

$$S^*_j[\psi](x) = \int_\Sigma \psi_j(y) \wedge d_y[\Gamma_{ij}(x-y,\omega)], \qquad j = 1, 2, 3.$$

Theorem 4.2 implies that the integral equation (19) has a solution $\varphi \in [L^p(\Sigma)]^3$ if and only if the compatibility conditions

$$\int_\Sigma \psi_j \wedge w_j = 0, \qquad j = 1, 2, 3$$

hold for any $\psi \in [L^q_1(\Sigma)]^3$ such that $S^*\psi = 0$. Let $\psi \in [L^q_1(\Sigma)]^3$ such that $S^*\psi = 0$, i.e.

$$\int_\Sigma \psi_j(y) \wedge d_y[\Gamma_{kj}(x-y,\omega)] = 0, \qquad \text{a.e. on } \Sigma.$$

For any $p \in [C^\lambda(\Sigma)]^3$, we have

$$\begin{aligned} 0 &= \int_\Sigma p_k(x) d\sigma_x \int_\Sigma \psi_j(y) \wedge d_y\, [\Gamma_{kj}(x-y,\omega)] \\ &= \int_\Sigma \psi_j(y) \wedge d_y \int_\Sigma p_k(x)\Gamma_{kj}(x-y,\omega) d\sigma_x = \int_\Sigma \psi_j \wedge du_j, \end{aligned}$$

where $u(y) = \int_\Sigma \Gamma(x-y,\omega)p(x)\,d\sigma_x$.

Thanks to Theorem 5.2, we can say that

$$\int_\Sigma \psi_j \wedge du_j = 0 \tag{20}$$

for any $u \in [C^{1,\lambda}(\Sigma)]^3$ such that

$$\int_\Sigma uT(\partial_x, n)v\,d\sigma = 0, \quad \forall\, v \in \mathcal{V}_0\,.$$

If ω^2 is not a Dirichlet eigenvalue of (5), $\mathcal{V}_0 = \{0\}$ and (20) holds for any $u \in [C^{1,\lambda}(\Sigma)]^3$. This means that $\psi \in [W_1^{1,q}(\Sigma)]^3$ and $d\psi = 0$.

If ω^2 is a Dirichlet eigenvalue of (5), there exist $v^1, \dots, v^{m_D}$ linearly independent eigensolutions of (5). Therefore, we may say that (20) holds for any $u \in [C^{1,\lambda}(\Sigma)]^3$ such that

$$\int_\Sigma u_j(x)T_{kj}(\partial_x, n(x))v_k^l(x)\,d\sigma_x = 0, \qquad l = 1, \dots, m_D.$$

Observe that $T(\partial_x, n)v^1, \dots, T(\partial_x, n)v^{m_D}$ are also linearly independent functions of $[L^q(\Sigma)]^3$. In fact, let $c_1, \dots, c_{mD}$ be complex constants such that

$$c_1 T(\partial_x, n)v^1 + \ldots + c_{m_D} T(\partial_x, n)v^{m_D} = 0$$

on Σ. Setting $U = c_1 v^1 + \ldots + c_{m_D} v^{m_D}$, we find $A(\partial_x, \omega)U = 0$ in Ω and $U = T(\partial_x, n)U = 0$ on Σ. Then, from Green's formula it follows that $U \equiv 0$ in $\overline{\Omega}$ and thus $c_1 = \ldots = c_{m_D} = 0$. Hence, for any fixed $j = 1, 2, 3$, the functions $T_{kj}(\partial_x, n)v_k^1, \dots, T_{kj}(\partial_x, n)v_k^{m_D}$ are linearly independent. Now, applying Lemma 5.3 to u_j with $g_{j,l} = T_{kj}(\partial_x, n)v_k^l$, $(l = 1, \dots, m_D)$ we get $\psi_j \in W_1^{1,q}(\Sigma)$ and $d\psi_j = \sum_{l=1}^{m_D} c_l T_{kj}(\partial_x, n)v_k^l d\sigma$ for some complex constants c_l.

Conversely, let $\psi \in [W_1^{1,q}(\Sigma)]^3$ be such that $d\psi = T(\partial_x, n)v\,d\sigma$ with $v \in \mathcal{V}_0$. Then

$$\int_\Sigma \Gamma(x-y,\omega)T(\partial_y, n(y))v(y)\,d\sigma_y = 0, \qquad \forall\, x \in \mathbb{R}^3 \setminus \overline{\Omega}$$

in view of Green's formulas. It follows that

$$\int_\Sigma \Gamma(x-y,\omega)T(\partial_y, n(y))v(y)\,d\sigma_y = 0, \qquad \text{a.e. on } \Sigma$$

and this means $S^*\gamma = 0$.

□

The next theorem provides the representability of solutions of the Dirichlet problem for elastostatic oscillations with data in $[W^{1,p}(\Sigma)]^3$ by means of a simple layer potential.

Theorem 5.5: *Let $f \in [W^{1,p}(\Sigma)]^3$ $(1 < p < \infty)$. There exists a solution of the Dirichlet problem*

$$\begin{cases} u \in \mathcal{S}^p \\ A(\partial_x, \omega)u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma \end{cases} \tag{21}$$

if and only if f satisfies the compatibility conditions (10).

Proof: The necessity of conditions (10) follows from Green's formula, as already remarked. For the sufficiency, we first prove that the singular integral system (14) is solvable in $[L^p(\Sigma)]^3$. In view of Theorem 5.4, there exists a solution $\varphi \in [L^p(\Sigma)]^3$ if and only if $\int_\Sigma \gamma_j \wedge df_j = 0$, $(j = 1,2,3)$ for every $\gamma \in [W_1^{1,q}(\Sigma)]^3$ such that $d\gamma = T(\partial_x, n)v d\sigma$ with $v \in \mathcal{V}_0$. Thus the simple layer potential

$$z(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi(y)\, d\sigma_y$$

satisfies $A(\partial_x,\omega)z = 0$ in Ω and $dz = df$ on Σ. Therefore there exists a complex constant c such that $z = f + c$ on Σ. Moreover, for every $v \in \mathcal{V}_0$, we have

$$\int_\Sigma cT(\partial_x,n)v d\sigma_x = \int_\Sigma (z-f)T(\partial_x,n)v\, d\sigma_x = 0.$$

Hence, there exists a solution of

$$\begin{cases} w \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 \\ A(\partial_x,\omega)w = 0 & \text{in } \Omega \\ w = c & \text{on } \Sigma\,, \end{cases}$$

and it can be represented by a simple layer potential (see Theorem 5.2). Then the simple layer potential $u = z - w$ satisfies the Dirichlet problem (21), so the claim is proved. □

References

[1] Cialdea, A. On oblique derivate problem for Laplace equation and connected topics. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* **12** (1988), no. 1, 181–200.

[2] Cialdea, A. The Simple-Layer Potential Approach to the Dirichlet Problem: An Extension to Higher Dimensions of Muskhelishvili Method and Applications, in: C. Constanda, M. Dalla Riva, P.D. Lamberti, P. Musolino (Eds.), *Integral Methods in Science and Engineering, Volume 1*, Birkhäuser, (2017), 59–69 .

[3] Cialdea, A., Hsiao, G.C. Regularization for some boundary integral equations of the first kind in Mechanics. *Rend. Accad. Naz. Sci. XL, Serie 5, Mem. Mat. Parte I* (19)**1** (1995), 25–42.

[4] Cialdea, A., Leonessa, V., Malaspina, A. Integral representations for solutions of some BVPs for the Lamé system in multiply connected domains. *Boundary Value Problems 2011*, **2011:53**, 25 pages.

[5] Cialdea, A., Leonessa, V., Malaspina, A. The Dirichlet problem for second-order divergence form elliptic operators with variable coefficients: the simple layer potential ansatz. *Abstr. Appl. Anal. 2015* Art. ID 276810 (2015) 11 pages.

[6] Cialdea, A., Leonessa, V., Malaspina, A. On the simple layer potential ansatz for the n-dimensional Helmholtz equation, in: P. Boggiatto et al. (Eds.), *Advances in Microlocal and Time-Frequency Analysis, Applied and Numerical Harmonic Analysis*, (2020), 157–168.

[7] Fichera, G. Una introduzione alla teoria delle equazioni integrali singolari *Rend. Mat. Roma, XIX*, 17 (1958), 82–191.

[8] Kupradze, V.D., Gegelia, T.G., Bašeleišvili, M.O. and Burculadze, T.V. *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity* North-Holland Series in Applied Mathematics and Mechanics, 25, North-Holland Publishing Co., Amsterdam, 1979.

[9] Mikhlin, S.G., Prössdorf, S. *Singular Integral Operators*. Springer-Verlag, Berlin, 1986.