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Structures for Corner Boundary Value Problems

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Boutet de Monvel's calculus [1] of pseudo-differential boundary value problems (BVPs) may be extended from the smooth case to corner manifolds with boundary. We outline here the approach of [7] which is to some extent parallel to the corner analysis off the boundary. Here we refer to new progress from [42].

Keywords: Mellin operators, corner-degenerate boundary value problems, symbol hierarchies, Kegel spaces, weighted corner spaces

1. Introduction

Boundary value problems (BVPs) on corner manifolds suggest applying Boutet de Monvel's calculus [1] of operators on a smooth manifold with the transmission property at the boundary to the corresponding case with corner singularities. We give an outline of such an approach for regular singularities, characterized by transversal intersections of the strata close to the singular points. Another aspect are symbol hierarchies of the operators, determined by the strata of the configuration, similarly to those in BVPs. Singularities of first order concern conical points, see [34, 35] or edges, cf. [33], [19, 20, 23], [5, 7]. Smoothness up to the boundary corresponds to singularity of order zero. The present exposition refers to tools from [1], combined with Mellin operator techniques developed in [36], [10, 11], [4] for the case with an empty boundary. Since the technical details for BVPs are a voluminous program we mainly illustrate specific corner aspects for the case without boundary. Here we focus on a manageable concept of weighted corner Sobolev spaces in the sense of [42]. More details for the case with non-trivial boundary will be studied elsewhere, cf., [24].

2. General Orientation

Boutet de Monvel's approach [1] to studying boundary value problems (BVPs) on manifolds with a smooth boundary has been organized in terms of pseudo-differential operators with the transmission property at the boundary, cf., also articles and monographs on this topic [9], [28], [12], [40], [41], [25]. More background information, also on the case with violated transmission property, may be

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found in [46], [9], [29]. The edge calculus from [36] and subsequent contributions, see more references below, is not only a kind of calculus for singular manifolds and degenerate operators, the idea of specific weighted Sobolev spaces form a specific chapter, cf., also the papers [14], [43], [15], and many other contributions. have stimulated to a large extent the progress of corner analysis. Similarly to the interior calculus, i.e., the case with empty boundary, a crucial idea is the relationship between operators $\mathcal{A}$ and symbols $\sigma(\mathcal{A})$, also referred to as a quantization of the respective symbol information. In BVPs we have $\sigma(\mathcal{A}) = \big(\sigma_\psi(\mathcal{A}), (\sigma_\partial(\mathcal{A})\big)$ with ψ indicating the interior and ∂ the boundary symbol of $\mathcal{A}$. One of the merits of Boutet de Monvel has been to integrating boundary operators as well as potential operators and Green's function into the symbol framework with interior and boundary symbols, together with their algebra properties. These objects are in general 2×2 block matrices; they may also have the form of rows or columns, such as in Dirichlet or Neumann problems for Laplacians, or corresponding inverse operators (or parametrices). Compositions concern the case when the numbers of rows and columns in the middle fit together. Ellipticity in Boutet de Monvel's space $\mathcal{B}^{\mu,e}(N)$ of operator of order $\mu \in \mathbb{Z}$ and type $e \in \{0, 1, 2, \dots, \}$ on a manifold N with smooth boundary ∂N is a bijection condition both on $\sigma_\psi(\mathcal{A})$ over $N_{\text{int}} = N \setminus \partial N$ and $\sigma_\partial(\mathcal{A})$ over ∂N. This entails the existence of a parametrix $\mathcal{A}^{(-1)}$ belonging to the pair of symbols $\sigma(\mathcal{A}^{(-1)}) = \big(\sigma_\psi(\mathcal{A})^{-1}, (\sigma_\partial(\mathcal{A})^{-1}\big)$. For compact N this is equivalent to the Fredholm property of $\mathcal{A}$ between the involved (direct sums) of Sobolev spaces.

The case of BVPs with conical singularities or edges on the boundary or with violated transmission property, including asymptotics of solutions, is much more complex. Many investigations are devoted to this case, cf., in particular, [9], [29], [33–35], [13], [22], [5, 7]. This belongs to the development of singular analysis in the past decades, originally motivated by models of applications and represented by schools of researchers worldwide, who developed different analytic approaches for studying concrete problems or general structure properties, with ideas from Geometry or Topology. The singular analysis has a particularly glorious tradition in Georgia, Russia and many other countries, belonging to a network of research groups and also international conferences. The Vekua Institute of Applied Mathematics of Tbilisi State University belongs to the active centers, with the specialist and organizer Prof. George Jaiani as a leading personality since many years. It is a great honor for the authors of this exposition, see also [22], to present an article on the occasion of his 75. birthday, with the best wishes for health and further scientific productivity.

In Section 3 we specify categories of singular manifolds $\mathfrak{M}$ or $\mathfrak{N}$ consisting of subcategories

$$\mathfrak{M}_k,\ \mathfrak{N}_l, \quad \text{for} \quad k, l = 0, 1, 2, \dots, \tag{1}$$

with k or l being singularity orders, where $\mathfrak{M}$ indicates the case with empty and $\mathfrak{N}$ with nontrivial boundary.

Section 4 will illustrate classes of degenerate differential operators and also relations to singular boundary conditions.

Section 5 is devoted to BVPs both on smooth and singular manifolds with boundary. We shall see immediately why it is reasonable referring to the classes of underlying spaces $\mathfrak{M}$ or $\mathfrak{N}$ at the same time, although we have, for instance, $\mathfrak{N}_0 \subset \mathfrak{M}_1$.

In the present consideration manifolds M with singularities are assumed to be regular in the sense of some transversality conditions close to points or subsets where smoothness is violated, see more notation below, where BVPs refer to singularities on boundaries and also to conical exits to infinity. But even for standard prototypes of such configurations M the operators A in the present exposition act in weighted corner Sobolev spaces and specific degenerate symbols $\sigma(A)$ are closely related to weighted cone spaces. The motivation of the general approach consists of characterizing solutions to elliptic equations on M or N, belonging to categories of singular manifolds (1), cf., notation in Section 3, below. In particular, singular spaces $M \in \mathfrak{M}_k$ for $k = 0$ are oriented smooth manifolds, $k = 1$ indicates conic singularities or smooth edges. Larger k correspond to higher singular corners or edges. Pseudo-differential analysis for $k = 0$ from the very beginning relies on coordinate invariance under diffeomorphisms. For $k > 0$ we have to take into account many specific constructions, both on the level of singular manifolds, of corner analogues of Sobolev spaces, and also of degenerate symbol structures, including various kinds of quantizations. Numerous analytic details in the iterative description of operator structures for increasing k, l, and also interactions of objects from the closed case to that with boundary and vice versa require a careful management of notation in the context of categories (1). Therefore, for future purposes, in the present exposition we briefly recall notation and methods concerning general structures of singular analysis and outline classes of boundary value problems in the corner context. Those play a role of examples of possibilities of treating more general models of applications, cf., motivation from [13] or [18] and methods from [17] [16].

3. Singular Manifolds

Examples of singular manifolds $M \in \mathfrak{M}_k$ or $N \in \mathfrak{N}_l$ for different k, l, concern spaces with conical points, edges, or corners, cf., other examples below. Although we do not aspire to utmost generality, we need the freedom of choosing objects in $\mathfrak{M}_k$ or $\mathfrak{N}_l$, but it is advisable to have in mind situations when the respective spaces are piecewise smooth manifolds such as cubes in Euclidean spaces or their faces of different dimension. Let us start with the closed case $\mathfrak{M}_k$, $k = 0, 1, 2, \ldots,$ of singular spaces M, where $k = 0$ indicates oriented smooth manifolds, $k = 1$ spaces with conic singularities or smooth edges, and $k > 1$ the case with corners or higher edges. As noted before, we employ here the terminology of other papers, see, also [3, 4], but the intuitive idea is that the spaces $\mathfrak{M}_k$ are characterized by repeatedly forming (regular) cones or wedges, combined with global constructions of gluing together such local models to larger configurations. We often refer to notation

$$X^{\Delta} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \tag{2}$$

which is an infinite straight cone, and

$$X^{\wedge} := \mathbb{R}_+ \times X, \tag{3}$$

the corresponding stretched version with X belonging to suitable categories of topological spaces, e.g., closed oriented manifolds. Later on those will assumed to be manifolds with smooth boundary ∂X. In such a case (3) $X^{\wedge}$ has the smooth boundary $(\partial X)^\wedge$. The subspace $(\{0\} \times X) \subset (\overline{\mathbb{R}}_+ \times X)$ in (2) represents a conical singularity where we tacitly impose some regular behavior. In simple cases we mean the regularity close to the vertex of a cone

$$V^\Delta \subset \mathbb{R}^{1+N}$$

for some sufficiently large N, where V is a (say, closed) sub manifold of the unit sphere S^N in $\mathbb{R}^{1+N}$ and $\dim V = n$ for some $n \leq N$, where

$$V^\Delta := \{rx : r \in \overline{\mathbb{R}}_+, x \in V\}. \tag{4}$$

For $n = 0$ where V is only a single point we have $V^\Delta = \overline{\mathbb{R}}_+$. Otherwise, if V contains two points $a \neq b$ then the generated closed half-lines have a transversal intersection at the origin of $\mathbb{R}^{1+N}$, and this is an aspect of the regular behavior we are talking about. Also when V is a space with singularities, iteratively generated by the definitions below, for convenience we may assume that for some sufficiently large N there is chosen an embedding $V \hookrightarrow S^N$ and that the regular cone V^Δ having the link V is of the form (4).

Remark 1 : There is another role of using the closed half-axis $\overline{\mathbb{R}}_+$ or the open interior $\mathbb{R}_+$ in formulating singular manifolds. For instance, let M be a Riemannian manifold with smooth boundary ∂M. There is then a collar neighborhood $V(M)$ of ∂M, often identified with a trivial $[0,1)$-bundle over ∂M. As we shall see below it makes sense also to talk about the trivial $\overline{\mathbb{R}}_+$-bundle over ∂M which is the inner normal bundle. In this context we also interpret M as a manifold with edge $s_1(M) = \partial M$. We often say that $s_1(M)$ has a neighborhood $V(M)$ in M with the structure of the respective cone bundle with fiber $\overline{\mathbb{R}}_+$ although the neighborhood itself in M is only the respective $[0,1)$-bundle.

We hope this will not cause confusion, but the control of distributions on the half-axis up to ∞ will be important for the philosophy of edge Sobolev spaces in considerations for higher singularities, and also in the interpretation of homogeneity properties of edge- and corner-symbols.

Remark 2 : The point of view of Remark **1** is more visible when we generalize the terminology for a manifold M with edge $s_1(M) = Y$. In this case there is a neighborhood $V(M)$ of $s_1(M)$ with the structure of a locally trivial X^Δ-bundle over $s_1(M)$ for some (oriented closed) manifold X. The neighborhood $V(M)$ itself may be identified with a corresponding $X^{\flat\Delta}$-bundle over Y for

$$X^{\flat\Delta} := ([0,1) \times X)/(\{0\} \times X). \tag{5}$$

In this case we do not ignore either the identification of $V(M)$ with an X^Δ-bundle where we treat $X^\Delta \setminus X^{\flat\Delta}$ as a manifold with a conical exit to ∞, not only as a cylinder $[1,\infty) \times X$. Moreover, $V(M)$ induces an X-bundle $V_{\mathbb{O}}(M)$ over $s_1(M)$ with fibers $\{1\} \times X$, identified with X, using the fact that the half-axis as a component of the fiber X^Δ only contributes a trivial line bundle (and we may refer to a fixed trivialization). Let us invariantly attach $V_{\mathbb{O}}(M)$ to $V(M) \setminus s_1(M)$ and denote the resulting stretched space by $\mathbb{V}(M)$, equipped with the splitting of variables

$$\tilde{m} = (r, m_{\mathbb{O}}) \in \overline{\mathbb{R}}_+ \times V_{\mathbb{O}}(M) \tag{6}$$

into a vertical and a horizontal component.

By definition a space $M \in \mathfrak{M}_k$ for $k \geq 1$ contains a subspace $s_k(M) \in \mathfrak{M}_0$ such that $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$ and a neighborhood of $s_k(M)$ in M can locally be identified with a (locally trivial) cone bundle $V(M)$ over $s_k(M)$ with fibers X^Δ_{k-1}, with compact links $X_{k-1} \in \mathfrak{M}_{k-1}$. Transition maps between fibers are induced by the notion of isomorphisms between elements $X_{k-1} \in \mathfrak{M}_{k-1}$ which are a natural consequence of those on the step for $k-1$ and required homogeneity of order 0 in the axial variable $t \in \mathbb{R}_+$ contained in X^Δ_{k-1}. A successive procedure then gives rise to a finite sequence

$$s(M) = \{(s_0(M), s_1(M), \ldots, s_k(M)\}. \tag{7}$$

For convenience we assume that for $M \in \mathfrak{M}_k$ the components of (7) satisfy the conditions

$$0 \leq \dim s_k(M) < \dim s_j(M) < \dim s_{j-1}(M) < \dim s_0(M), \tag{8}$$

for all $0 \leq j \leq k$, and we set $\dim M := \dim s_0(M)$. Another reasonable property of concrete examples is that

$$\dim s_j(M) + 1 + \dim (X_{j-1}) = \dim M \quad \text{for all} \quad 0 < j \leq k. \tag{9}$$

Let us form the stretched space

$$\mathbb{M} := \big((M \setminus s_k(M)) \cup V_{\mathbb{O}}(M)\big)/\sim, \tag{10}$$

associated with M, obtained by invariantly attaching the above-mentioned $V_{\mathbb{O}}(M)$ to $M \setminus s_k(M)$; this is just the meaning of notation in (10). There is then a double

$$2\,\mathbb{M} \in \mathfrak{M}_{k-1} \tag{11}$$

defined by gluing together two copies $\mathbb{M}_+ := \mathbb{M}$ and $\mathbb{M}_-$ of $\mathbb{M}$ along the common subspace $V_{\mathbb{O}}(M)$.

The cone bundle $V(M)$ over $s_k(M)$ with fibers X^Δ_{k-1} contains $V_{\mathbb{O}}(M)$ like a "horizontal part" which is complementary to a "vertical part", a trivial $\overline{\mathbb{R}}_+$-bundle over $s_k(M)$. Variables $\tilde{m}$ on the stretched manifold $\mathbb{M}$ locally close to $V_{\mathbb{O}}(M) = \mathbb{M}_- \cap \mathbb{M}_+$ admit a splitting of the form

$$\tilde{m} = (t, m_{\mathbb{O}}) \in \overline{\mathbb{R}}_+ \times V_{\mathbb{O}}(M). \tag{12}$$

This reminds of the splitting $\tilde{m} = (t, m)$ of local variables $\tilde{m}$ on a smooth manifold $\mathbb{M}$ with boundary close to $\partial\mathbb{M}$ into the inner normal t to the boundary (with respect to a Riemannian metric) and the component m tangent to $\partial\mathbb{M}$.

As a simple example we consider a closed manifold X and form a wedge $M := X^\Delta \times \mathbb{R}^q$ which belongs to $\mathfrak{M}_1$ with edge $s_1(M) = \mathbb{R}^q$. Then we have $V_{\mathbb{O}}(M) = X \times \mathbb{R}^q$ and $\mathbb{M} = \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$.

It is often convenient to employ global cut-off functions on $\mathbb{M}$ for $M \in \mathfrak{M}_k$, often denoted by ω_{glob_k} or simply ω. Such a function ω on $\overline{\mathbb{R}}_+$ is a smooth real-valued function such that $\omega(t) \equiv 1$ for $0 < t < \varepsilon_0$ and $\omega(t) \equiv 0$ for $t > \varepsilon_1$ for some $0 < \varepsilon_0 < \varepsilon_1$. Globally on $\mathbb{M}$ we write ω_{glob_k} when the cut off function is interpreted as a function on $\mathbb{M}$ which locally close to $V_{\mathbb{O}}(M)$ in a splitting (12) only depends on t and behaves like the former ω for sufficiently small $0 < \varepsilon_0 < \varepsilon_1$.

Prototypes of elements in $\mathfrak{M}_k$ are polyhedral configurations, embedded in $\mathbb{R}^N$, for instance, the unit cube Q in $\mathbb{R}^3$ which belongs to $\mathfrak{M}_3$. In this case the 8 isolated corner points form the subset $s_3(Q)$, and it is easy to identify the other strata $s_j(Q)$ for $j = 0, \dots, 2$, where $s_0(Q)$ is just the open interior. Moreover, we have $\partial Q \in \mathfrak{M}_2$. Such examples are of particular interest in models of mechanics or other applications, formulated in terms of (e.g., elliptic or parabolic) partial differential equations (PDEs), given in $s_0(M)$, with symbols of a specific corner-degenerate behavior in stretched variables. The singular geometry also gives rise to a corresponding degenerate behavior of boundary conditions. It is then desirable to express parametrices and regularity of solutions within a pseudo-differential approach.

The general calculus of (pseudo-differential) operators A - say, classical ones - on a space $M \in \mathfrak{M}_k$ refers to a symbol structure, connected with (7), in this case a principal symbol hierarchy

$$\sigma(A) = \{(\sigma_0(A), \sigma_1(A), \dots, \sigma_k(A))\}, \tag{13}$$

where $\sigma_0(A)$ is the standard principal symbol of A on $s_0(M)$, though degenerate in local stretched variables and covariables close to the singularities, while the other components, say, for $\dim s_k(M) > 0$, are families of operators, also with a specific dependence on variables and covariables on $s_j(M)$, $1 \le j \le k$. These operators act between weighted Kegel spaces over the open stretched cones $X^\wedge_{j-1} := \mathbb{R}_+ \times X_{j-1}$ to be studied below, with $X^\wedge_{j-1}$ being interpreted as a space with (in general, singular) conical exit to ∞, analogously to Boutet de Monvel's calculus [1] in the smooth case, we need the categories (1) for the case without or with boundary at the same time.

Definitions for $\mathfrak{N}$ are similar to those for $\mathfrak{M}$ but transition maps are required to be smooth up to boundaries. The closed half space

$$\overline{\mathbb{R}}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n \in \overline{\mathbb{R}}_+\} \tag{14}$$

belongs both to $\mathfrak{M}_1$ and $\mathfrak{N}_0$, where $\mathfrak{N}_0$ is defined to be category of oriented smooth manifolds with boundary. We have the proper inclusion $\mathfrak{N}_0 \subset \mathfrak{M}_1$ and a similar relation below for higher singularities. Isomorphisms in $\mathfrak{N}_0$ are orientation preserving diffeomorphisms up to the boundary.

More generally, $\mathfrak{N}_k$ for $k > 0$ is furnished by topological spaces N containing an $s_k(N) \in \mathfrak{M}_0$ such that $N \setminus s_k(N) \in \mathfrak{N}_{k-1}$, and N contains a neighborhood $V(N)$ of $s_k(N)$ with the structure of a locally trivial E^Δ_{k-1}- bundle over $s_k(N)$ for some compact $E_{k-1} \in \mathfrak{N}_{k-1}$. Then $V(N) \setminus s_k(N)$ just has the fibers $E^\wedge_{k-1} = \mathbb{R}_+ \times E_{k-1}$. The transition maps for the fibers E^Δ_{k-1} of $V(N)$ are controlled up to the corner points as follows. Denote by $V_{\mathbb{O}}(N)$ the E_{k-1}-bundle over $s_k(N)$ obtained by restricting the above-mentioned fibers $\mathbb{R}_+ \times E_{k-1}$ of $V(N) \setminus s_k(N)$ with variables

$(t, n_{\mathbb{O}})$ to $t = 1$. Let us attach in an invariant way $V_{\mathbb{O}}(N)$ to $V(N) \setminus s_k(N)$. This gives us a bundle $\mathbb{V}_+(N)$ over $s_k(N)$ with fibers $\overline{\mathbb{R}}_+ \times E_{k-1}$. An analogous process gives us the negative counterpart $\mathbb{V}_-(N)$ which is an $\overline{\mathbb{R}}_- \times E_{k-1}$-bundle over $s_k(N)$, where

$$\mathbb{V}_-(N) \cap \mathbb{V}_+(N) = V_{\mathbb{O}}(N) \tag{15}$$

and

$$\mathbb{V}(N) := \big(\mathbb{V}_-(N) \cup \mathbb{V}_+(N)\big)/\sim \tag{16}$$

is an $\mathbb{R} \times N_{k-1}$-bundle over $s_k(N)$. Here $/\sim$ indicates a natural gluing construction between the $\mathbb{V}_-(N)$ and $\mathbb{V}_+(N)$ along $V_{\mathbb{O}}(N)$ in (15). By virtue of $E_{k-1} \in \mathfrak{N}_{k-1}$ it follows that $\mathbb{R} \times E_{k-1} \in \mathfrak{N}_{k-1}$, and hence we may talk about transition maps of fibers

$$\mathbb{R}_+ \times E_{k-1} \to \mathbb{R}_+ \times E_{k-1} \quad \text{in the category} \quad \mathfrak{N}_{k-1} \tag{17}$$

since by induction assumption those features are known on the more regular level $k-1$, where in the case of straight cylinders $\mathbb{R}_+ \times E_{k-1}$ points are denoted by $(t, n_{\mathbb{O}})$. In our case, representing the compact link E_{k-1} as an embedded space $E_{k-1} \hookrightarrow S^{\Xi}$ similarly to (4), with S^{Ξ} being the unit sphere in $\mathbb{R}^{1+\Xi}$ for sufficiently large Ξ, we have an isomorphism of the form

$$T^{\wedge} : \mathbb{R}_+ \times E_{k-1} \to \mathbb{R}_+ \times E_{k-1}, \quad T^{\wedge} : \big((t, n_{\mathbb{O}}\big) \mapsto \big(t, tT_h(n_{\mathbb{O}})\big) \tag{18}$$

for any isomorphism T_h in $\mathfrak{N}_{k-1}$. The multiplication of T_h by $t \in \mathbb{R}_+$ is well-defined as an operation on points in $\mathbb{R}^{1+\Xi} \setminus \{0\}$. Relation (18) induces a bijective map

$$T^{\Delta} : E^{\Delta}_{k-1} \to E^{\Delta}_{k-1} \tag{19}$$

between the respective corner configurations, including their fixed vertex which is the origin on $\mathbb{R}^{1+\Xi}$. We also can invariantly attach $V_{\mathbb{O}}(N)$ to $N \setminus s_k(N)$ and obtain the stretched space $\mathbb{N}$, also called $\mathbb{N}_+$, and a corresponding negative counterpart $\mathbb{N}_-$ and we get the double

$$2\,\mathbb{N} = \big(\mathbb{N}_- \cup \mathbb{N}_+\big)/\sim \in \mathfrak{N}_{k-1}, \quad \mathbb{N}_- \cap \mathbb{N}_+ = V_{\mathbb{O}}(N). \tag{20}$$

For $k > 1$ successive procedure allows us to form

$$s_{k-1}(N) = s_{k-1}(N \setminus s_k(N)) \in \mathfrak{M}_0 \tag{21}$$

when $k-1 > 1$, otherwise it follows that $s_{k-1}(N) \in \mathfrak{N}_0$ and we get altogether a sequence

$$\boldsymbol{s}(N) := (\boldsymbol{s}_0(N), s_1(N), \ldots, s_k(N)), \tag{22}$$

where $\boldsymbol{s}_0(N) \in \mathfrak{N}_0$, $s_j(N) \in \mathfrak{M}_0$ for $j > 0$. The successive construction of $\boldsymbol{s}(N)$ is performed analogously to $s(M)$ in (7) for the closed case. This procedure is motivated by the iterative ideas for building up sequences of algebras of pseudo-differential operators, here of BVPs with the transmission property along the smooth parts of the boundary, beginning with the "most singular subset" $s_k(N) \subset N$ when $k > 0$. Nevertheless, in order to understand the behavior of transition maps of the above-mentioned E^{Δ}_{k-1}-bundle close to the respective corner boundaries it is also instructive to construct spaces $N \in \mathfrak{N}_k$ by a successive process the other way around, beginning with an $E_0 \in \mathfrak{N}_0$, then pass to a cone E^{Δ}_0 and wedges $E^{\Delta}_0 \times \mathbb{R}^q$ with boundary, etc..In order to reach $N \in \mathfrak{N}_k$ for $k = 1$ it suffices to work with such local cones or wedges for some arbitrary $q > 0$ and then to construct global spaces $N_1 \in \mathfrak{N}_1$ such that $Y = s_1(N_1)$ for a chosen manifold $Y \in \mathfrak{M}_0$ by gluing together local wedges with edge Y using a partition of unity on Y which yields N_1. where $\boldsymbol{s}_0(N) \in \mathfrak{N}_0$, $s_j(N) \in \mathfrak{M}_0$ for $j > 0$.

Let us now turn to the principal symbol structure of BVPs on singular manifolds $N \in \mathfrak{N}_k$, motivated by Boutet de Monvel's calculus [1] in the case $k = 0$. Recall that other aspects of boundary value problems, partly without the transmission property, have been investigated by Vishik and Eskin [46, 47] and [9] or [28], [12], [2], [25]. In BVPs A on compact $N \in \mathfrak{N}_0$ we first consider $\boldsymbol{s}(N) = \{(\boldsymbol{s}_0(N)\}$.

At the same time we have $N \in \mathfrak{M}_1$ and in "$\mathfrak{M}_1$"-notation $s(N) = (\operatorname{int} N, \partial N)$. An operator A in Boutet de Monvel's calculus which is an upper left corner of a corresponding 2×2 block matrix has a pair of principal symbols

$$\sigma(A) = \{(\sigma_0(A), \sigma_1(A))\} =: \{(\sigma_\psi(A), \sigma_\partial(A))\} \tag{23}$$

with $\sigma_\psi(A)$ being the homogeneous principal interior symbol and $\sigma_\partial(A)$ the (twisted homogeneous) principal boundary symbol of the operator A. This pair can be identified with $\boldsymbol{\sigma}(A)$. More generally, if an A of order $\mu \in \mathbb{Z}$ and type $\boldsymbol{a}$ which is a BVP on a corner manifold $N \in \mathfrak{N}_k$ with boundary, for $k > 1$ we have to expect a similar picture as in the closed case, and we have principal symbols $\sigma_j(A)$ associated with $s_j(N)$ consisting of families of operators

$$\sigma_j(A)(\cdot) : \mathcal{K}^{s,\beta,\gamma}(E^{\wedge}_{j-1}) \to \mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(E^{\wedge}_{j-1}), \tag{24}$$

continuous for $s > \boldsymbol{a} - 1/2$. Here $\mathcal{K}^{s,\beta,\gamma}(E^{\wedge}_{j-1})$ are Kegel spaces over $E^{\wedge}_{j-1}$ where $N \in \mathfrak{N}_k$ is locally close to $s_j(N)$ is modeled on $\mathbb{R}^{q_j} \times E^{\wedge}_{j-1}$ for a compact link $E_{j-1} \in \mathfrak{N}_{j-1}$. Moreover, $\beta = \{\beta_1, \ldots \beta_{j-1}\} \in \mathbb{R}^{j-1}$, $\gamma \in \mathbb{R}$ are weights. More information on such Kegel spaces may be found in in [24].

Interpreting A as an operator on $N \in \mathfrak{N}_0$, we identify $\boldsymbol{\sigma}_0(A)$ with (23) For a general BVP A on $N \in \mathfrak{N}_k$ we write

$$\boldsymbol{\sigma}(A) = \{(\boldsymbol{\sigma}_0(A), \sigma_1(A), \ldots, \sigma_k(A)\}, \tag{25}$$

where the component (24) are associated with $s_j(N)$, $j = 1, \ldots, k$. In the following sections we will study the structure of corner BVPs for the cases $k = 0, 1$ while $k = 0$ corresponds to the known case of BVPs in Boutet de Monvel's calculus on a smooth manifold with boundary $N \in \mathfrak{N}_0$.

4. Degenerate Operators

As noted before the corner calculus of BVPs on $N \in \mathfrak{N}_k$ induces the calculus of lower right corners which are pseudo-differential operators on a corner manifold $M := \partial N \in \mathfrak{M}_k$ without boundary. Conversely, the ideas of the latter calculus just extend to the case of BVPs on $N \in \mathfrak{N}_k$, consisting of upper left corners. Other ingredients are entries of trace and potential type, similarly to Boutet de Monvel's algebra in the smooth case. In the present exposition we focus on some crucial elements of such a voluminous program. Because of the assumed situation of regular corner geometry, encoded by the above-mentioned iterative definition of spaces in the categories (1), we formulate an iterative approach of pseudo-differential operators with degenerate symbols in stretched variables. The constructions start with compact $X \in \mathfrak{M}_0$ and we recall constructions for spaces of parameter-dependent edge operators on a space $B \in \mathfrak{M}_1$ with edge $Y := s_1(B)$ of dimension $q > 0$. The space of those operators will be denoted by

$$L^\mu(B, \boldsymbol{b}; \mathbb{R}^d_\lambda). \tag{26}$$

Here $\boldsymbol{b} := (\beta, \beta - \mu)$ has the meaning of weight data which correspond to mapping properties in weighted Sobolev spaces. By the condition of vanishing principal symbol pairs, we can also pass to order $\mu - 1$ and then, successively to $\mu - m$ for a natural number m. For simplicity we consider here the case $m = 0$, also later on in other operator spaces of similar structure. Recall that $B \in \mathfrak{M}_1$ is locally close to its edge Y modeled on $X^\Delta \times \mathbb{R}^q$. Later on, when we manage operators on a space $M \in \mathfrak{M}_2$ with a corresponding "corner edge" $Z := s_2(M)$, the parameter λ plays the role of $(\tilde{\tau}, \tilde{\zeta}) \in \mathbb{R}^{1+l}$, i.e., $d = 1 + l$, with $\tau \in \mathbb{R}$ being the covariable belonging to an extra corner axis variable $t \in \mathbb{R}_+$ and $\zeta \in \mathbb{R}^l$ the covariable belonging to Z of dimension l. At the same time λ will be replaced by $(t\tau, t\zeta)$, which gives rise to corner-degenerate families of operators. In addition locally near $t = 0$ operators will be composed with the factor $t^{-\mu}$. Concerning the structure of operator families in (26) for closed X we refer to the paper [6, formulas (3.1), (3.15), (3.16)]. However, in order to keep the ideas self-contained we outline some essential aspects for the present exposition. In particular,we employ the formalism connected with the new Mellin-edge quantization, elaborated in [11].

We formulate several families of operators of the form of parameter-dependent operator-valued symbols close to the edge $Y = s_1(B)$. First let $\mathcal{A}(\mathbb{C}, E)$ denote the space of all holomorphic functions in $w \in \mathbb{C}$ with values in a Fréchet space E, here $E = L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{q+d}_{\tilde{\eta}, \tilde{\lambda}})$. Let

$$M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^{q+d}_{\tilde{\eta}, \tilde{\lambda}}) \tag{27}$$

denote the space of all

$$\tilde{h}(w, \tilde{\eta}, \tilde{\lambda}) \in \mathcal{A}(\mathbb{C}_w, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{q+d}_{\tilde{\eta}, \tilde{\lambda}})) \text{ such that } h(\alpha + i\rho, \tilde{\eta}, \tilde{\lambda}) \in L^\mu_{\mathrm{cl}}(X; \Gamma_\alpha \times \mathbb{R}^{q+d}_{\tilde{\eta}, \tilde{\lambda}}) \tag{28}$$

for every α uniformly in compact α-intervals, where

$$\Gamma_\alpha := \{w \in \mathbb{C} : \operatorname{Re} w = \alpha\} \tag{29}$$

is identified with an extra one-dimensional component of the parameter space $\Gamma_\alpha \times \mathbb{R}^{q+d}_{\tilde{\eta},\tilde{\lambda}}$ with parameters $(\rho, \tilde{\eta}, \tilde{\lambda})$, $\rho = \operatorname{Im} w$. The space (27) is Fréchet, and we can talk about C^∞-functions of variables $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$ with values in (27). Later on, we employ notation Γ_α in similar meaning in another complex v-plane. Moreover, we set $h(r, y, w, \eta, \lambda) := \tilde{h}(r, y, w, r\eta, r\lambda)$ for some

$$\tilde{h}(r, y, w, \tilde{\eta}, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^{q+d}_{\tilde{\eta},\tilde{\lambda}})), \tag{30}$$

with w being the complex Mellin covariable belonging to r. For any weight $\Xi \in \mathbb{R}$ we set

$$\operatorname{Op}^\Xi_M(h)(y, \eta, \lambda)u(r, \cdot) := \int_{\Gamma_{\frac{\dim X+1}{2}-\Xi}} \int \left(\frac{r}{r'}\right)^{-w} h(r, y, w, \eta, \lambda)u(r', \cdot)\, dr' đw, \tag{31}$$

$đw := (2\pi i)^{-1} dw$, in the present case for

$$\Xi := \beta - \frac{n}{2},\ n := \dim X. \tag{32}$$

Moreover, let

$$p_{\mathrm{int}}(r, y, \rho, \eta, \lambda) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^q, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q+d}_{\rho,\eta,\lambda})),$$

satisfy the condition

$$p_{\mathrm{int}} = \sigma p_{\mathrm{int}} \tilde{\sigma} \tag{33}$$

for some cut-off functions $\sigma(r), \tilde{\sigma}(r)$. Symbols of the edge pseudo-differential calculus locally close to the edge Y have the form

$$\begin{aligned} a_{\mathrm{sing}}(y, \eta, \lambda) &= \sigma_1(r) r^{-\mu} \operatorname{Op}^{\beta-\frac{n}{2}}_{M_r}(h)(y, \eta, \lambda)\sigma_0(r) \\ &\quad + (1 - \sigma_1(r)) r^{-\mu} \operatorname{Op}_r(p_{\mathrm{int}})(y, \eta, \lambda)(1 - \sigma_2(r)) \\ &\quad + g_{\mathrm{M+G}}(y, \eta, \lambda), \end{aligned} \tag{34}$$

$X \in \mathfrak{M}_0$, where $\sigma_2 \prec \sigma_0 \prec \sigma_1$ are cut-off functions in r. In addition $g(y, \eta, \lambda)$ are Mellin plus Green symbols in the covariables (η, λ) and the edge-variable $y \in \mathbb{R}^q$, interpreted as local coordinates on Y of dimension q. In this explanation we suppress notation for the smoothing Mellin symbols; they have a similar structure as those in [37, Definition 3.2.6], [11]. Off Y the operators in (26) are simply standard parameter-dependent pseudo-differential operators p_{int} on $B \setminus Y$ added to those with symbols in (34) and localized by using a partition of unity on B subordinate to an open covering of the stretched manifold $\mathbb{B}$ which is of analogous meaning as $\mathbb{M}$ in relation (10) containing neighborhoods intersecting $\partial\mathbb{B}$ and interior neighborhoods disjoint to $\partial\mathbb{B}$.

In the following we systematically employ weighted Sobolev spaces $\mathcal{H}^{s,\beta}(X^\wedge)$ defined on $X^\wedge = \mathbb{R}_+ \times X$ of dimension n, first for closed $X \in \mathfrak{M}_0$ of dimension

n. Those are immediate analogues of "cylindical" Sobolev spaces $H^s(\mathbb{R} \times X)$ of smoothness s on $\mathbb{R} \times X$ locally on X related to $\mathcal{H}^{s,\beta}(X^\wedge)$ by isomorphisms

$$S_{\beta-n/2} : \mathcal{H}^{s,\beta}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^s(\mathbb{R}^{1+n}), \quad (S_\alpha u)(\boldsymbol{r}, \cdot) := e^{-(1/2-\alpha)\boldsymbol{r}} u(e^{-\boldsymbol{r}}, \cdot), \boldsymbol{r} \in \mathbb{R} \tag{35}$$

for $\alpha := \beta - n/2$. We have $r^\theta \mathcal{H}^{s,\beta}(X^\wedge) = \mathcal{H}^{s,\beta+\theta}(X^\wedge)$ for any real θ. Moreover, we also employ modified spaces defined by

$$\mathcal{K}^{s,\beta}(X^\wedge) = \{u = \omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\beta}(X^\wedge), u_\infty \in H^s_{\mathrm{cone}}(X^\wedge)\} \tag{36}$$

where $H^s_{\mathrm{cone}}(X^\wedge)$ locally for large $|\boldsymbol{x}|$, with $\boldsymbol{x} \in \mathbb{R}^{1+n}$ belonging to an infinite subcone determined by the condition $\{\boldsymbol{x}/|\boldsymbol{x}| \in V\}$ for a coordinate neighborhood V of the unit sphere S^{1+n}, is modeled on $(1-\omega)H^s(\mathbb{R}^{1+n})|_{V^\wedge}$. Here ω is a cut-off function, smooth and of bounded support, with $\omega(r) \equiv 1$ close to $r = 0$. The spaces (36), also called Kegel spaces, are independent of the choice of ω; for any fixed ω they are Hilbert spaces with a scalar product determined by a non-direct sum of spaces close to 0 and for large r, cf., [11, Definition 2.1]. The spaces (36) will also be called Kegel spaces; those are equipped with the group action

$$(\kappa_\delta u)(r, \cdot) := \delta^{\frac{n+1}{2}} u(\delta r, \cdot), \quad \delta \in \mathbb{R}_+. \tag{37}$$

Definition 4.1: We define spaces of operator-valued symbols $R^\mu_{\mathrm{edge}_G}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta})_\varepsilon)$

$$g(y, \eta, \lambda) \in \bigcap_{s,s',e,e' \in \mathbb{R}} S^\mu_{\mathrm{cl}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; \mathcal{K}^{s,\beta;e}(X^\wedge), \mathcal{K}^{s',\tilde{\beta}+\varepsilon;e'}(X^\wedge)), \tag{38}$$

$$g^*(y, \eta, \lambda) \in \bigcap_{s,s',e,e' \in \mathbb{R}} S^\mu_{\mathrm{cl}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; \mathcal{K}^{s,-\tilde{\beta};e}(X^\wedge), \mathcal{K}^{s',-\beta+\varepsilon;e'}(X^\wedge)) \tag{39}$$

for some $\varepsilon > 0$. Moreover, let $R^\mu_{\mathrm{edge}_G}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta})_\infty)$ be the space of those $g(y, \eta, \lambda)$ such that (83) and (84) hold for all $\varepsilon > 0$.

Furthermore, $R^\mu_{\mathrm{edge}_{\mathrm{M+G}}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta})_\varepsilon)$ is defined to be the set of all operator families of the form

$$g_{\mathrm{M+G}}(y, \eta, \lambda) = g_{\mathrm{M}}(y, \eta, \lambda) + g_{\mathrm{G}}(y, \eta, \lambda) \tag{40}$$

for

$$g_{\mathrm{M}}(y, \eta, \lambda) = \omega(r[\eta, \lambda]) r^{-\mu} \mathrm{Op}^{\beta-n/2}_M(f)(y) \omega'(r[\eta, \lambda]) \tag{41}$$

for some cut-off functions $\omega(r), \omega'(r)$ and $f(y, w) \in C^\infty(\mathbb{R}^q, M^{-\infty}_{\mathrm{As}}(X))$ and

$$g_{\mathrm{G}}(y, \eta, \lambda) \in R^\mu_{\mathrm{edge}_{\mathrm{G}}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta})_\varepsilon). \tag{42}$$

We use the fact that

$$a_{\text{sing}}(y,\eta,\lambda) \in S^{\mu}(\mathbb{R}^q \times \mathbb{R}^{q+d}; \mathcal{K}^{s,\beta}(X^{\wedge}), \mathcal{K}^{s-\mu,\beta-\mu}(X^{\wedge})), \tag{43}$$

for the symbols (34), between weighted Kegel spaces $\mathcal{K}^{s,\beta}(X^{\wedge})$ and $\mathcal{K}^{s-\mu,\beta-\mu}(X^{\wedge})$, respectively, both equipped with the group action (37). Then the operators

$$\operatorname{Op}_y(a_{\text{sing}})(\lambda) : \mathcal{W}^s_{\text{comp}}(\mathbb{R}^q, \mathcal{K}^{s,\beta}(X^{\wedge})) \to \mathcal{W}^{s-\mu}_{\text{loc}}(\mathbb{R}^q, \mathcal{K}^{s-\mu,\beta-\mu}(X^{\wedge})). \tag{44}$$

are continuous for all s.

The spaces (26) of the edge operator calculus consist of (families of) operators

$$A(\lambda) := \omega_{\text{glob}} A_{\text{sing}}(\lambda) \omega'_{\text{glob}} + (1-\omega_{\text{glob}}) A_{\text{int}}(\lambda)(1-\omega''_{\text{glob}}) + C(\lambda) \tag{45}$$

for global cut-off functions $\omega''_{\text{glob}} \prec \omega_{\text{glob}} \prec \omega'_{\text{glob}}$ on B that are $\equiv 1$ in a small neighborhood of Y and vanish off another neighborhood of Y. In (45) we assume $A_{\text{int}}(\lambda) \in L^{\mu}_{\text{cl}}(B \setminus Y; \mathbb{R}^d_{\lambda})$. The definition implies that

$$A(\lambda) : H^{s,\beta}(B) \to H^{s-\mu,\beta-\mu}(B) \tag{46}$$

is continuous for every s, cf., remarks on the involved spaces below. In this connection we employ that $A_{\text{sing}}(\lambda)$ is locally close to the edge Y a pseudo-differential operator (90)

The edge calculus is already a complicated structure. and also the weighted Sobolev spaces $H^{s,\beta}(B)$ deserve separate consideration. By definition those spaces locally close to Y coincide with

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\beta}(X^{\wedge})) \tag{47}$$

referring to (37), and we have $H^{s,\beta}(B)|_{B\setminus Y} \subset H^s_{\text{loc}}(B \setminus Y)$. Recall that the latter property is just a consequence of the special choice of the group action (37) on the involved Kegel spaces, in particular, of the exponent $(n+1)/2$ of δ on the right-hand side of (37) where $n = \dim X$, cf. also corresponding information in [37, Proposition 3.1.21].

In addition let $L^{-\infty}(B, \boldsymbol{b})$ be the space of all operators C which induce continuous operators

$$C : H^{s,\beta}(B) \to H^{\infty,\beta-\mu+\varepsilon}(B),\ C^* : H^{s,-\beta+\mu}(B) \to H^{\infty,-\beta+\varepsilon}(B) \tag{48}$$

for all $s \in \mathbb{R}$ and some $\varepsilon > 0$ and set

$$L^{-\infty}(B, \boldsymbol{b}; \mathbb{R}^d_{\lambda}) := \mathcal{S}(\mathbb{R}^d_{\lambda}, L^{-\infty}(B, \boldsymbol{b})) \tag{49}$$

which are specific smoothing elements of the edge pseudo-differential calculus $L^{\mu}(B, \boldsymbol{b}; \mathbb{R}^d_{\lambda})$. In (26) we assume $C(\lambda) \in .L^{-\infty}(B, \boldsymbol{b}; \mathbb{R}^d_{\lambda})$

The contributions $A_{\text{int}}(\lambda)$ in (45) are parameter-dependent classical pseudo-differential operators on $B \setminus Y$ and $C(\lambda)$ are smoothing operators of the edge calculus, characterized by their mapping properties, including those of formal adjoints.

Remark 1 : In connection with (26) it makes sense also to consider the chain of subspaces

$$L^{-\infty}(B, \boldsymbol{b}; \mathbb{R}^d_\lambda) \subset L^\mu_{\mathrm{G}}(B, \boldsymbol{b}; \mathbb{R}^d_\lambda) \subset L^\mu_{\mathrm{M+G}}(B, \boldsymbol{b}; \mathbb{R}^d_\lambda) \subset L^\mu(B, \boldsymbol{b}; \mathbb{R}^d_\lambda), \tag{50}$$

where $L^\mu_{\mathrm{M+G}}$ and L^μ_{G} are locally close to Y determined by symbols $g_{\mathrm{M+G}}(y, \eta, \lambda)$ and $g_{\mathrm{G}}(y, \eta, \lambda)$, respectively, up to globally smoothing remainders in $L^{-\infty}$, cf. notation around (40), (41), (83).

The space $B \in \mathfrak{M}_1$ has the sequence of strata

$$s(B) = (s_0(B), s_1(B)), \quad s_0(B) = B \setminus Y, \, s_1(B) = Y. \tag{51}$$

Elements A of (26) then have a tuple of parameter-dependent principal symbols

$$\begin{aligned} \sigma(A) = (\sigma_0(A), \sigma_1(A)), \quad & \sigma_0(A)(\boldsymbol{x}, \boldsymbol{\xi}, \lambda) := \sigma_\psi(A|_{B \setminus Y})(\boldsymbol{x}, \boldsymbol{\xi}, \lambda), \\ & \sigma_1(A)(y, \eta, \lambda) := \mathrm{Op}_M^{\beta - n/2}(h_0)(y, \eta, \lambda) + \sigma_1(M+G)(y, \eta, \lambda)), \end{aligned} \tag{52}$$

where $\sigma_1(M+G) := g_{(\mu)}(y, \eta, \lambda), (\eta, \lambda) \neq 0$, $G = \mathrm{Op}_y(g)$, and $\sigma_0(A)(\boldsymbol{x}, \boldsymbol{\xi}, \lambda)$ is homogeneous of order μ for $(\boldsymbol{\xi}, \lambda) \neq 0$ with $\boldsymbol{\xi}$ being the covariable of the cotangent bundle of the smooth manifold $B \setminus Y$, moreover,

$$h_0(r, y, w, \eta, \lambda) := \tilde{h}(0, y, w, r\eta, r\lambda), \tag{53}$$

$$\sigma_1(A)(y, \eta, \lambda) : \mathcal{K}^{s,\beta}(X^\wedge) \to \mathcal{K}^{s-\mu, \beta-\mu}(X^\wedge), \tag{54}$$

$(\eta, \lambda) \neq 0$. The operator family $\sigma_1(A)(y, \eta, \lambda)$ is twisted homogeneous in $(\eta, \lambda) \neq 0$ of order μ, cf., generalities on spaces of operator-valued symbols in [37, 38].

Remark 2 : Ellipticity of A is a bijectivity condition on both symbol components when we have in mind operators of "upper left corner" type. Otherwise a more complete concept of edge operators refers to 2×2 block matrices with extra conditions expressed by trace and potential contributions and an element in the lower right corner which is a parameter-dependent pseudo-differential operator of order μ on Y. Such a structure is rather similar to Boutet de Monvel's calculus where the half-axis $\mathbb{R}_+$ plays the role of $X^\wedge$, and spaces $H^s(\mathbb{R}_+)$ are considered rather than Kegel spaces on $X^\wedge$. The edge calculus for $\dim X = 0$ corresponds to a calculus of BVPs under violated transmission property, see also Eskin's book [9]. Originally such phenomena have been initiated by [36] where the present kind of edge theory has been created, also using [28–30], and then developed in [33–35] and [10, 11].

An operator family $A(\lambda) \in L^\mu(B, \boldsymbol{b}; \mathbb{R}^d_\lambda)$ is called parameter-dependent elliptic if $\sigma_0(A)(\boldsymbol{x}, \boldsymbol{\xi}, \lambda) \neq 0$ for $(\boldsymbol{\xi}, \lambda) \neq 0$ and (54) is bijective for $(\eta, \lambda) \neq 0$. We then have

Theorem 4.2 : *Let $B \in \mathfrak{M}_1$ and $\dim Y > 0$. Let $A(\lambda) \in L^\mu(B, \boldsymbol{b}; \mathbb{R}^d_\lambda)$ be parameter-dependent elliptic. Then there is a parameter-dependent parametrix*

$$P(\lambda) \in L^{-\mu}(B, \boldsymbol{b}^{-1}; \mathbb{R}^d_\lambda) \tag{55}$$

for weight data $\boldsymbol{b}^{-1} := (\beta - \mu, \beta)$, *such that*

$$P(\lambda)A(\lambda) - 1 = C_{\mathrm{L}}(\lambda), \quad A(\lambda)P(\lambda) - 1 = C_{\mathrm{R}}(\lambda) \tag{56}$$

for remainders $C_{\mathrm{L}}(\lambda) \in L^{-\infty}(B, \boldsymbol{b}_{\mathrm{L}}; \mathbb{R}^d)$, $C_{\mathrm{R}}(\lambda) \in L^{-\infty}(B, \boldsymbol{b}_{\mathrm{R}}; \mathbb{R}^d)$ *where* $\boldsymbol{b}_{\mathrm{L}} = (\beta, \beta)$, $\boldsymbol{b}_{\mathrm{R}} = (\beta - \mu, \beta - \mu)$. *In addition for compact* B *the operators* (46) *are Fredholm and for* $d > 0$ *there is a constant* $c > 0$ *such that* (46) *is a family of isomorphisms for* $|\lambda| > c$.

The proof is a consequence of the edge pseudo-differential calculus, developed in [36], [10, 11].

Remark 3 : The family of operators (54) on the infinite stretched cone $X^\wedge$ has a conormal symbol

$$\sigma_{\mathrm{cn}}(a_{\mathrm{sing}})(y, w) := \tilde{h}(0, y, w, 0, 0) + f(y, w) : H^s(X) \to H^{s-\mu}(X), \quad s \in \mathbb{R}. \tag{57}$$

In the present case (57) is a family of isomorphisms. However, this property only entails the Fredholm property of (54) for every fixed y and $(\eta, \lambda) \neq 0$. In such a case we have to add extra edge conditions along Y. This requires more formalism in the sense of elliptic edge conditions, analogies of elliptic boundary conditions in the case $\dim s_1(B) = 1$. We do not deepen this aspect here. Concerning K-theoretic structures in connection with families of Fredholm operators, cf. also [17], or [13, Subsection 3.3.4]. Other generalizations of the edge calculus concern operators, acting between spaces of distributional sections in (say, complex) vector bundles.

Let us now pass to a space $M \in \mathfrak{M}_2$ with $s_2(M) = Z$ of dimension $l > 0$. Assume that M close to Z is modeled on $B^\wedge \times Z$ for a compact $B \in \mathfrak{M}_1$ with $Y = s_1(B)$ of dimension $q > 0$. Local edge variables of $B^\wedge \times Z \in \mathfrak{M}_1$ are varying on $\mathbb{R}_{t,+} \times s_1(B) \times Z$. The (stretched) model cone of local wedges is $X^\wedge$ for compact $X \in \mathfrak{M}_0$. Let us now consider operator families

$$\tilde{p}(t, z, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^l_z, L^\mu(B, \boldsymbol{b}; \mathbb{R}^{1+l}_{\tilde{\tau},\tilde{\zeta}})) \tag{58}$$

and pass to the edge-degenerate form

$$p(t, z, \tau, \zeta) := \tilde{p}(t, z, t\tau, t\zeta). \tag{59}$$

Later an extra multiplicative factor $t^{-\mu}$ will also be included. The motivation is similar to the case of closed $X \in \mathfrak{M}_0$ rather than $B \in \mathfrak{M}_1$, where conical singularities of X^Δ involved in the "closed edge" calculus over $X^\Delta \times \mathbb{R}^q$ are translated to the corresponding degenerate behavior in covariables. Also here we pass to a Mellin representation, modulo adequate smoothing (here edge-Green) remainders, cf., [10, 11]. In other words in the present situation we pass to $L^\mu(B, \dots)$-valued holomorphic Mellin symbols. Let $M^\mu_{\mathcal{O}_v}(B, \boldsymbol{b}; \mathbb{R}^l_{\tilde{\zeta}})$ defined to be the space of all

$$\tilde{h}(v, \tilde{\zeta}) \in \mathcal{A}(\mathbb{C}_v, L^\mu(B, \boldsymbol{b}; \mathbb{R}^l_{\tilde{\zeta}})) \tag{60}$$

such that

$$\tilde{h}(v,\tilde{\zeta})|_{\Gamma_\alpha\times\mathbb{R}^l_{\tilde{\zeta}}} \in L^\mu(B,\boldsymbol{b};\Gamma_\alpha\times\mathbb{R}^l_{\tilde{\zeta}}) \tag{61}$$

for every $\alpha \in \mathbb{R}$, uniformly in compact α-intervals. We employ here the Mellin transform on the t half-axis $\mathbb{R}_+$. Recall that parameter dependence on the right-hand side of (61) means that the parameters range over $\operatorname{Im} v \times \mathbb{R}^d_{\tilde{\zeta}}$ for the weight line $\Gamma_\alpha \ni v$, where "uniformly" means that the associated operators run over a bounded set in the Fréchet topology of the chosen subspace of elements when $(v,\tilde{\zeta})$ varies over $\Gamma_\alpha \times \mathbb{R}^l_{\tilde{\zeta}}$ for α in any compact interval. Consider

$$\tilde{h}(t,z,v,\tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^l_z, M^\mu_{\mathcal{O}_v}(B,\boldsymbol{b};\mathbb{R}^l_{\tilde{\zeta}})) \tag{62}$$

and then set

$$h(t,z,v,\zeta) := \tilde{h}(t,z,v,t\zeta). \tag{63}$$

The way of constructing h in terms of p is close to the corresponding method in [10], cf., also [6]. Precise formulations on the nature of remainders depend on (more or less straightforward but voluminous) details of weighted corner Sobolev spaces, to be employed later on, see, considerations below. In any case, when a Mellin symbol h is associated with p we say that h is a Mellin quantization of p, and a consequence is that

$$\operatorname{Op}_t(p)(z,\zeta) - \operatorname{Op}^{\Xi}_M(h)(z,\zeta) \in C^\infty(\mathbb{R}^l_z, L^{-\infty}(B,\boldsymbol{b};\mathbb{R}^l_\zeta)), \tag{64}$$

where

$$\operatorname{Op}^{\Xi}_M(h)(z,\zeta)u(t,\cdot) := \int_{\Gamma_{\frac{N+1}{2}-\Xi}} \int \Big(\frac{t}{t'}\Big)^{-v} h(t,z,v,\zeta)u(t',\cdot)\,dt'đv$$

for $đv = (2\pi i)^{-1}dv$, $N := \dim B$ and any weight $\Xi \in \mathbb{R}$. The method is to fix some $\varphi \in C^\infty_0(\mathbb{R}_+)$ which is equal to 1 in a neighborhood of 1. Then we form

$$\operatorname{Op}_t((1-\varphi(\frac{t'}{t}))p)(z,\zeta) \tag{65}$$

containing a cut-off operation at $t = t'$, and this is at the same time a cut-off close to the diagonal with respect to all variables. This causes the smoothing behaviour of (65). The Mellin symbol h is obtained via the same method as in [10, Theorem 2.3] and it is directly constructed from

$$\operatorname{Op}_t(\varphi(\frac{t'}{t})p)(z,\zeta). \tag{66}$$

In fact, we have

$$\operatorname{Op}_t(p)(z,\zeta) = \operatorname{Op}_t(\varphi(\frac{t'}{t})p)(z,\zeta) + \operatorname{Op}_t((1-\varphi(\frac{t'}{t}))p)(z,\zeta)$$

and

$$\mathrm{Op}_t(\varphi(\frac{t'}{t})p)(z,\zeta) = \mathrm{Op}_M^{\Xi}(h)(z,\zeta)$$

for some $\tilde{h}(t,z,v,\tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^l_z, M^\mu_{\mathcal{O}_v}(B,\boldsymbol{b};\mathbb{R}^l_{\tilde{\zeta}}))$, cf., (62), (63), and

$$\mathrm{Op}_t((1-\varphi(\frac{t'}{t}))p)(z,\zeta) \in C^\infty(\mathbb{R}^l_z, L^{-\infty}(B,\boldsymbol{b};\mathbb{R}^l_\zeta)) \tag{67}$$

which explains relation (64). The computation may be performed first for a special weight, say $\Xi = \frac{1}{2}$, but then because of the holomorphy of h in the complex variable v as a consequence of Cauchy's theorem, we just conclude that

$$\mathrm{Op}_M^{\frac{1}{2}}(h)(z,\zeta) = \mathrm{Op}_M^{\Xi}(h)(z,\zeta). \tag{68}$$

The nature of the remainder (67) plays a role in connection with smoothing properties of

$$\sigma_1(t)(1-\omega_1(t[\zeta]))\mathrm{Op}_t(p)(z,\zeta)\sigma_2(t)(1-\omega_2(t[\zeta])) \tag{69}$$

for some cut-off functions $\sigma_2 \prec \sigma_1$ and $\omega_2 \prec \omega_1$. The factors $\sigma_i(t)(1-\omega_i(t[\zeta]))$, $i = 1,2$, localize operators close to $t = 0$ and also produce ζ-dependent localizations off $t = 0$ both in variables t and $t' \in \mathbb{R}_+$. Similar observations may be applied in the case of higher singularities.

From now on we pass again to formulating the calculus including parameters, i.e., we replace everywhere ζ by $(\zeta,\lambda) \in \mathbb{R}^{l+d}$. Clearly the constructions so far have an obvious parameter-dependent version and those will be applied. Similarly to (26) we now study spaces of parameter-dependent corner operators

$$L^\mu(M,\boldsymbol{b},\boldsymbol{g};\mathbb{R}^d_\lambda) \tag{70}$$

of order $\mu \in \mathbb{R}$ on a space $M \in \mathfrak{M}_2$ with corner $Z := s_2(M)$ of dimension $l > 0$ and weight data

$$\boldsymbol{b} := (\beta,\beta-\mu),\ \boldsymbol{g} := (\gamma,\gamma-\mu). \tag{71}$$

Because of the extent of auxiliary material in most general form we focus on operators of the type of upper left corners. The general case, i.e., including trace, and potential operators is analogous, though we would have to take care of the orders of the involved trace and potential operators. Structures of this kind have been discussed in [13, Subsection 5.4.2]. Recall that the edge space $B \in \mathfrak{M}_1$ locally close to $Y = s_1(B)$ of dimension $q > 0$ is modeled on $X^\Delta \times \mathbb{R}^q$ for a compact $X \in \mathfrak{M}_0$, $\dim X = n$

Similarly to (34) we first consider corner amplitude functions

$$\begin{aligned} a_{\mathrm{sing}}(z,\zeta,\lambda) := {} & \sigma_1(t)t^{-\mu}\mathrm{Op}_{M_t}^{\gamma-\frac{\dim B}{2}}(h)(z,\zeta,\lambda)\sigma_0(t) \\ & + (1-\sigma_1(t))t^{-\mu}\mathrm{Op}_t(p_{\mathrm{int}})(z,\zeta,\lambda)(1-\sigma_2(t)) + g_{\mathrm{M+G}}(z,\zeta,\lambda) \end{aligned} \tag{72}$$

where the Mellin symbol h is given by (63) which is associated with (59). Moreover,

$$p_{\mathrm{int}}(t, z, \tau, \zeta, \lambda) \in C^\infty(\mathbb{R}_{t,+} \times \mathbb{R}^l_z, L^\mu(B, \boldsymbol{b}; \mathbb{R}^{1+l+d}_{\tau,\zeta,\lambda}))$$

is assumed to satisfy the condition

$$p_{\mathrm{int}} = \sigma p_{\mathrm{int}} \tilde{\sigma} \tag{73}$$

for some cut-off functions $\sigma(t), \tilde{\sigma}(t)$, cf. relation (33) The space (70) is defined as the set of all families of operators

$$A(\lambda) = \omega_{\mathrm{glob}} A_{\mathrm{sing}}(\lambda) \omega'_{\mathrm{glob}} + (1 - \omega_{\mathrm{glob}}) A_{\mathrm{int}}(\lambda)(1 - \omega''_{\mathrm{glob}}) + C(\lambda) \tag{74}$$

where $A_{\mathrm{sing}}(\lambda)$ is a locally close to Z determined by

$$\mathrm{Op}_z(a_{\mathrm{sing}}(z, \zeta, \lambda)) \tag{75}$$

Moreover, recall the fact that the corner space $M \in \mathfrak{M}_2$ is locally close to Z modeled on $B^\Delta \times \mathbb{R}^l$ for a compact manifold $B \in \mathfrak{M}_1$, $\dim B = N$, with edge $Y := s_1(B)$ of dimension $q > 0$.

The following notation will employ spaces

$$L^\mu(2\,\mathbb{M}, \boldsymbol{b}, \mathbb{R}^d)) \tag{76}$$

with $\mathbb{M}$ being the stretched manifold belonging to $M \in \mathfrak{M}_2$ and its double $2\mathbb{M} \in \mathfrak{M}_1$, i.e., $2\mathbb{M}$ has an edge, cf., (11), such that notation (26) works. Let us set

$$L^\mu_{\mathrm{int}}(M, \boldsymbol{b}; \mathbb{R}^d) := L^\mu(2\,\mathbb{M}, \boldsymbol{b}; \mathbb{R}^d)|_{\mathbb{M} \setminus V_0(M)}. \tag{77}$$

In (74) we assume

$$A_{\mathrm{int}}(\lambda) \in L^\mu_{\mathrm{int}}(M, \boldsymbol{b}; \mathbb{R}^d). \tag{78}$$

Denote the space of operators $C(\lambda)$ in (74) by

$$L^{-\infty}(M, \boldsymbol{b}, \boldsymbol{g}; \mathbb{R}^d) \tag{79}$$

For convenience we assume M to be compact; otherwise we have to work with several variants of "comp" or "loc" Sobolev spaces. In the compact case we employ weighted Sobolev spaces over M, cf. [42], denoted by

$$H^{s,\beta,\gamma}(M) \tag{80}$$

of smoothness $s \in \mathbb{R}$ and with weights β, γ from the involved weight data. The complete definition of (79) contains global smoothing operators, expressed in terms of mapping properties between spaces (80), where $C \in L^{-\infty}(M, \boldsymbol{b}, \boldsymbol{g})$ is asked to induce continuous operators

$$C : H^{s,\beta,\gamma}(M) \longrightarrow H^{\infty,\beta-\mu+\varepsilon,\gamma-\mu+\varepsilon}(M) \tag{81}$$

for all s and some $\varepsilon > 0$, and the formal adjoint C^* with respect to the scalar product of $H^{0,0,0}$ induces continuous operators

$$C^* : H^{s,-\beta+\mu,-\gamma+\mu}(M) \longrightarrow H^{\infty,-\beta+\varepsilon,-\gamma+\varepsilon}(M) \tag{82}$$

for some $\varepsilon > 0$. The operators are compact in the sense $C : H^{s,\beta,\gamma}(M) \to H^{s-\mu,\beta-\mu,\gamma-\mu}(M)$, and $C^* : H^{s,-\beta+\mu,-\gamma+\mu}(M) \to H^{s+\mu,-\beta,-\gamma}(M)$. Then $C(\lambda) \in \mathcal{S}(\mathbb{R}^d_\lambda, L^{-\infty}(M,\boldsymbol{b},\boldsymbol{g}))$.

Definition 4.3: We define spaces of operator-valued symbols $R^\nu_{\mathrm{corner}_G}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta,\gamma,\tilde{\beta},\tilde{\gamma})_{\varepsilon_0,\varepsilon})$ for $\tilde{\beta} := \beta - \mu$, $\tilde{\gamma} := \gamma - \mu$,

$$g(z,\zeta,\lambda) \in \bigcap_{s,s',e,e'\in\mathbb{R}} S^\nu_{\mathrm{cl}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; \mathcal{K}^{s,\beta,\gamma;e}(B^\wedge), \mathcal{K}^{s',\tilde{\beta}+\varepsilon_0,\tilde{\gamma}+\varepsilon;e'}(B^\wedge)), \tag{83}$$

$$g^*(z,\zeta,\lambda) \in \bigcap_{s,s',e,e'\in\mathbb{R}} S^\nu_{\mathrm{cl}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; \mathcal{K}^{s,-\tilde{\beta},-\tilde{\gamma};e}(B^\wedge), \mathcal{K}^{s',-\beta+\varepsilon_0,-\gamma+\varepsilon;e'}(B^\wedge)) \tag{84}$$

for some $\varepsilon > 0$. Moreover, let $R^\mu_{\mathrm{corner}_G}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta,\gamma,\tilde{\beta},\tilde{\gamma})_{\varepsilon_0,\infty}$ be the space of those $g(z,\zeta,\lambda)$ such that (83) and (84) hold for all $\varepsilon > 0$.

Furthermore, $R^\mu_{\mathrm{corner}_{\mathrm{M+G}}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta,\tilde{\beta},\gamma,\tilde{\gamma})_{\varepsilon_0,\varepsilon})$ is defined to be the set of all operator families of the form

$$g_{\mathrm{M+G}}(z,\zeta,\lambda) = g_{\mathrm{M}}(z,\zeta,\lambda) + g_{\mathrm{G}}(z,\zeta,\lambda) \tag{85}$$

for

$$g_{\mathrm{M}}(z,\zeta,\lambda) = \omega(t[\zeta,\lambda])t^{-\mu}\mathrm{Op}_M^{\beta-N/2}(f)(z)\omega'(t[\zeta,\lambda]), \tag{86}$$

$N = \dim B$, for some cut-off functions $\omega(t), \omega'(t)$ and $f(z,v) \in C^\infty(\mathbb{R}^l, M^{-\infty}_{\mathrm{As}}(B))$ and

$$g_{\mathrm{G}}(z,\zeta,\lambda) \in R^\mu_{\mathrm{corner}_{\mathrm{G}}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta,\gamma,\tilde{\beta},\tilde{\gamma})_{\varepsilon_0,\varepsilon}). \tag{87}$$

Remark 4: In connection with (70) it makes sense also to consider the chain of subspaces

$$L^{-\infty}(B,\boldsymbol{b};\mathbb{R}^d_\lambda) \subset L^\mu_{\mathrm{G}}(B,\boldsymbol{b};\mathbb{R}^d_\lambda) \subset L^\mu_{\mathrm{M+G}}(B,\boldsymbol{b};\mathbb{R}^d_\lambda) \subset L^\mu(B,\boldsymbol{b};\mathbb{R}^d_\lambda), \tag{88}$$

where $L^\mu_{\mathrm{M+G}}$ and L^μ_{G} are locally close to Y determined by symbols $g_{\mathrm{M+G}}(y,\eta,\lambda)$ and $g_{\mathrm{G}}(y,\eta,\lambda)$, respectively, up to globally smoothing remainders in $L^{-\infty}$, cf. notation around (85), (86), (87).

We use the fact that

$$a_{\rm sing}(z,\zeta,\lambda) \in S^{\mu}(\mathbb{R}^{l}_{z} \times \mathbb{R}^{l+d}_{\zeta,\lambda}; \mathcal{K}^{s,\beta,\gamma}(B^{\wedge}), \mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(B^{\wedge})), \tag{89}$$

for the symbols (72), between weighted Kegel spaces $\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})$ and $\mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(B^{\wedge})$, respectively, both equipped with the group action

$$(\kappa_{\delta}u)(t,\cdot) := \delta^{\frac{N+1}{2}}u(\delta t,\cdot), \quad \delta \in \mathbb{R}_{+}.$$

Then the operators

$$\mathrm{Op}_{z}(a_{\rm sing})(\lambda) : \mathcal{W}^{s}_{\rm comp}(\mathbb{R}^{l}, \mathcal{K}^{s,\beta,\gamma}(B^{\wedge})) \to \mathcal{W}^{s-\mu}_{\rm loc}(\mathbb{R}^{l}, \mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(B^{\wedge})) \tag{90}$$

are continuous for all s.

Definition 4.4: An $A(\lambda) \in L^{\mu}(M, \boldsymbol{b}, \boldsymbol{g}; \mathbb{R}^{d}_{\lambda})$ on $M \in \mathfrak{M}_2$ is called parameter-dependent elliptic of order μ if

$$A(\lambda)|_{\mathbb{M}\setminus V_{\mathbb{O}}(M)}$$

is elliptic of order μ on $\mathbb{M} \setminus V_{\mathbb{O}}(M) \in \mathfrak{M}_1$ in the space $L^{\mu}(\mathbb{M} \setminus V_{\mathbb{O}}(M), \boldsymbol{b}; \mathbb{R}^{d}_{\lambda})$ and if in addition

$$\sigma_2(a_{\rm sing})(z,\zeta,\lambda) = t^{-\mu}\mathrm{Op}_{M_t}^{\gamma-\frac{\dim B}{2}}(h_0)(z,\zeta,\lambda) + \sigma_2(g_{\rm M+G})(z,\zeta,\lambda) \tag{91}$$

for $h_0(t,z,v,\zeta,\lambda) := \tilde{h}(0,z,v,t\zeta,t\lambda)$ and induces a family of isomorphisms

$$\sigma_2(a_{\rm sing})(z,\zeta,\lambda) : \mathcal{K}^{s,\beta,\gamma}(B^{\wedge}) \longrightarrow \mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(B^{\wedge}) \tag{92}$$

for all $z \in Z$ and all $(\zeta,\lambda) \neq 0$.

Note that the bijectivity of (92) entails conormal ellipticity, namely, that

$$\sigma_{2,\rm cn}(a_{\rm sing})(z,v) := \tilde{h}(0,z,v,0,0) + f(z,v) \tag{93}$$

induces a family of isomorphisms

$$\sigma_{2,\rm cn}(a_{\rm sing})(z,v) : H^{s,\beta}(B) \to H^{s-\mu,\beta-\mu}(B) \tag{94}$$

for all s and $z \in Z, v \in \Gamma_{(N+1)/2-\gamma}$.

Proposition 4.5: *Let $M \in \mathfrak{M}_2$ be compact, and let $l = \dim Z > 0$. Then operators of upper left corner type*

$$A(\lambda) \in L^{\mu}(M, \boldsymbol{b}, \boldsymbol{g}; \mathbb{R}^{d}) \tag{95}$$

for any fixed λ induce continuous maps

$$A(\lambda) : H^{s,\beta,\gamma}(M) \to H^{s-\mu,\beta-\mu,\gamma-\mu}(M) \tag{96}$$

for all s, except for a discrete system of exceptional weights γ, determined by the poles of the involved (meromorphic) Mellin symbols.

Theorem 4.6: *Let $M \in \mathfrak{M}_2$, and let $A(\lambda)$ in (95) be parameter-dependent elliptic of order μ. Then there is a parameter-dependent parametrix*

$$P(\lambda) \in L^{-\mu}(M, \boldsymbol{b}^{-1}, \boldsymbol{g}^{-1}; \mathbb{R}^d) \tag{97}$$

with weight data $\boldsymbol{b}^{-1} := (\beta - \mu, \beta), \boldsymbol{g}^{-1} := (\gamma - \mu, \gamma)$, such that

$$P(\lambda)A(\lambda) - 1 = C_{\mathrm{L}}(\lambda), \quad A(\lambda)P(\lambda) - 1 = C_{\mathrm{R}}(\lambda) \tag{98}$$

for remainders

$$C_{\mathrm{L}}(\lambda) \in L^{-\infty}(M, \boldsymbol{b}_{\mathrm{L}}, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^d),\, C_{\mathrm{R}}(\lambda) \in L^{-\infty}(M, \boldsymbol{b}_{\mathrm{R}}, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^d) \tag{99}$$

where $\boldsymbol{b}_{\mathrm{L}} = (\beta, \beta)$, $\boldsymbol{g}_{\mathrm{L}} = (\gamma, \gamma)$, $\boldsymbol{b}_{\mathrm{R}} = (\beta - \mu, \beta - \mu)$, $\boldsymbol{g}_{\mathrm{R}} = (\gamma - \mu, \gamma - \mu)$. In addition For compact N the operators (96) are λ-wise Fredholm and for $d > 0$ there is a constant $c > 0$ such that (96) is a family of isomorphisms for $|\lambda| > c$.

5. Singular Boundary Value Problems

The iterative approach of studying pseudo-differential boundary value problems BVPs on a singular space $N \in \mathfrak{N}_l$ is of similar structure as the calculus of degenerate operators developed before on a space $M \in \mathfrak{M}_k$ concerning the closed case. As noted in the beginning we refer to the work of Boutet de Monvel [1] concerning the smooth case $N \in \mathfrak{N}_0$ and to a series of joint papers, especially, [19], [22], [21], [7]. Known structures from the closed case, see the references in [7], give an impression on the complexity of operator structures to be established in terms of algebras, symbols and quantizations, for preparing the program of constructing parametrices, say, in the elliptic case. It is just the main idea of the present considerations to consolidate the respective structures. Using the approach on closed M from the preceding sections we will pass to the case of spaces N with boundary by replacing parameter-dependent operators in

$$L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^d_\lambda),\ L^{\mu}(B, \boldsymbol{b}; \mathbb{R}^d_\lambda),\ L^{\mu}(M, \boldsymbol{b}, \boldsymbol{g}; \mathbb{R}^d_\lambda),\ L^{\mu}(P, \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{k}; \mathbb{R}^d_\lambda), \dots, \tag{100}$$

for $X \in \mathfrak{M}_0$, $B \in \mathfrak{M}_1$, $M \in \mathfrak{M}_2$, $P \in \mathfrak{M}_3, \dots$ by corresponding parameter-dependent spaces of BVPs, see, formulas (101) below.

First note that the dimension d of the space of parameters λ in sequences (100) depends on the corresponding reference links X, B, M, $P, \dots$. However, for brevity, we avoid additional notation such as $d(X)$, $d(B)$, $\dots$, or some numeration, and the relation would be $d(X) = 1 + \dim s_2(M) + d(B)$, $d(B) = 1 + \dim s_3(P) + d(M)$, etc. In other words, we keep in mind the position of the respective singular calculus according to the reached step of iteration. In BVPs we have the operator spaces

$$\mathcal{B}^{\mu,\boldsymbol{a}}(D; \mathbb{R}^d_\lambda),\ \mathcal{B}^{\mu,\boldsymbol{a}}(E, \boldsymbol{b}; \mathbb{R}^d_\lambda),\ \mathcal{B}^{\mu,\boldsymbol{a}}(F, \boldsymbol{b}, \boldsymbol{g}; \mathbb{R}^d_\lambda),\ \mathcal{B}^{\mu,\boldsymbol{a}}(Q, \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{k}; \mathbb{R}^d_\lambda), \dots, \tag{101}$$

for $D \in \mathfrak{N}_0$, $E \in \mathfrak{N}_1$, $F \in \mathfrak{N}_2$, $Q \in \mathfrak{N}_3, \dots$. For the parameter dimensions in (101) we also keep in mind that those depend on the refernce links. Since constructions for higher orders of singularity in BVPs are to some extent parallel to the closed case in the sequel we mainly look at the step from $\mathcal{B}^{\mu,\boldsymbol{a}}(D;\mathbb{R}^d_\lambda)$ to $\mathcal{B}^{\mu,\boldsymbol{e}}(E,\boldsymbol{b};\mathbb{R}^d_\lambda)$. with $\mathcal{B}^{\mu,\boldsymbol{a}}(D;\mathbb{R}^d_\lambda)$ being the space of (upper left corners) of parameter-dependent BVPs in Boutet de Monvel's calculus of order $\mu\in\mathbb{Z}$ and type $\boldsymbol{a}\in\mathbb{N}$, while the other operator spaces are corresponding spaces of BVPs of order μ and type $\boldsymbol{a}$, also being upper left corners of a more general operator block-matrix set-up. Clearly the full operator block matrices require much more machinery and we hope to come back to those questions with a corresponding calculus. Also here we abbreviate notation for involved dimensions.

Recall from the closed case, i.e., operator spaces in (100), the starting point have been spaces of holomorphic operator functions associated with

$$L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q+d}_{\tilde\rho,\tilde\eta,\tilde\lambda})|_{(\tilde\rho,\tilde\eta,\tilde\lambda)=(r\rho,r\eta,r\lambda)}\,,\ L^\mu(B,\boldsymbol{b};\mathbb{R}^{1+l+d}_{\tilde\tau,\tilde\zeta,\tilde\lambda})|_{(\tilde\tau,\tilde\zeta,\tilde\lambda)=(t\tau,t\zeta,t\lambda)}\,,\ldots. \tag{102}$$

We also consider spaces

$$C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q_y,\,L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q+d}_{\tilde\rho,\tilde\eta,\tilde\lambda}))|_{(\tilde\rho,\tilde\eta,\tilde\lambda)=(r\rho,r\eta,r\lambda)}\,, \tag{103}$$

$$C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^l_z,\,L^\mu(B,\boldsymbol{b};\mathbb{R}^{1+l+d}_{\tilde\tau,\tilde\zeta,\tilde\lambda}))|_{(\tilde\tau,\tilde\zeta,\tilde\lambda)=(t\tau,t\zeta,t\lambda)}\,. \tag{104}$$

Via Mellin quantizations from the latter operator families we obtain spaces

$$C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q_y,M^\mu_{\mathcal{O}_w}(X;\mathbb{R}^{q+d}_{\tilde\eta,\tilde\lambda}))\ni\tilde h_X(r,y,w,\tilde\eta,\tilde\lambda), \tag{105}$$

$$C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^l_z,M^\mu_{\mathcal{O}_v}(B,\boldsymbol{b};\mathbb{R}^{l+d}_{\tilde\zeta,\tilde\lambda}))\ni\tilde h_B(t,z,v,\tilde\zeta,\tilde\lambda), \tag{106}$$

cf., relations (30), (62). Then we set

$$h_X(r,y,\eta,\lambda))=\tilde h_X(r,y,w,r\eta,r\lambda),\ h_B(t,z,v,\zeta,\lambda)=\tilde h_B(t,z,v,t\zeta,t\lambda). \tag{107}$$

Subscripts at Mellin symbols indicate the links of involved model cones. In sections before we outlined the structure of operator classes (102). Recall that those always contain weight factors r^μ, $t^{-\mu},\ldots$, while in Mellin symbols (62) they are added in the corresponding Mellin operators.

Moreover, in the case of BVPs (101) we consider compact spaces $D\in\mathfrak{N}_0$, $E\in\mathfrak{N}_1$ and we form operator functions

$$C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q_y,\,\mathcal{B}^{\mu,\boldsymbol{a}}(D;\mathbb{R}^{1+q+d}_{\tilde\rho,\tilde\eta,\tilde\lambda}))|_{(\tilde\rho,\tilde\eta,\tilde\lambda)=(r\rho,r\eta,r\lambda)}\,, \tag{108}$$

$$C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^l_z,\,\mathcal{B}^{\mu,\boldsymbol{a}}(E,\boldsymbol{b};\mathbb{R}^{1+l+d}_{\tilde\tau,\tilde\zeta,\tilde\lambda}))|_{(\tilde\tau,\tilde\zeta,\tilde\lambda)=(t\tau,t\zeta,t\lambda)}\,. \tag{109}$$

Then Mellin quantization gives us spaces

$$C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q_y, \mathcal{B}M^{\mu;\boldsymbol{a}}_{\mathcal{O}_w}(D; \mathbb{R}^{q+d}_{\tilde\eta,\tilde\lambda})) \ni \tilde h_D(r, y, w, \tilde\eta, \tilde\lambda), \tag{110}$$

$$C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^l_z, \mathcal{B}M^{\mu,\boldsymbol{a}}_{\mathcal{O}_v}(E, \boldsymbol{b}; \mathbb{R}^{l+d}_{\tilde\zeta,\tilde\lambda})) \ni \tilde h_E(t, z, v, \tilde\zeta, \tilde\lambda). \tag{111}$$

For

$$h_D(r, y, w, \eta, \lambda) := \tilde h_D(r, y, w, r\eta, r\lambda), \quad h_E(t, z, v, \zeta, \lambda) := \tilde h_E(t, z, v, t\zeta, t\lambda), \tag{112}$$

we set

$$h_{D,0}(r, y, w, \eta, \lambda) := \tilde h_D(0, y, w, r\eta, r\lambda), \quad h_{E,0}(t, z, v, \zeta, \lambda) := \tilde h_E(0, z, v, t\zeta, t\lambda). \tag{113}$$

Similarly to (34) symbols of the edge pseudo-differential calculus of BVPs locally close to the edge Y have the form

$$\begin{aligned} a_{D,\mathrm{sing}}(y, \eta, \lambda) = \sigma_1(r) r^{-\mu} \mathrm{Op}^{\beta-\frac{n}{2}}_{M_r}(h_D)(y, \eta, \lambda)\sigma_0(r) \\ + (1 - \sigma_1(r)) r^{-\mu} \mathrm{Op}_r(p_{D,\mathrm{int}})(y, \eta, \lambda)(1 - \sigma_2(r)) \\ + g_{\mathrm{D;M+G}}(y, \eta, \lambda), \end{aligned} \tag{114}$$

The corresponding analogies of (72) for corner amplitude functions of BVPs locally close to the edge Z are

$$\begin{aligned} a_{E,\mathrm{sing}}(z, \zeta, \lambda) := \sigma_1(t) t^{-\mu} \mathrm{Op}^{\gamma-\frac{N}{2}}_{M_t}(h_E)(z, \zeta, \lambda)\sigma_0(t) \\ + (1 - \sigma_1(t)) t^{-\mu} \mathrm{Op}_t(p_{E,\mathrm{int}})(z, \zeta, \lambda)(1 - \sigma_2(t)) + g_{\mathrm{E;M+G}}(z, \zeta, \lambda). \end{aligned} \tag{115}$$

The meaning of the other ingredients such as cut-off functions in the corresponding context, or $p_{D,\mathrm{int}}$, $g_{\mathrm{D;M+G}}$ and $p_{E,\mathrm{int}}$, $g_{\mathrm{E;M+G}}$ is similar to the corresponding operator functions in (34) and (72), respectively.

To be more precise for $D \in \mathfrak{N}_0$ we employ here edge Green symbols $g_{\mathrm{D,G}}(y, \eta, \lambda) \in \mathcal{B}R^{\mu,\boldsymbol{a}}_{\mathrm{edge}_G}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde\beta)_\varepsilon)$ of order μ and type $\boldsymbol{a}$ for $\beta := \tilde\beta - \mu$ and some $\varepsilon > 0$. Those are operator-valued symbols

$$g_{\mathrm{D,G}}(y, \eta, \lambda) \in \bigcap_{s > \boldsymbol{a} - 1/2, s', e, e' \in \mathbb{R}} S^\mu_{\mathrm{cl}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; \mathcal{K}^{s,\beta;e}(D^\wedge), \mathcal{K}^{s',\tilde\beta+\varepsilon;e'}(D^\wedge)), \tag{116}$$

with additional "dual" properties coming from standard manipulations in terms of Green symbols in Boutet de Monvel's calculus of BPVs. Moreover, let $\mathcal{B}R^{\mu,\boldsymbol{a}}_{\mathrm{edge}_G}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde\beta)_\infty)$ be the space of those $g_{\mathrm{D,G}}(y, \eta, \lambda)$ belonging to the former spaces for all $\varepsilon > 0$.

Remark 1 : The above-mentioned "dual" properties of operator functions (120) already occur in symbols of Green operators in upper left corners of 2×2 block-matrices on a manifold with smooth boundary when they are of type $\boldsymbol{a} > 0$. The notion "dual" here is a substitute of properties of pointwise formal adjoints to be imposed when $\boldsymbol{a} = 0$. The case $\boldsymbol{a} > 0$ is characterized by the presence of involved derivatives in the variable transversal to ∂D of order $m = 1, \dots, \boldsymbol{a}$ composed with Green symbols of type zero and of order $\mu - m$, see, for instance material around [25, Propositon 2.2.28] concerning the nature of Green symbols on the level of boundary symbols close to a smooth boundary. Similar descriptions may be found in other systematic descriptions of Boutet de Monvel's calculus, also in connection with trace entries of 2×2 block-matrices. The full story requires several voluminous details. Therefore, here we simply talk about correspondig "dual" requirements.

Furthermore, $\mathcal{B}R^{\mu,\boldsymbol{a}}_{\mathrm{edge}_{\mathrm{M+G}}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta})_\varepsilon)$ is defined to be a set of operator families of the form

$$g_{D,\mathrm{M+G}}(y, \eta, \lambda) = g_{D,\mathrm{M}}(y, \eta, \lambda) + g_{D,\mathrm{G}}(y, \eta, \lambda) \tag{117}$$

for

$$g_{D,\mathrm{M}}(y, \eta, \lambda) = \omega(r[\eta, \lambda]) r^{-\mu} \mathrm{Op}_M^{\beta - n/2}(f_D)(y) \omega'(r[\eta, \lambda]) \tag{118}$$

for some cut-off functions $\omega(r), \omega'(r)$ and $f_D(y, w) \in C^\infty(\mathbb{R}^q, \mathcal{B}M^{-\infty,\boldsymbol{a}}_{\mathrm{As}}(D))$ and

$$g_{D,\mathrm{G}}(y, \eta, \lambda) \in \mathcal{B}R^{\mu,\boldsymbol{a}}_{\mathrm{edge}_{\mathrm{G}}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta})_\varepsilon). \tag{119}$$

The space $\mathcal{B}M^{-\infty,\boldsymbol{a}}_{\mathrm{As}}(D)$ consisting of y-wise values of Mellin symbols f_D in w is formed in terms of families $\mathcal{B}^{-\infty,\boldsymbol{a}}(D; \Gamma_{(n+1)/2-\beta})$ for $n = \dim D$ which extend to the complex w-plane to an operator-valued function, meromorphic in simplest cases or with continuous asymptotic types of a similar kind as is described in corresponding chapters in [18]. Such smoothing Mellin symbols also occur in the closed case and are studied in detail in [37]. Subscript "As" just indicates this situation.

We also define spaces of corner Green symbols $g_{E,\mathrm{G}}(z, \zeta, \lambda) \in \mathcal{B}R^{\mu,\boldsymbol{a}}_{\mathrm{corner}_G}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta, \gamma, \tilde{\beta}, \tilde{\gamma})_{\varepsilon_0,\varepsilon})$ for $\tilde{\beta} := \beta - \mu, \tilde{\gamma} := \gamma - \mu$. These are operator functions

$$g_{E,\mathrm{G}}(z, \zeta, \lambda) \in \bigcap_{s > \boldsymbol{a} - 1/2, s', e, e' \in \mathbb{R}} S^\mu_{\mathrm{cl}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; \mathcal{K}^{s,\beta,\gamma;e}(E^\wedge), \mathcal{K}^{s',\tilde{\beta}+\varepsilon_0,\tilde{\gamma}+\varepsilon;e'}(E^\wedge)), \tag{120}$$

together with specific properties of duals, similarly as before for edge Green symbols in (120). Moreover, $\mathcal{B}R^{\mu,\boldsymbol{a}}_{\mathrm{corner}_G}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta, \gamma, \tilde{\beta}, \tilde{\gamma})_{\varepsilon_0,\infty})$ for $\tilde{\beta} := \beta - \mu, \tilde{\gamma} := \gamma - \mu$ is the space of all those symbols with the mapping properties in (120) and also of their duals for all $\varepsilon > 0$. Furthermore, $\mathcal{B}R^{\mu}_{\mathrm{corner}_{\mathrm{M+G}}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta, \tilde{\beta}, \gamma, \tilde{\gamma})_{\varepsilon_0,\varepsilon})$ is defined to be the set of all operator families of the form

$$g_{E,\mathrm{M+G}}(z, \zeta, \lambda) = g_{E,\mathrm{M}}(z, \zeta, \lambda) + g_{E,\mathrm{G}}(z, \zeta, \lambda) \tag{121}$$

for

$$g_{E,\mathrm{M}}(z, \zeta, \lambda) = \omega(t[\zeta, \lambda]) t^{-\mu} \mathrm{Op}_M^{\gamma - \dim E/2}(f_E)(z) \omega'(t[\zeta, \lambda]), \tag{122}$$

for some cut-off functions $\omega(t), \omega'(t)$ and $f_E(z,v) \in C^\infty(\mathbb{R}^l, \mathcal{B}M_{\mathrm{As}}^{-\infty,\boldsymbol{a}}(E))$ and

$$g_{E,\mathrm{G}}(z,\zeta,\lambda) \in R^{\mu,\boldsymbol{a}}_{\mathrm{corner}_\mathrm{G}}(\mathbb{R}^l_z \times \mathbb{R}^l_\zeta \times \mathbb{R}^d_\lambda; (\beta,\gamma,\tilde{\beta},\tilde{\gamma})_{\varepsilon_0,\varepsilon}). \tag{123}$$

Let $N \in \mathfrak{N}_1$, and let $\mathcal{B}^{-\infty,\boldsymbol{a}}(N,\boldsymbol{b})$ be the space of all operators C which induce continuous operators

$$C : H^{s,\beta}(N) \to H^{\infty,\beta-\mu+\varepsilon}(N), \tag{124}$$

for all $s > \boldsymbol{a} - 1/2$ and some $\varepsilon > 0$ and satisfying a corresponding dual condition, associated with the involved type $\boldsymbol{a}$ (via standard structures in BVPs). Let us set

$$\mathcal{B}^{-\infty,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda) := \mathcal{S}(\mathbb{R}^d_\lambda, \mathcal{B}^{-\infty,\boldsymbol{a}}(N,\boldsymbol{b})) \tag{125}$$

which are global smoothing elements of the edge pseudo-differential calculus. In addition let $M \in \mathfrak{N}_2$ and let $\mathcal{B}^{-\infty,\boldsymbol{a}}(M,\boldsymbol{b},\boldsymbol{g})$ be the space of all operators C which induce continuous operators

$$C : H^{s,\beta,\gamma}(M) \to H^{\infty,\beta-\mu+\varepsilon,\gamma-\mu+\varepsilon}(M), \tag{126}$$

for all $s > \boldsymbol{a} - 1/2$ and some $\varepsilon > 0$ and satisfying a corresponding dual condition, associated with the involved type $\boldsymbol{a}$. Let us set

$$\mathcal{B}^{-\infty,\boldsymbol{a}}(M,\boldsymbol{b},\boldsymbol{g};\mathbb{R}^d_\lambda) := \mathcal{S}(\mathbb{R}^d_\lambda, \mathcal{B}^{-\infty,\boldsymbol{a}}(M,\boldsymbol{b},\boldsymbol{g})) \tag{127}$$

which are global smoothing elements of the edge pseudo-differential calculus.

We now consider the space of (families of) edge BVPs of order $\mu \in \mathbb{Z}$ and type $\boldsymbol{a}$ and weight data $\boldsymbol{b} = (\beta, \beta-\mu)$ on a space $N \in \mathfrak{N}_1$ with edge Y of dimension $q > 0$ and local model cones D^Δ for $D \in \mathfrak{N}_0$, denoted by

$$\mathcal{B}^{\mu,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda), \tag{128}$$

The space (128) is defined as follows: it consists of all

$$A(\lambda) := \omega_{\mathrm{glob}} A_{\mathrm{sing}}(\lambda)\omega'_{\mathrm{glob}} + (1-\omega_{\mathrm{glob}})A_{\mathrm{int}}(\lambda)(1-\omega''_{\mathrm{glob}}) + C(\lambda) \tag{129}$$

for global cut-off functions $\omega''_{\mathrm{glob}} \prec \omega_{\mathrm{glob}} \prec \omega'_{\mathrm{glob}}$ on N that are $\equiv 1$ in a small neighborhood of Y and vanish off another neighborhood of Y, where A_{sing} locally close to Y is determined by amplitude functions (114). Moreover, in (129) we assume $A_{\mathrm{int}}(\lambda) \in \mathcal{B}^{\mu,\boldsymbol{a}}(N \setminus Y; \mathbb{R}^d_\lambda)$ and $C(\lambda) \in \mathcal{B}L^{-\infty,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda)$. The definition implies that

$$A(\lambda) : H^{s,\beta}(N) \to H^{s-\mu,\beta-\mu}(N) \tag{130}$$

is continuous for every $s > \boldsymbol{a} - 1/2$.

Remark 2 : For $N \in \mathfrak{N}_1$ we have a chain of subspaces

$$\mathcal{B}^{-\infty,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda) \subset \mathcal{B}^{\mu,\boldsymbol{a}}_{\mathrm{G}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda) \subset \mathcal{B}^{\mu,\boldsymbol{a}}_{\mathrm{M+G}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda) \subset \mathcal{B}^{\mu,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda), \tag{131}$$

where $\mathcal{B}^{\mu,\boldsymbol{a}}_{\mathrm{M+G}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda)$ and $\mathcal{B}^{\mu,\boldsymbol{a}}_{\mathrm{G}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda)$ are locally close to Y determined by symbols $g_{D,\mathrm{M+G}}(y,\eta,\lambda)$ and $g_{D,\mathrm{G}}(y,\eta,\lambda)$, respectively, up to globally smoothing remainders in $\mathcal{B}^{-\infty,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda)$, cf., also notation around (117), (118), (119).

Moreover, there is the space of (families of) edge BVPs of order $\mu\in\mathbb{Z}$ and type $\boldsymbol{a}$ and weight data $\boldsymbol{b}=(\beta,\beta-\mu),\boldsymbol{g}=(\gamma,\gamma-\mu)$ on a space $M\in\mathfrak{N}_2$ with edge Z of dimension $l>0$ and local model cones E^Δ for $E\in\mathfrak{N}_1$, denoted by

$$\mathcal{B}^{\mu,\boldsymbol{a}}(M,\boldsymbol{b},\boldsymbol{g};\mathbb{R}^d_\lambda). \tag{132}$$

The space (132) is defined as follows: it consists of all

$$A(\lambda):=\omega_{\mathrm{glob}}A_{\mathrm{sing}}(\lambda)\omega'_{\mathrm{glob}}+(1-\omega_{\mathrm{glob}})A_{\mathrm{int}}(\lambda)(1-\omega''_{\mathrm{glob}})+C(\lambda) \tag{133}$$

for global cut-off functions $\omega''_{\mathrm{glob}}\prec\omega_{\mathrm{glob}}\prec\omega'_{\mathrm{glob}}$ on M that are $\equiv 1$ in a small neighborhood of Z and vanish off another neighborhood of Z, where A_{sing} locally close to Z is determined by amplitude functions (114). Moreover, in (133) we assume $A_{\mathrm{int}}(\lambda)\in\mathcal{B}^{\mu,\boldsymbol{a}}(M\setminus Z,\boldsymbol{b};\mathbb{R}^d_\lambda)$ and $C(\lambda)\in\mathcal{B}^{-\infty,\boldsymbol{a}}(M,\boldsymbol{b},\boldsymbol{g};\mathbb{R}^d_\lambda)$. The definition implies that

$$A(\lambda):H^{s,\beta,\gamma}(M)\to H^{s-\mu,\beta-\mu,\gamma-\mu}(M) \tag{134}$$

is continuous for every $s>\boldsymbol{a}-1/2$.

Definition 5.1: An operator family $A(\lambda)\in\mathcal{B}^{\mu,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda)$ is called parameter-dependent elliptic if $A(\lambda)|_{N\setminus Y}$ is parameter-dependent elliptic in $\mathcal{B}^{\mu,\boldsymbol{a}}(N\setminus Y;\mathbb{R}^d_\lambda)$ and if

$$\sigma_1(A)(y,\eta,\lambda):\mathcal{K}^{s,\beta}(D^\wedge)\to\mathcal{K}^{s-\mu,\beta-\mu}(D^\wedge) \tag{135}$$

is a family of isomorphisms for all y and $(\eta,\lambda)\neq 0$, for all $s>\boldsymbol{a}-1/2$ with $s-\mu>\boldsymbol{a}-1/2$.

Let us set $\nu^+=\max\{\nu,0\}$ for any real ν. We then have

Theorem 5.2: *Let $N\in\mathfrak{N}_1$, $\dim Y>0$ and let $A(\lambda)\in\mathcal{B}^{\mu,\boldsymbol{a}}(N,\boldsymbol{b};\mathbb{R}^d_\lambda)$, $\boldsymbol{a}\leq\mu^+$, be parameter-dependent elliptic. Then there is a parameter-dependent parametrix*

$$\mathcal{B}^{-\mu,(-\mu)^+}(N,\boldsymbol{b}^{-1};\mathbb{R}^d_\lambda) \tag{136}$$

for weight data $\boldsymbol{b}^{-1}:=(\beta-\mu,\beta)$, such that

$$P(\lambda)A(\lambda)-1=C_{\mathrm{L}}(\lambda),\quad A(\lambda)P(\lambda)-1=C_{\mathrm{R}}(\lambda) \tag{137}$$

for remainders $C_{\mathrm{L}}(\lambda)\in\mathcal{B}^{-\infty,\mu^+}(N,\boldsymbol{b}_{\mathrm{L}};\mathbb{R}^d)$, $C_{\mathrm{R}}(\lambda)\in\mathcal{B}^{-\infty,(-\mu)^+}(N\boldsymbol{b}_{\mathrm{R}};\mathbb{R}^d)$ where $\boldsymbol{b}_{\mathrm{L}}=(\beta,\beta)$, $\boldsymbol{b}_{\mathrm{R}}=(\beta-\mu,\beta-\mu)$. In addition for compact N (130) is a family of Fredholm operators and for $d>0$ there is a constant $c>0$ such that (130) are isomorphisms for $|\lambda|>c$ and $s,s-\mu>\mu^+-1/2$.

The proof is a consequence of the edge pseudo-differential calculus, developed in [36], [10, 11], combined with information on modifications for BVPs, see also the tools outlined in [25].

Remark 3 : Note that the bijectivity of (135) entails conormal ellipticity, namely, that

$$\sigma_{1,\mathrm{cn}}(a_{D,\mathrm{sing}})(y,w) := \tilde{h}_D(0,y,w,0,0) + f_D(y,w) \tag{138}$$

induces a family of isomorphisms

$$\sigma_{1,\mathrm{cn}}(a_{D,\mathrm{sing}})(y,w) : H^s(D) \to H^{s-\mu}(D) \tag{139}$$

for all $s > \max\{\boldsymbol{a} - 1/2, \boldsymbol{a} + \mu - 1/2\}$ and $y \in Y, w \in \Gamma_{(\dim D+1)/2-\beta}$.

Definition 5.3: An $A(\lambda) \in \mathcal{B}^{\mu,\boldsymbol{a}}(M, \boldsymbol{b}, \boldsymbol{g}; \mathbb{R}^d_\lambda)$ on $M \in \mathfrak{N}_2$ is called parameter-dependent elliptic of order μ if

$$A(\lambda)|_{\mathbb{M}\setminus V_{\mathbb{O}}(M)}$$

is elliptic of order μ on $\mathbb{M} \setminus V_{\mathbb{O}}(M) \in \mathfrak{N}_1$ in the space $\mathcal{B}^{\mu,\boldsymbol{a}}(\mathbb{M} \setminus V_{\mathbb{O}}(M), \boldsymbol{b}; \mathbb{R}^d_\lambda)$ and if in addition

$$\sigma_2(A)(z,\zeta,\lambda) := \sigma_2(a_{\mathrm{sing}})(z,\zeta,\lambda) = t^{-\mu}\mathrm{Op}_{M_t}^{\gamma-\frac{\dim N}{2}}(h_0)(z,\zeta,\lambda) + \sigma_2(g_{\mathrm{M+G}})(z,\zeta,\lambda) \tag{140}$$

for $h_0(t,z,v,\zeta,\lambda) := \tilde{h}(0,z,v,t\zeta,t\lambda)$ and induces a family of isomorphisms

$$\sigma_2(a_{E,\mathrm{sing}})(z,\zeta,\lambda) : \mathcal{K}^{s,\beta,\gamma}(E^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(E^\wedge) \tag{141}$$

for all $z \in Z$ and all $(\zeta,\lambda) \neq 0$.

Note that the bijectivity of (141) entails conormal ellipticity, namely, that

$$\sigma_{2,\mathrm{cn}}(a_{E,\mathrm{sing}})(z,v) := \tilde{h}_E(0,z,v,0,0) + f_E(z,v) \tag{142}$$

induces a family of isomorphisms

$$\sigma_{2,\mathrm{cn}}(a_{\mathrm{sing}})(z,v) : H^{s,\beta}(N) \to H^{s-\mu,\beta-\mu}(N) \tag{143}$$

for all $s > \max\{\boldsymbol{a} - 1/2, \boldsymbol{a} + \mu - 1/2\}$ and $z \in Z, v \in \Gamma_{(\dim N+1)/2-\gamma}$.

The ideas of arranging corner operator theories on spaces in $\mathfrak{M}_k$ suggest a program of dealing with BVPs on $\mathfrak{N}_k$ for higher k. Since the scheme is expected to be similar, we stop here the discussion. More information on the program is given in [7].

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