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Computing Matrix Roots by 2nd Kind Pseudo-Chebyshev Functions and Dunford-Taylor Integral

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The problem of finding matrix roots for a wide class of non-singular complex matrices has been solved by using the 2nd kind pseudo-Chebyshev functions and the Dunford-Taylor integral. For an n-th root of an $r \times r$ matrix we find in general n^r roots, depending on the chosen determination of the numerical roots appearing in the considered equation. Of course the exceptional cases for which there are infinite many roots, or no roots at all are excluded by the introduced technique.

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1. Introduction

Applications of hypergeometric functions appear in many frameworks of mathematical physics, engineering, biomedical sciences and in other fields of natural sciences. Multivariate generalizations of hypergeometric functions have been studied in the literature, even through an extension of the Pochhammer symbol [36–40].
Recently, a special set of univariate hypergeometric functions, called the pseudo-Chebyshev functions, have been introduced [26] and studied [27].
The problem of finding roots of matrices has been considered by several authors in past time [10, 15, 33, 41] and even more recently [6, 13]. In the classical book of Gantmacher [10] this problem has not been examined in his generality, and only a simple example is shown. Actually this subject has been considered either from an abstract point of view - as in the case of Hermitian matrices, since it appears in the theory of positive compact operators [32] - or in computational aspects [11], since it has several applications in engineering, for example, in the theory of vibrations of structures with damping or in the analysis of linear stability of viscoelastic materials, etc.
In old articles, explicit equations for computing matrix powers have been considered,

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in connection with the introduction of multivariate 2nd kind Chebyshev polynomials [23–25]. Since these articles, dating back to the seventies of the previous Century, were written in Italian, they were mostly ignored by the mathematical community. An unusual application of Special functions seems to be the solution of the above mentioned problem of computing, in many cases, the roots of non-singular complex matrices. In fact, in what follows, we consider two different rechniques for computating matrix roots.
The first one is based on the 2nd kind Chebyshev polynomials and their extension to the pseudo-Chebyshev functions recently introduced [26] and studied [27].
This is an algebraic-type technique as it is mainly based on properties of linear recurrence relations [4, 31] and their connection with matrix functions [1, 10, 11, 24]. This method, described in [27] for a 2×2 non-singular complex matrix, is reported here for completeness, and have been extended to the general case by using the multivariate 2nd kind pseudo-Chebyshev functions of fractional degree considered in [28].
The second technique is based on the direct use of the Dunford-Taylor integral (a result which traces back to the works of F. Riesz [29] and L. Fantappiè [9]). Exploiting this powerful analytic instrument, the computation of matrix roots can be performed even in the case of higher order matrices with random entries.
Worked examples are reported in the last Section, in order to show the effectiveness of the methods described here, for computing roots of non-singular complex matrices.

2. Basic definitions

Definition. – Given the $r \times r$, matrix $\mathcal{A} = (a_{ij})$, its *characteristic polynomial* is given by

$$P(\lambda) := det(\lambda \mathcal{I} - \mathcal{A}) = \lambda^r - u_1 \lambda^{r-1} + u_2 \lambda^{r-2} + \ldots + (-1)^r u_r \,. \tag{1}$$

and its *invariants* are:

$$\begin{cases} u_1 := tr\ \mathcal{A} = a_{11} + a_{22} + ... + a_{rr} \\ u_2 := \sum_{i<j}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ \cdots \\ u_r := det\ \mathcal{A} \end{cases} \tag{2}$$

2.1. *Recalling the $F_{k,n}$ functions*

It is known [1, 22] that a basis for the r-dimensional vectorial space of solutions of the homogeneous linear bilateral recurrence relation with constant coefficients u_k $(k = 1, 2, \ldots, r)$, with $u_r \neq 0$,

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \cdots + (-1)^{r-1} u_r X_{n-r} \,, \qquad (n \in \mathbf{Z}) \,, \tag{3}$$

is given by the functions $F_{k,n} = F_{k,n}(u_1, u_2, \ldots, u_r)$, $(k = 1, 2, \ldots, r,\ n \geq -1)$, defined by the initial conditions:

$$F_{r-k+1,h-2}(u_1, u_2, \ldots, u_r) = \delta_{k,h}\,, \qquad (k, h = 1, 2, \ldots, r)\,, \tag{4}$$

where δ is the Kronecker symbol.
Since $u_r \neq 0$, the $F_{k,n}$ functions can be defined even if $n < -1$, by means of the so called *reflection properties*:

$$\begin{gathered} F_{k,n}(u_1, \ldots, u_r) = F_{r-k+1,-n+r-3}\left(\frac{u_{r-1}}{u_r}, \ldots, \frac{u_1}{u_r}, \frac{1}{u_r}\right)\,, \\ (k = 1, 2, \ldots, r\,;\, n < -1)\,. \end{gathered} \tag{5}$$

It has been show by É Lucas [1, 19] that all $\{F_{k,n}\}_{n\in\mathbf{Z}}$ functions are expressed through the only bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$, corresponding to the initial conditions in (4). More precisely, the following equations hold

$$\begin{cases} F_{k,n} = (-1)^{k-1} u_k F_{1,n-1} + F_{k+1,n-1}\,, & (k = 1, 2, \ldots, r-1)\,, \\ F_{r,n} = (-1)^{r-1} u_r F_{1,n-1}\,. \end{cases} \tag{6}$$

Therefore, the bilateral sequence $\{F_{1,n}\}_{n\in\mathbf{Z}}$ is called the *fundamental solution* of the recursion (3) (*"fonction fondamentale"* by É. Lucas [19]).

Remark *2.1* **:** The $F_{k,n}$ functions are the standard basis for the recursion (3) which is a different from the usual one, which uses the roots of the characteristic equation [4]. This basis does not imply the knowledge of roots and does not depend on their multiplicity, so that it is sometimes more convenient.

The functions $F_{1,n}(u_1, \ldots, u_r)$ are called in literature [22] *generalized Lucas polynomials of the second kind*, and are related to the multivariate Chebyshev polynomials (see e.g. R. Lidl - C. Wells [16], R. Lidl [17], M. Bruschi - P.E. Ricci [2], K.B. Dunn - R. Lidl [8], R.J. Beerends [3]).

2.2. *Matrix powers representation*

In preceding articles [1, 24], the following result is proved:

Theorem 2.2 : – *Given an $r \times r$ matrix $\mathcal{A}$, putting by definition $u_0 := 1$, and denoting by $P(\lambda)$ its characteristic polynomial* (or possibly its minimal polynomial, if this is known)*, the matrix powers $\mathcal{A}^n$, with integral exponent n, are given by the equation:*

$$\begin{gathered} \mathcal{A}^n = F_{1,n-1}(u_1, \ldots, u_r)\,\mathcal{A}^{r-1} + F_{2,n-1}(u_1, \ldots, u_r)\,\mathcal{A}^{r-2} + \\ + \cdots + F_{r,n-1}(u_1, \ldots, u_r)\,\mathcal{I}\,, \end{gathered} \tag{7}$$

where the functions $F_{k,n}(u_1, \ldots, u_r)$ are defined in Section 2.1.
Moreover, if $\mathcal{A}$ is non-singular, i.e. $u_r \neq 0$, equation (7) *still works for negative integers n, assuming the definition* (5) *for the $F_{k,n}$ functions.*

It is worth to recall that the knowledge of eigenvalues is equivalent to that of invariants, since the second ones are the elementary symmetric functions of the first ones.

Remark *2.3***:** Note that, as a consequence of the above result, the higher powers of matrix $\mathcal{A}$ are always expressible in terms of the lowers ones (at most up to the dimension of $\mathcal{A}$).

2.3. *The Dunford-Taylor integral*

Theorem 2.4: – *Consider an $r \times r$ matrix $\mathcal{A} = \{a_{h,k}\}$, with characteristic polynomial* (1) *and invariants given by equation* (2)*. Let f be a function holomorphic in an open set $\mathcal{O}$, containing all the eigenvaues of $\mathcal{A}$. Then, the matrix functions $f(\mathcal{A})$ are given by the Dunford-Taylor* (*also called Riesz-Fantappiè*) *integral* [14]*:*

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_\gamma \frac{f(\lambda)\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\; \mathcal{A}^{r-k} \right] \;, \tag{8}$$

where γ denotes a simple contour encircling all the zeros of $P(\lambda)$.

In particular the integer powers of $\mathcal{A}$ are given by the equation:

$$\mathcal{A}^n = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_\gamma \frac{\lambda^n\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\; \mathcal{A}^{r-k} \right] \;. \tag{9}$$

Remark *2.5***:** If the eigenvalues of $\mathcal{A}$, are known, equation (9), by the residue theorem, gives back the Lagrange-Sylvester representation. However, for computing the integrals appearing in equation (9) it is sufficient the knowledge of a circular domain D, $(\gamma := \partial D)$, encircling the spectrum of $\mathcal{A}$. By the Gerschgorin theorem, this can be done by using the entries of $\mathcal{A}$, without computing its eigenvalues. Therefore, this approach is computationally more convenient with respect to the Lagrange-Sylvester formula.

Remark *2.6***:** Note that, for the matrix exponential function, equation (8) writes:

$$e^{\mathcal{A}} = \sum_{h=0}^{r-1} \left[\frac{1}{2\pi i} \sum_{j=0}^{r-h-1} (-1)^j u_j \oint_\gamma \frac{e^\lambda \lambda^{r-h-j-1}}{P(\lambda)}\, d\lambda \right] \mathcal{A}^h \,, \tag{10}$$

so that the usual definition by series expansions [12, 34] shoud be avoided (see [20, 21]).

Remark *2.7***:** By comparing equations (7) and (9) it follows that for any $n \in \mathbf{N}$ the $F_{k,n}$ function is represented by the integral:

$$F_{k,n}(u_1, \ldots, u_r) = \frac{1}{2\,\pi\,\mathrm{i}} \sum_{\ell=1}^{r} \oint_\gamma \frac{\lambda^{n+1}\, \sum_{h=0}^{\ell-1} (-1)^h u_h\, \lambda^{\ell-h-1}}{P(\lambda)}\, d\lambda \,, \qquad (k = 1, 2, \ldots, r)\,. \tag{11}$$

3. Using multivariate 2nd kind Chebyshev polynomials

Note that, by exploiting equations (6), the result of Theorem 1 writes in terms of the fundamental solution $F_{1,n}(u_1,\dots,u_r)$.

Theorem 3.1: – *Putting for shortness* $\mathbf{u} := (u_1,\dots,u_r)$*, the integer powers of a non-singular* $r\times r$ *matrix* $\mathcal{A}$ *can be written in terms of the sequence* $F_{1,n}(\mathbf{u})$ *as follows:*

$$\begin{aligned}\mathcal{A}^n = F_{1,n-1}(\mathbf{u})\mathcal{A}^{r-1} + [-u_2\,F_{1,n-2}(\mathbf{u}) + u_3\,F_{1,n-3}(\mathbf{u}) + \cdots \\ + (-1)^{r-2}u_{r-1}\,F_{1,n-r+1}(\mathbf{u}) + (-1)^{r-1}\,F_{1,n-r}(\mathbf{u})\big]\,\mathcal{A}^{r-2} + \cdots \\ + \big[(-1)^{r-2}u_{r-1}\,F_{1,n-2}(\mathbf{u}) + (-1)^{r-1}\,F_{1,n-3}(\mathbf{u})\big]\,\mathcal{A} + (-1)^{r-1}\,F_{1,n-2}(\mathbf{u})\,\mathcal{I}\,.\end{aligned} \tag{12}$$

However, it is possible to reduce the number of variables in the last equation, since as it has been noticed in [2], the fundamental solution of the recursion (3) - that is the sequence $F_{1,n}(u_1,u_2,\dots,u_{r-1},u_r)$ - assuming $u_r = 1$, defines the set of $r-1$ variate 2nd kind Chebyshev polynomials:

$$U_n^{(r-1)}(u_1,u_2,\dots,u_{r-1}) := F_{1,n}(u_1,u_2,\dots,u_{r-1},1)\,. \tag{13}$$

In what follows, when $u_r = 1$, in order to simplify the notation it is convenient to put, by definition:

$$\Delta(\lambda) := \lambda^r - u_1\,\lambda^{r-1} + \cdots + (-1)^{r-1}u_{r-1}\,\lambda + (-1)^r\,, \tag{14}$$

and $\tilde{\mathbf{u}} = (u_1,u_2,\dots,u_{r-1})$.
The $(r-1)$variable 2nd kind Chebyshev polynomials [2],

$$U_n^{(r-1)}(\tilde{\mathbf{u}}) \equiv U_n^{(r-1)}(u_1,u_2,\dots,u_{r-1})\,,$$

are defined by the recursion:

$$\begin{aligned}U_n^{(r-1)}(\tilde{\mathbf{u}}) = u_1\,U_{n-1}^{(r-1)}(\tilde{\mathbf{u}}) - u_2\,U_{n-2}^{(r-1)}(\tilde{\mathbf{u}}) + \cdots + \\ + (-1)^{r-2}u_{r-1}\,U_{n-r+1}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\,U_{n-r}^{(r-1)}(\tilde{\mathbf{u}})\,,\end{aligned} \tag{15}$$

with initial conditions:

$$\begin{aligned}U_0^{(r-1)}(\tilde{\mathbf{u}}) = U_1^{(r-1)}(\tilde{\mathbf{u}}) = \cdots = U_{r-3}^{(r-1)}(\tilde{\mathbf{u}}) = 0\,, \\ U_{r-2}^{(r-1)}(\tilde{\mathbf{u}}) = 1\,, \quad U_{r-1}^{(r-1)}(\tilde{\mathbf{u}}) = u_1\,.\end{aligned} \tag{16}$$

Remark *3.2***:** Note that, when $r = 2$ the polynomials $U_n^{(1)}(u_1)$ reduce essentially to the 2nd kind Chebyshev polynomials, since it results: $U_n^{(1)}(u_1) = U_n(u_1/2)$.

Then, by equation (11), the integral representation of the multivariate 2nd kind

Chebyshev polynomials follows:

$$U_n^{(r-1)}(u_1, u_2, \ldots, u_{r-1}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n+1}}{\Delta(\lambda)}\, d\lambda\,. \tag{17}$$

Let $\mathcal{A} = [a_{h,k}]_{r\times r}$ be a non-singular complex matrix of order r, put:

$$v_1 = u_1 u_r^{-1/r}\,, \quad v_2 = u_2 u_r^{-2/r}\,, \quad \cdots\,, \quad v_{r-1} = u_{r-1} u_r^{-(r-1)/r}\,, \tag{18}$$

and, for shortness, $\mathbf{v} = (v_1, v_2, \cdots, v_{r-1})$.
Consider the $(r-1)$variable Chebyshev polynomials [2],

$$U_n^{(r-1)}(\mathbf{v}) \equiv U_n^{(r-1)}(v_1, v_2, \cdots, v_{r-1})\,, \tag{19}$$

defined by the recursion (15)-(16), where ve have assumed the notation: $v_k \equiv u_k$, $(k = 1, 2, \ldots, r-1)$, that is: $\mathbf{v} \equiv \tilde{\mathbf{u}}$.
Then, from Theorem 1, we can derive the result [24]:

Theorem 3.3: *The integer powers of the matrix $\mathcal{A}$ are given by the equation:*

$$\begin{aligned} \mathcal{A}^n = U_{n-1}^{(r-1)}(\mathbf{v})\, u_r^{\frac{n-r+1}{r}} \mathcal{A}^{r-1} + \Big[-u_2\, U_{n-2}^{(r-1)}(\mathbf{v}) + u_3\, U_{n-3}^{(r-1)}(\mathbf{v}) + \cdots \\ + (-1)^{r-2} u_{r-1}\, U_{n-r+1}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-r}^{(r-1)}(\mathbf{v})\Big]\, u_r^{\frac{n-r+2}{r}} \mathcal{A}^{r-2} + \cdots \\ + \Big[(-1)^{r-2} u_{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-3}^{(r-1)}(\mathbf{v})\Big]\, u_r^{\frac{n-1}{r}} \mathcal{A} + (-1)^{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v})\, u_r^{\frac{n}{r}} \mathcal{I}\,. \end{aligned} \tag{20}$$

Remark *3.4***:** Note that equation (20) can be simplified assuming the condition $\det \mathcal{A} = u_r = 1$, which is not a restriction. In fact, letting:

$$\mathcal{A} = u_r\, \tilde{\mathcal{A}}\,, \qquad \text{with} \qquad \det \tilde{\mathcal{A}} = 1\,,$$

it results: $\mathcal{A}^n = (u_r)^n\, \tilde{\mathcal{A}}^n$, and we have again $\tilde{\mathbf{u}} \equiv \mathbf{v}$.

By using a matrix $\tilde{\mathcal{A}}$, such that $\det \tilde{\mathcal{A}} = 1$, the equation (20) becomes:

$$\begin{aligned} \tilde{\mathcal{A}}^n = U_{n-1}^{(r-1)}(\mathbf{v})\, \tilde{\mathcal{A}}^{r-1} + \Big[-u_2\, U_{n-2}^{(r-1)}(\mathbf{v}) + u_3\, U_{n-3}^{(r-1)}(\mathbf{v}) + \cdots \\ + (-1)^{r-2} u_{r-1}\, U_{n-r+1}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-r}^{(r-1)}(\mathbf{v})\Big]\, \tilde{\mathcal{A}}^{r-2} + \cdots \\ + \Big[(-1)^{r-2} u_{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v}) + (-1)^{r-1}\, U_{n-3}^{(r-1)}(\mathbf{v})\Big]\, \tilde{\mathcal{A}} + (-1)^{r-1}\, U_{n-2}^{(r-1)}(\mathbf{v})\, \mathcal{I}\,. \end{aligned} \tag{21}$$

4. Extension to the rational case

It is worth to note that the integral representation (17) allows the possibility to extend the definition of the 2nd kind Chebyshev polynomials to the case of rational indexes.
Of course it should be necessary to consider the root multiplicity problem, but to narrow down the scope of our investigation, we will limit ourselves to work with the

principal value of the considered roots.
To this aim, we put the following

Definition 4.1: The multivariate 2nd kind pseudo-Chebyshev functions of rational degree p/q are defined by the integral:

$$U_{\frac{p}{q}}^{(r-1)}(\tilde{\mathbf{u}}) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}+1}}{\Delta(\lambda)}\, d\lambda \,, \tag{22}$$

where the principal value of the root of $\lambda^{(p/q)+1}$ has been fixed.

This definition generalizes that one introduced in [26] for the single value 2nd kind pseudo-Chebyshev functions.
As the more interesting case in that framework was the case of half-integer degree, in this case we put the definition:

$$U_{\frac{p}{2}}^{(r-1)}(\tilde{\mathbf{u}}) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{2}+1}}{\Delta(\lambda)}\, d\lambda\,, \quad (p \text{ odd number})\,. \tag{23}$$

In a similar way, the definition of matrix powers can be extended to the rational case, by choosing the principal value of the considered roots, and writing equation (20) in the form:

$$\begin{aligned} \mathcal{A}^{p/q} = {} & U_{\frac{p}{q}-1}^{(r-1)}(\tilde{\mathbf{u}})\, u_r^{\frac{p-q(r-1)}{qr}} \mathcal{A}^{r-1} + \Big[-u_2\, U_{\frac{p}{q}-2}^{(r-1)}(\tilde{\mathbf{u}}) + u_3\, U_{\frac{p}{q}-3}^{(r-1)}(\tilde{\mathbf{u}}) + \cdots \\ & + (-1)^{r-2} u_{r-1}\, U_{\frac{p}{q}-r+1}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\, U_{\frac{p}{q}-r}^{(r-1)}(\tilde{\mathbf{u}})\Big]\, u_r^{\frac{p-q(r-2)}{qr}} \mathcal{A}^{r-2} + \cdots \\ & + \Big[(-1)^{r-2} u_{r-1}\, U_{\frac{p}{q}-2}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\, U_{\frac{p}{q}-3}^{(r-1)}(\tilde{\mathbf{u}})\Big]\, u_r^{\frac{p-q}{qr}} \mathcal{A} \\ & + (-1)^{r-1}\, U_{\frac{p}{q}-2}^{(r-1)}(\tilde{\mathbf{u}})\, u_r^{\frac{p}{qr}} \mathcal{I}\,, \end{aligned} \tag{24}$$

where the 2nd kind pseudo-Chebyshev functions with rational indexes are defined by equation (22).
In particular, considering a matrix $\tilde{\mathcal{A}}$, according to Remark 6, we find, for the principal value of its square root:

$$\begin{aligned} \tilde{\mathcal{A}}^{1/2} = {} & U_{-\frac{1}{2}}^{(r-1)}(\tilde{\mathbf{u}})\, \tilde{\mathcal{A}}^{r-1} + \Big[-u_2\, U_{-\frac{3}{2}}^{(r-1)}(\tilde{\mathbf{u}}) + u_3\, U_{-\frac{5}{2}}^{(r-1)}(\tilde{\mathbf{u}}) + \cdots \\ & + (-1)^{r-2} u_{r-1}\, U_{\frac{3}{2}-r}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\, U_{\frac{1}{2}-r}^{(r-1)}(\tilde{\mathbf{u}})\Big]\, \tilde{\mathcal{A}}^{r-2} + \cdots \\ & + \Big[(-1)^{r-2} u_{r-1}\, U_{-\frac{3}{2}}^{(r-1)}(\tilde{\mathbf{u}}) + (-1)^{r-1}\, U_{-\frac{5}{2}}^{(r-1)}(\tilde{\mathbf{u}})\Big]\, \tilde{\mathcal{A}} + (-1)^{r-1}\, U_{-\frac{3}{2}}^{(r-1)}(\tilde{\mathbf{u}})\, \mathcal{I}\,. \end{aligned} \tag{25}$$

5. Direct use of the Dunford-Taylor integral

Theorem 5.1: *Suppose that $f(\lambda)$ is a holomorphic function in a domain $\mathcal{O}$ of the complex plane containing all the eigenvalues λ_h, of $\mathcal{A}$, and let $\gamma \subset \mathcal{O}$ be a simple* *closed smooth curve with positive direction enclosing all the λ_h in its interior. Then the matrix function $f(\mathcal{A})$ is defined by the Dunford-Taylor integral*

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma f(\lambda)\,(\lambda\mathcal{I} - \mathcal{A})^{-1} d\lambda\,, \tag{26}$$

where $(\lambda\mathcal{I} - \mathcal{A})^{-1}$ denotes the *resolvent* of $\mathcal{A}$.
As an example, given the natural number n, the n-th power of $\mathcal{A}$ can be found by the equation:

$$\mathcal{A}^n = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \lambda^n\,(\lambda\mathcal{I} - \mathcal{A})^{-1} d\lambda\,. \tag{27}$$

In [5], pp. 93–95, the following representation equation for the resolvent $(\lambda\mathcal{I} - \mathcal{A})^{-1}$, in terms of the invariants of $\mathcal{A}$ is proved:

$$(\lambda\mathcal{I} - \mathcal{A})^{-1} = \frac{1}{P(\lambda)} \sum_{k=0}^{r-1} \left[\sum_{h=0}^{r-k-1} (-1)^h\, u_h\, \lambda^{r-k-h-1} \right] \mathcal{A}^k\,. \tag{28}$$

By using equations (27) and (28), we find a representation formula for matrix functions, reported in [1].

Theorem 5.2 : *Let $f(\lambda)$ be a holomorphic function in a domain $\mathcal{O}$ of the complex plane containing the spectrum of $\mathcal{A}$, and denote by $\gamma \subset \mathcal{O}$ a simple contour enclosing all the zeros of $P(\lambda)$. Then the Dunford-Taylor integral writes:*

$$f(\mathcal{A}) = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_\gamma \frac{f(\lambda)\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\ \mathcal{A}^{r-k} \right]\ . \tag{29}$$

Note that, if $\mathcal{O}$ does not contain the origin, a simple choice of γ is a circle centered at the origin and radius greater then the spectral radius of $\mathcal{A}$. This radius can be determined, only using the entries of $\mathcal{A}$, as a consequence of the Gershgorin circle theorem.
Consider now the fonction $f(\lambda) = \lambda^{1/n}$, where n is a fixed integer number. As this function is always holomorphic in open set $\mathcal{O} := (\mathbf{C} - \{0\})$, i.e. in the whole plane excluding the origin, the preceding theorem becomes

Theorem 5.3 : *If $\mathcal{A}$, is a non-singular comlex matrix and $\gamma \subset \mathcal{O}$ is a simple contour enclosing all the zeros of $P(\lambda)$, then the roots of $\mathcal{A}$ are represented by*

$$\mathcal{A}^{1/n} = \frac{1}{2\,\pi\,\mathrm{i}} \left[\sum_{k=1}^{r} \oint_\gamma \frac{\lambda^{1/n}\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda\ \mathcal{A}^{r-k} \right]\ . \tag{30}$$

Recalling Cauchy's residue theorem [32], and denoting by $\Phi_k = \Phi_k(\lambda)$ the integrand in equation (30), the contour integral is given by:

$$\oint_\gamma \frac{\lambda^{1/n}\, \sum_{h=0}^{k-1} (-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)}\, d\lambda = (2\,\pi\,\mathrm{i}) \sum_{\ell=1}^{r} Res_{\Phi_k}(\lambda_\ell)\,. \tag{31}$$

Supposing, for simplicity, the eigenvalues are all distinct, and putting

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_r),$$

we find:

$$\sum_{\ell=1}^{r} Res_{\Phi_k}(\lambda_\ell) = \sum_{\ell=1}^{r} \lim_{\lambda\to\lambda_\ell} (\lambda - \lambda_\ell)\, \frac{\lambda^{1/n}\, \sum_{h=0}^{k-1}(-1)^h u_h\, \lambda^{k-h-1}}{P(\lambda)} =$$
$$= \sum_{\ell=1}^{r} \frac{\lambda_\ell^{1/n}\, \sum_{h=0}^{k-1}(-1)^h u_h\, \lambda_\ell^{k-h-1}}{(\lambda_\ell - \lambda_1)\cdots(\lambda_\ell - \lambda_{\ell-1})(\lambda_\ell - \lambda_{\ell+1})\cdots(\lambda_\ell - \lambda_r)}\,,$$

where we have put, by definition: $(\lambda - \lambda_0) = (\lambda - \lambda_{r+1}) := 1$.
Then, equation (30) becomes:

$$\mathcal{A}^{1/n} = \sum_{k=1}^{r}\sum_{\ell=1}^{r} \frac{\lambda_\ell^{1/n}\, \sum_{h=0}^{k-1}(-1)^h u_h\, \lambda_\ell^{k-h-1}}{(\lambda_\ell - \lambda_1)\cdots(\lambda_\ell - \lambda_{\ell-1})(\lambda_\ell - \lambda_{\ell+1})\cdots(\lambda_\ell - \lambda_r)}\, \mathcal{A}^{r-k}\,. \qquad (32)$$

A similar result can be found in case of multiple roots of the characteristic polynomial, by using the more general equation, which holds for a pole of order m at the point λ_ℓ:

$$Res_{\Phi_k}(\lambda_\ell) = \frac{1}{(m-1)!} \lim_{\lambda\to\lambda_\ell} \frac{d^{m-1}}{d\lambda^{m-1}} \left[(\lambda - \lambda_\ell)^m\, \Phi_k(\lambda)\right]\,.$$

Remark *5.4***:** Note that the knowledge of eigenvalues is not strictly necessary. It is mandatory if we compute the integral in equation (29) by Cauchy's residue theorem, but actually only the knowledge of the invariants is necessary, since we could compute the contour integral by choosing as γ a circle centered at the origin with radius greater then the spectral radius of $\mathcal{A}$.

6. Numerical examples

6.1. *Roots of a* 2×2 *non-singular matrix*

In [27] the second kind pseudo-Chebyshev functions $U_{k-\frac{1}{2}}$ have been used for finding the roots of a non-singular 2×2 complex matrix. The relative procedure is reported here for convenience of the reader.
Let $\mathcal{A} = \{a_{h,k}\}_{2\times 2}$ be such a matrix, and denote respectively by $J_1 := tr\,\mathcal{A}$ and $J_2 = det\,\mathcal{A} \neq 0$ the trace and the determinant of $\mathcal{A}$.
In [23], the representation formula for integer powers of $\mathcal{A}$, in terms of second kind Chebyshev polynomials, has been proven:

$$\mathcal{A}^n = J_2^{(n-1)/2} U_{n-1}\left(\tfrac{J_1}{2\sqrt{J_2}}\right)\mathcal{A} - J_2^{n/2} U_{n-2}\left(\tfrac{J_1}{2\sqrt{J_2}}\right)\mathcal{I}\,, \qquad (33)$$

where $\mathcal{I}$ denotes the 2×2 identity matrix.
Actually this equation still holds when n is replaced by $1/n$

$$\mathcal{A}^{1/n} = J_2^{-\frac{n-1}{2n}} U_{-\frac{n-1}{n}} \left(\frac{J_1}{2\sqrt{J_2}} \right) \mathcal{A} - J_2^{\frac{1}{2n}} U_{-\frac{2n-1}{n}} \left(\frac{J_1}{2\sqrt{J_2}} \right) \mathcal{I} \,, \tag{34}$$

provided that the pseudo-Chebyshev functions are used. Note that, as the roots appearing in the preceding equation are multi-valued, we have n^2 possible values for the n-th root of a 2×2 matrix.
For example, putting $n = 2$, we find the square root of $\mathcal{A}$ in the form:

$$\mathcal{A}^{1/2} = J_2^{-\frac{1}{4}} U_{-\frac{1}{2}} \left(\frac{J_1}{2\sqrt{J_2}} \right) \mathcal{A} - J_2^{\frac{1}{4}} U_{-\frac{3}{2}} \left(\frac{J_1}{2\sqrt{J_2}} \right) \mathcal{I} \,. \tag{35}$$

Recalling the initial conditions

$$U_{-\frac{3}{2}}(x) = -\sqrt{\frac{1}{2(1+x)}} \,, \qquad U_{-\frac{1}{2}}(x) = \sqrt{\frac{1}{2(1+x)}} \,, \tag{36}$$

we find

$$U_{-\frac{1}{2}} \left(\frac{J_1}{2\sqrt{J_2}} \right) = \frac{J_2^{\frac{1}{4}}}{\sqrt{J_1 + 2\sqrt{J_2}}} \,, \qquad U_{-\frac{3}{2}} \left(\frac{J_1}{2\sqrt{J_2}} \right) = -\frac{J_2^{\frac{1}{4}}}{\sqrt{J_1 + 2\sqrt{J_2}}} \,,$$

so that we recover the known equation (see [41]):

$$\pm\, \mathcal{A}^{1/2} = \frac{\mathcal{A} \pm \sqrt{J_2}\, \mathcal{I}}{\sqrt{J_1 \pm 2\sqrt{J_2}}} \,. \tag{37}$$

Since the roots of J_2 have two determinations, we find four possible values for the square root of $\mathcal{A}$.

6.2. *The three dimensional case*

Let $\mathcal{A}$ be a 3×3 non-singular complex matrix, and assume that $det\, \mathcal{A} \neq 0$. Denote by $\lambda_1, \lambda_2, \lambda_3$ the roots of $\Delta(\lambda)$, that is: $\Delta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$.
Then, according to the Remark 7, i.e assuming $det\, \mathcal{A} = 1$, and $\mathcal{A} \equiv \tilde{\mathcal{A}}$, by using the notations of Section 3, we have, for the pricipal value of its square root the equation:

$$\tilde{\mathcal{A}}^{1/2} = U^{(2)}_{-\frac{1}{2}}(u,v)\, \tilde{\mathcal{A}}^2 + \left[-v\, U^{(2)}_{-\frac{3}{2}}(u,v) + U^{(2)}_{-\frac{5}{2}}(u,v) \right] \tilde{\mathcal{A}} + U^{(2)}_{-\frac{3}{2}}(u,v)\, \mathcal{I} \,.$$

By using equations (21), we find:

$$\tilde{\mathcal{A}}^{1/2} = \frac{1}{2\,\pi\,\mathrm{i}} \left[\oint_\gamma \frac{\lambda^{1/2}\, d\lambda}{\Delta(\lambda)}\, \mathcal{A}^2 + \oint_\gamma \frac{(-v\, \lambda^{-1/2} + \lambda^{-3/2})\, d\lambda}{\Delta(\lambda)}\, \mathcal{A} + \oint_\gamma \frac{\lambda^{-1/2}\, d\lambda}{\Delta(\lambda)}\, \mathcal{I} \right] , \tag{38}$$

and by using Cauchy's residue theorem it follows:

$$\tilde{\mathcal{A}}^{\frac{1}{2}} = \frac{\lambda_1^{\frac{1}{2}}(\lambda_2-\lambda_3) - \lambda_2^{\frac{1}{2}}(\lambda_1-\lambda_3) + \lambda_3^{\frac{1}{2}}(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\,\tilde{\mathcal{A}}^2$$

$$+\,\frac{(-v\lambda_1^{-\frac{1}{2}}+\lambda_1^{-\frac{3}{2}})(\lambda_2-\lambda_3) - (-v\lambda_2^{-\frac{1}{2}}+\lambda_2^{-\frac{3}{2}})(\lambda_1-\lambda_3) + (-v\lambda_3^{-\frac{1}{2}}+\lambda_3^{-\frac{3}{2}})(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\,\tilde{\mathcal{A}} \qquad (39)$$

$$+\,\frac{\lambda_1^{-\frac{1}{2}}(\lambda_2-\lambda_3) - \lambda_2^{-\frac{1}{2}}(\lambda_1-\lambda_3) + \lambda_3^{-\frac{1}{2}}(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\,\mathcal{I}\,.$$

Remark *6.1***:** Note that the numerical root $\lambda^{1/2}$ appearing in each of the three contour integrals of equation (38), has two determinations, so that we can derive in total $2^3 = 8$ square roots of the matrix A. This remark can be extended to the general case: for the n-th root of an $r \times r$ matrix we can find n^r possible determinations.

6.3. *A worked example*

Consider the matrix

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{so that} \qquad \mathcal{A}^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \qquad (40)$$

The invariants are:

$$u_1 = 1\,, \quad u_2 = 1, \quad u_3 = 1\,.$$

The characteristic equation is:

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0\,,$$

and the roots are:

$$\lambda_1 = \mathrm{i}\,, \quad \lambda_2 = -\mathrm{i}\,, \quad \lambda_3 = 1\,.$$

According to equation (39), by choosing the determinations of $\sqrt{2}$ in order to get a matrix with real entries, we find:

$$\lambda_1-\lambda_2 = 2\,\mathrm{i}\,, \qquad \lambda_1-\lambda_3 = \mathrm{i}-1\,, \qquad \lambda_2-\lambda_3 = -\mathrm{i}-1\,,$$
$$(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3) = 4\,\mathrm{i}\,,$$
$$\lambda_1^{1/2} = \frac{1+\mathrm{i}}{\sqrt{2}}\,, \qquad \lambda_2^{1/2} = \frac{1-\mathrm{i}}{\sqrt{2}}\,, \qquad \lambda_3^{1/2} = 1\,,$$

$$\lambda_1^{1/2}(\lambda_2-\lambda_3) = \sqrt{2}\,\mathrm{i}\,, \qquad \lambda_2^{1/2}(\lambda_1-\lambda_3) = -\sqrt{2}\,\mathrm{i}\,, \qquad \lambda_3^{1/2}(\lambda_1-\lambda_2) = 2\,\mathrm{i}\,,$$
$$\lambda_1^{-1/2}(\lambda_2-\lambda_3) = -\sqrt{2}\,, \qquad \lambda_2^{-1/2}(\lambda_1-\lambda_3) = -\sqrt{2}\,, \qquad \lambda_3^{-1/2}(\lambda_1-\lambda_2) = 2\,\mathrm{i}\,,$$
$$\lambda_1^{-3/2}(\lambda_2-\lambda_3) = \sqrt{2}\,\mathrm{i}\,, \qquad \lambda_2^{-3/2}(\lambda_1-\lambda_3) = -\sqrt{2}\,\mathrm{i}\,, \qquad \lambda_3^{-3/2}(\lambda_1-\lambda_2) = 2\,\mathrm{i}\,,$$

so that the coefficient of $\mathcal{A}^2$ is: $(1-\sqrt{2})/2$.
Moreover, recalling that $u = v = 1$, by elementary computations, we find the other coefficients in equation (39).
The coefficient of $\mathcal{A}$ is: $\sqrt{2}/2$, and the coefficient of $\mathcal{I}$ is: $1/2$.
Then equation (24) writes:

$$\mathcal{A}^{1/2} = \frac{1-\sqrt{2}}{2}\,\mathcal{A}^2 + \frac{\sqrt{2}}{2}\,\mathcal{A} + \frac{1}{2}\,\mathcal{I}\,, \tag{41}$$

that is:

$$\mathcal{A}^{1/2} = \frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 1 & 1-\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1-\sqrt{2} & 1 \end{bmatrix}. \tag{42}$$

It is easily seen that by equation (42) it follows:

$$\left[\mathcal{A}^{1/2}\right]^2 = \mathcal{A}\,.$$

Remark *6.2***:** Note that in the preceding formulas we could change the determination of $\sqrt{2}$, so that we could find 8 possible values for the square root of $\mathcal{A}$.

6.4. *A few other examples*

• 1. Consider the matrix:

$$\mathcal{A} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{so that} \qquad \mathcal{A}^2 = \begin{bmatrix} 2 & 2 & 1 \\ -3 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}. \tag{43}$$

The invariants are:

$$u_1 = 2\,, \quad u_2 = 2, \quad u_3 = 1\,, \tag{44}$$

and the roots are:

$$\lambda_1 = \frac{1+\mathrm{i}\sqrt{3}}{2}\,, \quad \lambda_2 = \frac{1-\mathrm{i}\sqrt{3}}{2}\,, \quad \lambda_3 = 1\,.$$

A square root is:

$$\mathcal{A}^{1/2} = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 & \sqrt{3}-2 \\ -\sqrt{3} & 3-\sqrt{3} & 3-\sqrt{3} \\ \sqrt{3}-1 & \sqrt{3}-2 & \sqrt{3}-1 \end{bmatrix}. \tag{45}$$

• 2. Consider the matrix:

$$\mathcal{A} = \begin{bmatrix} 7 & 1 & 0 \\ -14 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix}, \qquad \text{so that} \qquad \mathcal{A}^2 = \begin{bmatrix} 35 & 7 & 1 \\ -90 & -14 & 0 \\ 56 & 8 & 0 \end{bmatrix}. \tag{46}$$

The invariants are:

$$u_1 = 7\,, \quad u_2 = 14, \quad u_3 = 8\,,$$

and the roots are:

$$\lambda_1 = 1\,, \quad \lambda_2 = 2\,, \quad \lambda_3 = 4\,.$$

A square root is:

$$\mathcal{A}^{1/2} = \begin{bmatrix} \dfrac{17-6\sqrt{2}}{3} & \dfrac{5-3\sqrt{2}}{3} & \dfrac{4-3\sqrt{2}}{6} \\ -18+10\sqrt{2} & -6+5\sqrt{2} & \dfrac{5\sqrt{2}-6}{2} \\ \dfrac{40-24\sqrt{2}}{3} & \dfrac{16-12\sqrt{2}}{3} & \dfrac{10-6\sqrt{2}}{3} \end{bmatrix} \simeq \tag{47}$$

$$\simeq \begin{bmatrix} 2.8382 & 0.25245 & -0.04044 \\ -3.8578 & 1.07106 & 0.53553 \\ 2.0196 & -0.32352 & 0.50490 \end{bmatrix}.$$

6.5. *Numerical examples with random matrices*

6.5.1. Fourth root of a $\mathbf{3} \times \mathbf{3}$ *non-singular matrix*

Consider the matrix

$$\mathcal{A} = \begin{bmatrix} 87 & 45 & -44 \\ -47 & -24 & 23 \\ 86 & 45 & -43 \end{bmatrix} \tag{48}$$

The invariants are:

$$u_1 = 20\,, \quad u_2 = 67\,, \quad u_3 = 48 \tag{49}$$

and the roots are:

$$\lambda_1 = 16\,, \quad \lambda_2 = 3\,, \quad \lambda_3 = 1\,. \tag{50}$$

We find:

$$\mathrm{A}^{1/4} = \begin{pmatrix} \frac{1}{78}\left(1229 - 621 \times 3^{1/4}\right) & -\frac{45}{13}\left(-2 + 3^{1/4}\right) & \frac{1}{78}\left(-749 + 459 \times 3^{1/4}\right) \\ \frac{1}{13}\left(-145 + 92 \times 3^{1/4}\right) & \frac{2}{13}\left(-27 + 20 \times 3^{1/4}\right) & \frac{1}{13}\left(97 - 68 \times 3^{1/4}\right) \\ \frac{1}{78}\left(1151 - 621 \times 3^{1/4}\right) & -\frac{45}{13}\left(-2 + 3^{1/4}\right) & \frac{1}{78}\left(-671 + 459 \times 3^{1/4}\right) \end{pmatrix}$$

and it results:

$$\left[\mathcal{A}^{1/4}\right]^4 - \mathcal{A} = \mathcal{O}_{3\times 3}\,,$$

that is the 3×3 zero matrix.

6.5.2. Fifth root of a $\mathbf{4} \times \mathbf{4}$ *non-singular matrix*

Consider the matrix

$$\mathcal{A} = \begin{bmatrix} -10 & 46 & 30 & -46 \\ -28 & 166 & 116 & -168 \\ 4 & -21 & -12 & 22 \\ -26 & 143 & 102 & -144 \end{bmatrix} \tag{51}$$

The invariants are:

$$u_1 = 0\,, \quad u_2 = -80\,, \quad u_3 = 0\,, \quad u_4 = 1024 \tag{52}$$

and the roots are:

$$\lambda_1 = -8\,, \quad \lambda_2 = 8\,, \quad \lambda_3 = -4\,, \quad \lambda_4 = 4\,. \tag{53}$$

We find:

$$A^{1/5} = \begin{pmatrix} \frac{-1+25\,(-2)^{1/5}-15\,(-1)^{1/5}+3\times 2^{1/5}}{6\times 2^{3/5}} & \frac{-8-85\,(-2)^{1/5}+24\,(-1)^{1/5}+69\times 2^{1/5}}{12\times 2^{3/5}} & \frac{1-25\,(-2)^{1/5}+3\,(-1)^{1/5}+21\times 2^{1/5}}{6\times 2^{3/5}} & \frac{5+40\,(-2)^{1/5}-9\,(-1)^{1/5}-36\times 2^{1/5}}{6\times 2^{3/5}} \\ \frac{-6+105\,(-2)^{1/5}-110\,(-1)^{1/5}+11\times 2^{1/5}}{6\times 2^{3/5}} & \frac{-48-357\,(-2)^{1/5}+176\,(-1)^{1/5}+253\times 2^{1/5}}{12\times 2^{3/5}} & \frac{6-105\,(-2)^{1/5}+22\,(-1)^{1/5}+77\times 2^{1/5}}{6\times 2^{3/5}} & \frac{5+28\,(-2)^{1/5}-11\,(-1)^{1/5}-22\times 2^{1/5}}{2^{3/5}} \\ -\frac{4+35\,(-2)^{1/5}-40\,(-1)^{1/5}+2^{1/5}}{12\times 2^{3/5}} & \frac{-32+119\,(-2)^{1/5}-64\,(-1)^{1/5}-23\times 2^{1/5}}{24\times 2^{3/5}} & \frac{4+35\,(-2)^{1/5}-8\,(-1)^{1/5}-7\times 2^{1/5}}{12\times 2^{3/5}} & \frac{5-14\,(-2)^{1/5}+6\,(-1)^{1/5}+3\times 2^{1/5}}{3\times 2^{3/5}} \\ \frac{-14+185\,(-2)^{1/5}-190\,(-1)^{1/5}+19\times 2^{1/5}}{12\times 2^{3/5}} & \frac{-112-629\,(-2)^{1/5}+304\,(-1)^{1/5}+437\times 2^{1/5}}{24\times 2^{3/5}} & \frac{14-185\,(-2)^{1/5}+38\,(-1)^{1/5}+133\times 2^{1/5}}{12\times 2^{3/5}} & \frac{35+148\,(-2)^{1/5}-57\,(-1)^{1/5}-114\times 2^{1/5}}{6\times 2^{3/5}} \end{pmatrix}$$

and it results:

$$\left[\mathcal{A}^{1/5}\right]^5 - \mathcal{A} = \mathcal{O}_{4\times 4}\,,$$

that is the 4×4 zero matrix.

6.5.3. Square root of a $\mathbf{5} \times \mathbf{5}$ *non-singular matrix*

Consider the matrix

$$\mathcal{A} = \begin{bmatrix} 1209 & 1210 & -1210 & -1211 & -1 \\ -360 & -359 & 360 & 357 & 1 \\ 1225 & 1226 & -1226 & -1229 & 0 \\ -400 & -400 & 400 & 400 & 0 \\ -1201 & -1201 & 1202 & 1201 & 1 \end{bmatrix} \tag{54}$$

The invariants are:

$$u_1 = 25\,, \quad u_2 = -17\,, \quad u_3 = -425\,, \quad u_4 = 16\,, \quad u_5 = 400 \tag{55}$$

and the roots are:

$$\lambda_1 = 25\,, \quad \lambda_2 = -4\,, \quad \lambda_3 = 4\,, \quad \lambda_4 = -1\,, \quad \lambda_5 = 1\,. \tag{56}$$

We find:

$$\mathsf{A}^{1/2} = \begin{pmatrix} \frac{6\,605\,629}{33\,930} - \frac{351\,881\,i}{7540} & \frac{6\,426\,619}{33\,930} - \frac{281\,681\,i}{7540} & -\frac{3\,291\,019}{16\,965} + \frac{176\,381\,i}{3770} & -\frac{3\,170\,513}{16\,965} + \frac{154\,447\,i}{3770} & -\frac{673}{522} - \frac{54\,i}{29} \\ -\frac{1\,492\,019}{26\,390} + \frac{91\,223\,i}{7540} & -\frac{1\,422\,469}{26\,390} + \frac{72\,503\,i}{7540} & \frac{743\,219}{13\,195} - \frac{44\,423\,i}{3770} & \frac{1\,392\,557}{26\,390} - \frac{17\,981\,i}{1885} & \frac{1221}{2030} + \frac{72\,i}{145} \\ \frac{3\,353\,144}{16\,965} - \frac{1\,169\,497\,i}{22\,620} & \frac{3\,280\,604}{16\,965} - \frac{936\,277\,i}{22\,620} & -\frac{3\,340\,783}{16\,965} + \frac{586\,447\,i}{11\,310} & -\frac{12\,981\,767}{67\,860} + \frac{205\,619\,i}{4524} & -\frac{2669}{2610} - \frac{299\,i}{145} \\ -\frac{1\,526\,920}{23\,751} + \frac{19\,180\,i}{1131} & -\frac{1\,491\,820}{23\,751} + \frac{15\,280\,i}{1131} & \frac{1\,521\,140}{23\,751} - \frac{18\,860\,i}{1131} & \frac{1\,475\,455}{23\,751} - \frac{15\,685\,i}{1131} & \frac{620}{1827} + \frac{20\,i}{29} \\ -\frac{3\,061\,757}{15\,834} + \frac{38\,737\,i}{754} & -\frac{2\,991\,557}{15\,834} + \frac{30\,937\,i}{754} & \frac{1\,529\,057}{7917} - \frac{19\,237\,i}{377} & \frac{1\,467\,538}{7917} - \frac{16\,439\,i}{377} & \frac{1229}{609} + \frac{60\,i}{29} \end{pmatrix}$$

and it results:

$$\left[\mathcal{A}^{1/2}\right]^2 - \mathcal{A} = \mathcal{O}_{5\times 5}\,,$$

that is the 5×5 zero matrix.

Remark *6.3***:** The results contained in Section 6.5, and many others roots of matrices with random entries, have been obtained by Dr. Diego Caratelli by using the computer algebra program Mathematica©.

7. Conclusion

By using a classical result about a representation formula for matrix functions and the basic solution of a linear recurrence relation, it has been shown that the Dunford-Taylor integral allows to define the 2nd kind pseudo-Chenbyshev functions of rational index. These functions can be used in order to compute matrix powers, according to the method presented in the case of the square root of particular matrices. General results relevant to higher order random matrices have been obtained by using the Mathematica© computer algebra program.

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