Lecture Notes of TICMI
Vol. 22, 2021, 73–90

Non-local Contact Problem for Linear Differential Equations with Partial Derivatives of Parabolic Type with Constant and Variable Coefficients

Tinatin Davitashvili[a*] and Hamlet Meladze[b]

[a] *I. Javakhishvili Tbilisi State University, 3 Ilia Tchavtchavadze Avenue, 0179, Tbilisi, Georgia*;
[b] *Muskhelishvili Institute of Computational Mathematics of Georgian Technical University, 4, Peradze St., 0131, Tbilisi, Georgia*

The present work is devoted to the formulation and investigation of a non-local contact problem for a parabolic-type linear differential equation with partial derivatives.

In the first part of the work, the linear parabolic equation with constant coefficients is considered. To solve a non-local contact problem, the variable separation method (Fourier method) is used. Analytical solutions are built for this problem.

One then elaborates on the non-local contact problem for parabolic equations with variable coefficients. Using the iterative method, the existence and uniqueness of the classical solution to the problem is proved. The proof of the existence and uniqueness of the solution is based on the use of the generalized Harnack theorem, which is also valid for linear differential equations with partial derivatives of parabolic type. The effectiveness of the method is confirmed by numerical calculations.

Keywords: Parabolic equation, Nonlocal problem, Contact problem, Fourier method, Iterative method.

AMS Subject Classification: 35K20, 65.

1. Introduction

The formulation and study of non-local problems of mathematical physics is an actual, theoretically and practically interesting direction of computational and applied mathematics [1]-[4]. The study of these problems has been carried out since the beginning of the last century [5]-[8].

After the appearance of remarkable works by J.R. Canon [9], A.V. Bitsadze and A.A. Samarskii [10], intensive studies of non-local problems and their various generalizations began (see [11]-[26] and the references autocited there). The present work is devoted to the formulation and investigation of a non-local contact problem for a parabolic-type linear differential equation with partial derivatives.

In the first part of the work, the linear parabolic equation with constant coefficients is considered. To solve a non-local contact problem, the variable separation method (Fourier method) is used. Analytical solutions are built for this problem.

*Corresponding author. Email: tinatin.davitashvili@tsu.ge, h_meladze@hotmail.com

ISSN: 1512-0511 print

One then elaborates on the non-local contact problem for parabolic equations with variable coefficients. Using the iterative method, the existence and uniqueness of the classical solution to the problem is proved. The proof of the existence and uniqueness of the solution is based on the use of the generalized Harnack theorem, which is also valid for linear differential equations with partial derivatives of parabolic type. The effectiveness of the method is confirmed by numerical calculations.

2. Method of separation of variables

2.1.

We will consider the nonlocal contact problem for one dimensional parabolic equation with constant coefficients.

Find the function

$$u(x,t) = \begin{cases} u^-(x,t), \ 0 \le x \le c, \ t \ge 0, \\ u^+(x,t), \ c \le x \le l, \ t \ge 0, \end{cases}$$

$0 < c < l$, which satisfies the following equations

$$\frac{\partial u^-}{\partial t} = a_1^2 \frac{\partial^2 u^-}{\partial x^2} + f^-(x,t) \ \ 0 < x < c, \ t > 0, \tag{1}$$

$$\frac{\partial u^+}{\partial t} = a_2^2 \frac{\partial^2 u^+}{\partial x^2} + f^+(x,t) \ \ c < x < l, \ t > 0, \tag{2}$$

the initial conditions

$$u^-(x,0) = 0, \ \ 0 \le x \le c, \tag{3}$$

$$u^+(x,0) = 0, \ \ c \le x \le l, \tag{4}$$

the boundary conditions

$$u^-(0,t) = 0, \ \ t \ge 0, \tag{5}$$

$$u^+(l,t) = 0, \ \ t \ge 0, \tag{6}$$

and the nonlocal contact condition

$$u^-(c,t) = u^+(c,t) = u(c,t) = \alpha_1 u^-(c^-,t) + \alpha_2 u^+(c^+,t) + \mu(t), \tag{7}$$

where $\alpha_1 > 0, \ \alpha_2 > 0, \ 0 < c^- < c < c^+ < l.$

2.2.

At first, we will consider the following problem: the equation (1), initial condition (3) and the boundary conditions:

$$u^{-}(0,t)=0,\ t\geq 0,\ \ u^{-}(c,t)=u(c,t),\ t\geq 0, \tag{8}$$

where $u(c,t)$ is so far an unknown function. We introduce the new unknown function [27]:

$$\begin{cases} u^{-}(x,0)=U^{-}(x,t)+z^{-}(x,t) \\ U^{-}(x,t)=\dfrac{x}{c}u(c,t) \end{cases} \tag{9}$$

Then for the function $z^{-}(x,t)$ we get the following problem

$$\begin{cases} \dfrac{\partial z^{-}}{\partial t}=a_1^2\dfrac{\partial^2 z^{-}}{\partial x^2}+\tilde{f}^{-}(x,t),\ \tilde{f}^{-}(x,t)=f^{-}(x,t)-\dfrac{x}{c}u'(c,t),\ 0<x<c, t>0, \\ z^{-}(x,0)=0,\ 0\leq x\leq c,\ \ z^{-}(0,t)=0,\ \ z^{-}(c,t)=0,\ t\geq 0. \end{cases} \tag{10}$$

The solution of problem (10) is the following function [27]:

$$z^{-}(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}\tilde{f}_n^{-}(\tau)d\tau\right\}\sin\frac{\pi n}{c}x,$$

where

$$\tilde{f}_n^{-}(t)=\frac{2}{c}\int_0^c \tilde{f}^{-}(\xi,t)\sin\frac{\pi n}{c}\xi d\xi.$$

If we transform the last expression, we can receive

$$\begin{aligned}\tilde{f}_n^{-}(t)&=\frac{2}{c}\int_0^c f^{-}(\xi,t)\sin\frac{\pi n}{c}\xi d\xi-\frac{2}{c}\int_0^c\frac{x}{c}u'(c,t)\sin\frac{\pi n}{c}\xi d\xi\\ &=\frac{2}{c}\int_0^c f^{-}(\xi,t)\sin\frac{\pi n}{c}\xi d\xi-\frac{2u'(c,t)}{c^2}\int_0^c\xi\sin\frac{\pi n}{c}\xi d\xi.\end{aligned}$$

Let us consider the integral $\int_0^c\xi\sin\frac{\pi n}{c}\xi d\xi$ and use the integration by parts

$$\int_0^c\xi\sin\frac{\pi n}{c}\xi d\xi=-\frac{c^2}{\pi n}\cos\pi n+\left(\frac{c}{\pi n}\right)^2\sin\frac{\pi n}{c}\ |_0^c=(-1)^{n+1}\frac{c^2}{\pi n}.$$

Then the function $\tilde{f}_n^{-}(t)$ can be written in the following form:

$$\tilde{f}_n^{-}(t)=\frac{2}{c}\int_0^c f^{-}(\xi,t)\sin\frac{\pi n}{c}\xi dx-(-1)^{n+1}\frac{2}{\pi n}u'(c,t).$$

Considering the last equality for the solution of problem (10), we obtain the following expression

$$z^-(x,t)=\sum_{n=1}^{\infty}\left\{\frac{2}{c}\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}\left[\int_0^c f^-(\xi,\tau)\sin\frac{\pi n}{c}\xi d\xi\right]dt\right\}\sin\frac{\pi n}{c}x$$

$$-\sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{2}{\pi n}\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}u'(c,\tau)d\tau\right]\sin\frac{\pi n}{c}x$$

$$=\int_0^t\int_0^c G^-(x,\xi,t-\tau)f^-(\xi,\tau)d\xi d\tau$$

$$-\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi n}\sin\frac{\pi n}{c}x\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}u'(c,\tau)d\tau,$$

where the function

$$G^-(x,\xi,t-\tau)=\frac{2}{c}\sum_{n=1}^{\infty}e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}\sin\frac{\pi n}{c}x\sin\frac{\pi n}{c}\xi \tag{11}$$

is called as function of an instant point-source [27].

Thus, for the solution of the problem (1), (3), (8) we received the following expression

$$u^-(x,t)=\frac{x}{c}u(c,t)+\int_0^t\int_0^c G^-(x,\xi,t-\tau)f^-(\xi^-,\tau)d\xi d\tau$$

$$-\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi n}\sin(\frac{\pi n}{c}x)\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}u'(c,\tau)d\tau.$$

Using the integration by parts regarding the integral $\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}u'(c,\tau)d\tau$ from the last expression we get

$$\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}u'(c,\tau)d\tau=u(c,t)-\left(\frac{\pi n}{c}\right)^2 a_1^2\int_0^t e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}u(c,\tau)d\tau.$$

Considering this equality in the expression of $u^-(x,t)$, we get

$$u^-(x,t)=\frac{x}{c}u(c,t)+\int_0^t\int_0^c G^-(x,\xi,t-\tau)f^-(\xi,\tau)d\xi d\tau$$

$$-\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi n}\sin\left(\frac{\pi n}{c}x\right)\right)u(c,t)+\int_0^t k^-(x,t-\tau)u(c,\tau)d\tau, \tag{12}$$

where

$$k^-(x,t-\tau) = \frac{2\pi a_1^2}{c^2}\sum_{n=1}^{\infty}\left((-1)^{n+1} n e^{-(\frac{\pi n}{c})^2 a_1^2 (t-\tau)} sin\left(\frac{\pi n}{c}x\right)\right). \tag{13}$$

The following sum is equal

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin\left(\frac{\pi n}{c}x\right) = \frac{x}{c},$$

(Wolfram Mathematica 10.4, the function Sum[ ]).

Note, that the sum $k^-(x,t-\tau)$ converges, as the sum $\sum_{n=1}^{\infty} n e^{-(\frac{\pi n}{c})^2 a_1^2 (t-\tau)}$ converges.

Finally, we get

$$u^-(x,t) = \int_0^t\int_0^c G^-(x,\xi,t-\tau)f^-(\xi,\tau)d\xi d\tau + \int_0^t k^-(x,t-\tau)u(c,\tau)d\tau. \tag{14}$$

2.3.

Analogously we can consider the problem: the equation (2), the initial condition (4) and the boundary conditions:

$$u^+(c,t) = u(c,t), \ \ u^+(l,t) = 0, \ \ t \geq 0, \tag{15}$$

where $u(c,t)$ is so far an unknown function.

Let us introduce the new unknown function

$$\begin{cases} u^+(x,t) = U^+(x,t) + z^+(x,t), \ \ c \leq x \leq l \\ U^+(x,t) = \dfrac{l-x}{l-c}u(c,t). \end{cases} \tag{16}$$

Then for the function $z^+(x,t)$ we get the following problem

$$\begin{cases} \dfrac{\partial z^+}{\partial t} = a_2^2\dfrac{\partial^2 z^+}{\partial x^2} + \tilde{f}^+(x,t), \ \ \tilde{f}^+(x,t) = f^+(x,t) - \dfrac{l-x}{l-c}u'(x,t) \\ z^+(x,0) = 0, \ c \leq x \leq l, \ \ z^+(c,t) = 0, \ \ z^+(l,t) = 0, \ t \geq 0. \end{cases} \tag{17}$$

The solution of problem (17) is the following function [27]:

$$z^+(x,t) = \sum_{n=1}^{\infty}\left\{\int_0^t e^{-(\frac{\pi n}{1-c})^2 a_2^2 (t-\tau)}\tilde{f}_n^+(\tau)d\tau\right\}\sin\frac{\pi n}{l-c}(l-x), \ \ c \leq x \leq l,$$

where

$$\tilde{f}_n^+(t) = \frac{2}{l-c}\int_c^l \tilde{f}^+(\xi,t)\sin\frac{\pi n}{l-c}(l-\xi)d\xi.$$

Let us convert the last expression for $\tilde{f}_n^+(t)$:

$$\tilde{f}_n^+(t) = \frac{2}{l-c}\int_c^l f^+(\xi,t)\sin\frac{\pi n}{l-c}(l-\xi)d\xi - \frac{2u'(c,t)}{(l-c)^2}\int_c^l (l-\xi)\sin\frac{\pi n}{l-c}(l-\xi)d\xi.$$

At first we consider the integral $\int_c^l (l-\xi)\sin\frac{\pi n}{l-c}(l-\xi)d\xi$ and use the integration by parts regarding it. Then this integral is equal to $(-1)^{n+1}\frac{(l-c)^2}{\pi n}$ and therefore,

$$\tilde{f}_n^+(t) = \frac{2}{l-c}\int_c^l f^+(\xi,t)\sin\frac{\pi n}{l-c}(l-\xi)d\xi - (-1)^{n+1}\frac{2}{\pi n}u'(c,t).$$

Given the last equality, for the function $u^+(x,t)$ we get the following expression:

$$u^+(x,t) = \frac{l-x}{l-c}u(c,t) + \int_0^t\int_c^l G^+(x,\xi,t-\tau)f^+(\xi,\tau)d\xi d\tau$$
$$- \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi n}\sin\frac{\pi n}{l-c}(l-x)\int_0^t e^{-(\frac{\pi n}{l-c})^2 a_2^2(t-\tau)}u'(c,\tau)d\tau,$$

where

$$G^+(x,\xi,t-\tau) = \frac{2}{l-c}\sum_{n=1}^{\infty} e^{-(\frac{\pi n}{c})^2 a_1^2(t-\tau)}\sin\frac{\pi n}{l-c}(l-\xi)\sin\frac{\pi n}{l-c}(l-x). \tag{18}$$

Let us consider the integral $\int_0^t e^{-(\frac{\pi n}{c})^2 a_2^2(t-\tau)}u'(c,\tau)d\tau$ and use the integration by parts. Then we get

$$\int_0^t e^{-(\frac{\pi n}{c})^2 a_2^2(t-\tau)}u'(c,\tau)d\tau = u(c,t) + \left(\frac{\pi n}{l-c}\right)^2 a_2^2\int_0^t e^{-(\frac{\pi n}{c})^2 a_2^2(t-\tau)}u'(c,\tau)d\tau.$$

Considering this equality, for the solution of problem (2), (4), (15) we get the following expression:

$$u^+(x,t) = \frac{l-x}{l-c}u(c,t) + \int_0^t\int_c^l G^+(x,\xi,t-\tau)f^+(\xi,\tau)d\xi d\tau$$
$$- \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi n}\sin\frac{\pi n}{l-c}(l-x)\right)u(c,t) + \int_0^t k^+(x,t-\tau)u(c,\tau)d\tau, \tag{19}$$

where

$$k^{+}(x,t-\tau)=\frac{2\pi a_2^2}{(l-c)^2}\sum_{n=1}^{\infty}(-1)^{n+1}ne^{-(\frac{\pi n}{1-c})^2a_2^2(t-\tau)}\sin\frac{\pi n}{l-c}(l-x). \tag{20}$$

Consider that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi n}\sin\frac{\pi n}{l-c}(l-x)=\frac{l-x}{l-c},$$

(Wolfram Mathematica 10.4, the function Sum[ ]).
Finally we get

$$u^{+}(x,t)=\int_0^t\int_c^l G^{+}(x,\xi,t-\tau)f^{+}(\xi,\tau)d\xi d\tau+\int_0^t k^{+}(x,t-\tau)u(c,\tau)d\tau. \tag{21}$$

2.4.

Let's define function $u(c,t)$, using a nonlocal contact condition (7):

$$u(c,t)=\alpha_1u^{-}(c^{-},t)+\alpha_2u^{+}(c^{+},t)+\mu(t).$$

Considering expressions (14) and (21), we can get

$$u(c,t)=\int_0^t \alpha_1k^{-}(c^{-},t-\tau)+\alpha_2k^{+}(c^{+},t-\tau)]u(c,t)d\tau+\Phi(t),$$

where

$$\Phi(t)=\alpha_1\int_0^t\int_0^c G^{-}(c^{-},\xi,t-\tau)f^{-}(\xi,\tau)d\xi d\tau$$
$$+\alpha_2\int_0^t\int_c^l G^{+}(c^{+},\xi,t-\tau)f^{+}(\xi,\tau)d\xi d\tau+\mu(t). \tag{22}$$

Thus, for the definition of the function $u(c,t)$ we receive an integral equation of Volterra of the second order

$$u(c,\tau)-\int_0^t k(t-\tau)u(c,\tau)d\tau=\Phi(t), \tag{23}$$

where

$$k(t-\tau)=\alpha_1k^{-}(c^{-},t-\tau)+\alpha_2k^{+}(c^{+},t-\tau).$$

As the core of the integral equation $k(t-\tau)$ and the right-side function $\Phi(t)$ are continuous functions, then equation (23) has a unique solution [28].

Thus, we proved the existence and uniqueness of a regular solution of problem (1)-(7), if the functions $f^-(x,t)$, $f^+(x,t)$ and $\mu(t)$ are sufficiently smooth functions. The solution of problem (1)-(7) can be written in the following form:

$$u(x,t)=\begin{cases}\int_0^t\int_0^c G^-(x,\xi,t-\tau)f^-(\xi,\tau)d\xi d\tau+\int_0^t k^-(x,t-\tau)u(c,\tau)d\tau, \\ \hfill 0\le x<c, \\ \\ \int_0^t\int_0^c G^+(x,\xi,t-\tau)f^+(\xi,\tau)d\xi d\tau+\int_0^t k^+(x,t-\tau)u(c,\tau)d\tau, \\ \hfill c<x\le l,\end{cases} \tag{24}$$

where the function $u(c,t)$ is a solution of equation (23), and the functions $G^-(x,\xi,t-\tau)$, $G^+(x,\xi,t-\tau)$, $k^-(x,t-\tau)$, $k^+(x,t-\tau)$ are defined using equalities (11), (18), (13), (20).

Note that the applied technique can be successfully extended in case of more general problems, but in this case the use of the spectral theory of linear operators will be necessary.

3. The problem with variable coefficients

The use of expression (24) for the numerical solution of problem (1)-(7) is related with rather bulky calculations: at first, you need to solve numerically the second-order Volterra integral equation (23) and then calculate the approximate sum of the Fourier series. Therefore, for the numerical solution of problem (1)-(7), the more rational way is to use the iterative method [25].

Let us set the following problem: Find the functions

$$u^-(x,t)\in C^{2,1}(-l<x<0,\ 0<t\le T)\cap C^{1,0}(-l\le x\le 0,\ 0\le t\le T), \tag{25}$$

$$u^+(x,t)\in C^{2,1}(0<x<l,\ 0<t\le T)\cap C^{1,0}(0\le x\le l,\ 0\le t\le T), \tag{26}$$

which satisfy the equations

$$\frac{\partial u^-}{\partial t}=\frac{\partial}{\partial x}\left(k^-(x,t)\frac{\partial u^-}{\partial x}\right)-q^-(x,t)u^-(x,t)=f^-(x,t),$$
$$-l<x<0,\ 0<t\le T, \tag{27}$$

$$\frac{\partial u^+}{\partial t}=\frac{\partial}{\partial x}\left(k^+(x,t)\frac{\partial u^+}{\partial x}\right)-q^+(x,t)u^+(x,t)=f^+(x,t),$$
$$0<x<l,\ 0<t\le T, \tag{28}$$

where $k^-(x,t)$, $k^+(x,t)$, $q^-(x,t)$, $q^+(x,t)$, $f^-(x,t)$, $f^+(x,t)$ are given sufficiently smooth real functions of x, t.

The coefficients $k^-(x,t)$, $k^+(x,t)$ are bounded above and below

$$0 < \sigma_1^- \le k^-(x,t) \le \sigma_2^-, \quad -l \le x \le 0, \quad 0 \le t \le T,$$
$$0 < \sigma_1^+ \le k^+(x,t) \le \sigma_2^+, \quad 0 \le x \le l, \quad 0 \le t \le T,$$
$$q^-(x,t) \ge 0, \quad q^+(x,t) \ge 0.$$

The functions $u^-(x,t)$, $u^+(x,t)$ satisfy the initial conditions

$$u^-(x,0) = u_0^-(x), \ -l \le x \le 0, \ \ u^+(x,0) = u_0^+(x), \ 0 \le x \le l, \tag{29}$$

and boundary conditions

$$u^-(-l,t) = \varphi^-(t), \quad u^+(l,t) = \varphi^+(t), \quad 0 \le t \le T, \tag{30}$$

where $u_0^-(x)$, $u_0^+(x,t)$, $\varphi^-(t)$, $\varphi^+(t)$ are given real functions, $\varphi^-(-l) = u_0^-(-l)$, $\varphi^+(l) = u_0^+(l)$.

The functions $u^-(x,t)$, $u^+(x,t)$ satisfy the following nonlocal contact condition as well:

$$u^-(0,t) = u^+(0,t) = u_0(t) = \sum_{i=1}^{m} \gamma_i^- u^-(\xi_i^-, t) + \sum_{j=1}^{n} \gamma_j^+ u^+(\xi_j^+, t) + \Phi_0(t),$$
$$0 \le t \le T, \tag{31}$$

where

$$-1 < \xi_m^- < \xi_{m-1}^- < \cdots < \xi_1^- < 0 < \xi_1^+ < \xi_2^+ < \cdots < \xi_n^+,$$
$$\gamma_i^- > 0, \ \gamma_i^+ > 0, \ \sum_{i=1}^{m} \gamma_i^- + \sum_{j=1}^{n} \gamma_j^+ \le 1 \tag{32}$$

and $\Phi_0(t)$ is a given real function.

When the conditions (8) and the corresponding requirements, imposed on the initial data of problem (27)-(31) are met, we prove theorems on the existence and uniqueness of a regular solution, as well as we formulate an algorithm for the numerical solution of this problem.

4. Uniqueness of a solution of Problem (27)-(31)

Theorem 4.1 : *Let the conditions, under which a regular solution to problem (27)-(31) exists and inequality (32) be met, then this solution is unique.*

Proof: Suppose the opposite, that is, suppose there are two solutions to the problem $u^-(x,t)$, $u^+(x,t)$ and $v^-(x,t)$, $v^+(x,t)$. Let us consider the functions

$$z^-(x,t) = u^-(x,t) - v^-(x,t), \;\; z^+(x,t) = u^+(x,t) - v^+(x,t).$$

Then these functions are solutions to the following problem

$$\frac{\partial z^-}{\partial t} = \frac{\partial}{\partial x}\left(k^-(x,t)\frac{\partial z^-}{\partial x}\right) - q^-(x,t)z^-(x,t) = 0, \;\; -l < x < 0, \; 0 < t \leq T, \quad (33)$$

$$\frac{\partial z^+}{\partial t} = \frac{\partial}{\partial x}\left(k^+(x,t)\frac{\partial z^+}{\partial x}\right) - q^+(x,t)z^+(x,t) = 0, \;\; 0 < x < l, \; 0 < t \leq T, \quad (34)$$

$$z^-(x,0) = 0, \; -l < x < 0, \;\; z^+(x,0) = 0, \; 0 < x < l, \quad (35)$$

$$z^-(-l,0) = 0, \;\; z^+(l,0) = 0, \;\; 0 \leq t \leq T, \quad (36)$$

$$z^-(0,t) = z^+(0,t) = z_0(t) = \sum_{i=1}^{m} \gamma_i^- z^-(\xi_i^-,t) + \sum_{j=1}^{n} \gamma_j^+ z^+(\xi_j^+,t), \; 0 \leq t \leq T. \quad (37)$$

Using equality (37), the following estimate can be obtained:

$$\max_{0\leq t\leq T} \mid z_0(t) \mid \leq \max_{\substack{1\leq i\leq m \\ 0\leq t\leq T}} \mid z^-(\xi_i^-,t) \mid \sum_{i=1}^{m} \gamma_i^- + \max_{\substack{1\leq j\leq n \\ 0\leq t\leq T}} \mid z^+(\xi_j^+,t) \mid \sum_{j=1}^{n} \gamma_j^+$$

$$\leq \max\left\{ \max_{\substack{1\leq i\leq m \\ 0\leq t\leq T}} \mid z^-(\xi_i^-,t) \mid ; \; \max_{\substack{1\leq j\leq n \\ 0\leq t\leq T}} \mid z^+(\xi_j^+,t) \mid \right\} \left(\sum_{i=1}^{m} \gamma_i^- + \sum_{j=1}^{n} \gamma_j^+ \right).$$

Given the condition (32), one can write

$$\max_{0\leq t\leq T} \mid z_0(t) \mid \leq \max_{\substack{1\leq i\leq m \\ 0\leq t\leq T}} \mid z^-(\xi_i^-,t) \mid \quad or \quad \max_{0\leq t\leq T} \mid z_0(t) \mid \leq \max_{\substack{1\leq j\leq n \\ 0\leq t\leq T}} \mid z^+(\xi_j^+,t) \mid .$$

And this means that either the function $z^-(x,t)$ does not reach its maximum at the border of region $(-l \leq x \leq 0, \; 0 \leq t \leq T)$, or the function $z^+(x,t)$ does not reach its maximum at the border of region $(0 \leq x \leq l, \; 0 \leq t \leq T)$, which contradicts the principle of maximum [29]. Then, given the conditions (35), 36), we can conclude that $z^-(x,t) = 0 \, (-l \leq x \leq 0, \; 0 \leq t \leq T)$ and $z^+(x,t) = 0 \, (0 \leq x \leq l, \; 0 \leq t \leq T)$.□

5. The iterative process for solving Problem (27)-(31)

First, assume that there exists a regular solution to problem (27)-(31), and to solve the problem, consider the following iterative process

$$\left[\frac{\partial u^-}{\partial t}\right]^{(k)} = \left[\frac{\partial}{\partial x}\left(k^-(x,t)\frac{\partial u^-}{\partial x}\right)\right]^{(k)} - q^-(x,t)\left[u^-\right]^{(k)} + f^-(x,t),$$
$$-l < x < 0,\ 0 < t \le T, \quad (38)$$

$$\left[\frac{\partial u^+}{\partial t}\right]^{(k)} = \left[\frac{\partial}{\partial x}\left(k^+(x,t)\frac{\partial u^+}{\partial x}\right)\right]^{(k)} - q^+(x,t)\left[u^+\right]^{(k)} + f^+(x,t),$$
$$0 < x < l,\ 0 < t \le T, \quad (39)$$

$$\left[u^-(x,0)\right]^{(k)} = u_0^-(x),\ -l \le x \le 0,\ \ \left[u^+(x,0)\right]^{(k)} = u_0^+(x),\ 0 \le x \le l, \quad (40)$$

$$\left[u^-(-l,t)\right]^{(k)} = \varphi^-(t),\ \ \left[u^+(l,t)\right]^{(k)} = \varphi^+(t),\ 0 \le t \le T, \quad (41)$$

$$\left[u^-(0,t)\right]^{(k)} = \left[u^+(0,t)\right]^{(k)} = [u_0(t)]^{(k)} = \sum_{i=1}^{m}\gamma_i^-\left[u^-(\xi_i^-,t)\right]^{(k-1)}$$
$$+\sum_{j=1}^{n}\gamma_j^+\left[u^+(\xi_j^+,t)\right]^{(k-1)} + \Phi_0(t),\ 0 \le t \le T, \quad (42)$$

where k=0,1,2, and $\left[u^-(\xi_i^-,t)\right]^{(-1)} = 0,\ i = \overline{1,m},\ \left[u^+(\xi_j^+,t)\right]^{(-1)} = 0,\ j = \overline{1,n}.$

Theorem 5.1: *If there exists a regular solution of problem (27)-(31) and conditions (32) are met, then the iterative process (38)-(42) converges to this solution at the rate of infinitely decreasing geometric progression.*

Proof: For the error

$$\left[z^-(x,t)\right]^{(k)} = \left[u^-(x,t)\right]^{(k)} - u^-(x,t),\ \ -l \le x \le 0,\ 0 \le t \le T,$$

$$\left[z^+(x,t)\right]^{(k)} = \left[u^+(x,t)\right]^{(k)} - u^+(x,t),\ \ 0 \le x \le l,\ 0 \le t \le T,$$

where $u^-(x,t),\ u^+(x,t)$ are the solutions of (27)-(31), we receive the following problem:

$$\left[\frac{\partial z^-}{\partial t}\right]^{(k)} = \left[\frac{\partial}{\partial x}\left(k^-(x,t)\frac{\partial z^-}{\partial x}\right)\right]^{(k)} - q^-(x,t)\left[z^-\right]^{(k)},$$

$$-l < x < 0,\ 0 < t \le T, \quad (43)$$

$$\left[\frac{\partial z^+}{\partial t}\right]^{(k)} = \left[\frac{\partial}{\partial x}\left(k^+(x,t)\frac{\partial z^+}{\partial x}\right)\right]^{(k)} - q^+(x,t)\left[z^+\right]^{(k)},$$

$$0 < x < l,\ 0 < t \le T, \quad (44)$$

$$\left[z^-(x,0)\right]^{(k)} 0,\ -l \le x \le 0,\ \ \left[z^+(x,0)\right]^{(k)} = 0,\ 0 \le x \le l,$$

$$\left[z^-(-l,t)\right]^{(k)} = 0,\ \ \left[z^+(l,t)\right]^{(k)} = 0,\ 0 \le t \le T, \quad (45)$$

$$\left[z^-(0,t)\right]^{(k)} = \left[z^+(0,t)\right]^{(k)} = [z_0(t)]^{(k)} = \sum_{i=1}^{m}\gamma_i^-\left[z^-(\xi_i^-,t)\right]^{(k-1)}$$

$$+\sum_{j=1}^{n}\gamma_j^+\left[z^+(\xi_j^+,t)\right]^{(k-1)},\ 0 \le t \le T, \quad (46)$$

If we use the Schwartz lemma, we can write

$$\max_{\substack{1\le i\le m\\ 0\le t\le T}} \mid [z^-(\xi_i^-,t)]^{(k-1)} \mid \le q^- \max_{0\le t\le T} \mid [z_0(t)]^{(k)} \mid,$$

$$\max_{\substack{1\le j\le n\\ 0\le t\le T}} \mid [z^+(\xi_j^+,t)]^{(k-1)} \mid \le q^+ \max_{0\le t\le T} \mid [z_0(t)]^{(k)} \mid,$$

where $0 < q^- < 1,\ 0 < q^+ < 1$.

We will use the non-local contact condition

$$\max_{0\le t\le T} \mid [z_0(t)]^{(k)} \mid \le \max_{\substack{1\le i\le m\\ 0\le t\le T}} \mid [z^-(\xi_i^-,t)]^{(k-1)} \mid \sum_{i=1}^{m}\gamma_i^-$$

$$+\max_{\substack{1\le j\le n\\ 0\le t\le T}} \mid [z^+(\xi_j^+,t)]^{(k-1)} \mid \sum_{j=1}^{n}\gamma_j^+ \le Q \max_{0\le t\le T} \mid [z_0(t)]^{(k-1)} \mid, \quad (47)$$

where $Q = q^- \sum_{i=1}^{m} \gamma_i^- + q^+ \sum_{j=1}^{n} \gamma_j^+$.

Given the condition (32), we can conclude that $0 \leq Q \leq 1$. Then we get

$$\lim_{k \to \infty} [z_0(t)]^{(k)} = 0.$$

Thus, if there exists a solution to problem (27)-(31), then based on the principle of maximum we get

$$\max_{\substack{-l \leq x \leq 0 \\ 0 \leq t \leq T}} \mid [u^-(x,t)]^{(k)} - u^-(x,t) \mid = O\left(Q^k\right),$$

$$\max_{\substack{0 \leq x \leq l \\ 0 \leq t \leq T}} \mid [u^+(x,t)]^{(k)} - u^+(x,t) \mid = O\left(Q^k\right).$$

□

Remark *5.2***:** The iterative process (38)-(42) gives us the opportunity to reduce the solution of the problem (27)-(31) to a sequence of solutions to the Cauchy-Dirichlet problems for equations of the parabolic type with variable coefficients. Thus, one can obtain an algorithm for the numerical solution of the problem (27)-(31).

6. Existence of a regular solution

Now we prove the existence of a regular solution to problem (27)-(31).

We will use the iterative process (38)-(42). Introduce the notations

$$\left[\varepsilon^-(x,t)\right]^{(k)} = \left[u^-(x,t)\right]^{(k)} - \left[u^-(x,t)\right]^{(k-1)},$$

$$\left[\varepsilon^+(x,t)\right]^{(k)} = \left[u^+(x,t)\right]^{(k)} - \left[u^+(x,t)\right]^{(k-1)},$$

It is evident that the functions $[\varepsilon^-(x,t)]^{(k)}$ and $[\varepsilon^+(x,t)]^{(k)}$ are the solutions to problem (43)-(46). Then, similar to the estimate (47), we get the following inequality

$$\max_{0 \leq t \leq T} \mid [\varepsilon_0(0,t)]^{(k)} \mid \leq Q \max_{0 \leq t \leq T} \mid [\varepsilon_0(0,t)]^{(k-1)} \mid, \;\; 0 < Q < 1.$$

And this means that

$$u_0^{(k)}(t) - u_0^{(k-1)} \to 0, \; if \; k \to \infty.$$

Thus, we get that the sequence $\left\{u_0^{(k)}(t)\right\}$ uniformly converges. Then based on Harnack's theorem [29] function $[u^-(x,t)]^{(k)}$ and $[u^+(x,t)]^{(k)}$, respectively in areas

$(-l \le x \le 0,\ 0 \le t \le T)$, $(0 \le x \le l,\ 0 \le t \le T)$, converges to functions $u^-(x,t)$ and $u^+(x,t)$, which are the solution to problem (27)-(31).

Thus, we proved the following theorem:

Theorem 6.1: *If $f^-(x,t)$, $f^+(x,t)$, $k^-(x,t)$, $k^+(x,t)$, $q^-(x,t)$, $q^+(x,t)$ sufficiently smooth functions, then there exists a regular solution to problem (27)-(31).*

7. Numerical example

We consider the following problem as an example: find the function $u(x,t)$ in the area $\{(x,t) \mid 0 \le x \le 1,\ 0 \le t \le T\}$, $T = 6$:

$$u(x,t) = \begin{cases} u^-(x,t),\ if\ 0 \le x \le 0.5,\ 0 \le t \le 6 \\ u^+(x,t),\ if\ 0.5 \le x \le 1,\ 0 \le t \le 6, \end{cases}$$

which satisfies the equations

$$\frac{\partial u^-}{\partial t} = \frac{\partial}{\partial x}\left((1+2x)\frac{\partial u^-}{\partial x}\right) + f^-(x,t),\ 0 < x < 0.5,\ 0 < t \le T,$$

$$\frac{\partial u^+}{\partial t} = \frac{\partial}{\partial x}\left((1+x)\frac{\partial u^+}{\partial x}\right) + f^+(x,t),\ 0.5 < x < 1,\ 0 < t \le T,$$

the conditions

$$u^-(x,0) = 0,\ 0 \le x \le 0.5,\ \ u^+(x,0) = 0,\ 0.5 \le x \le 1,$$

$$u^-(0,t) = 0,\ u^+(1,t) = 0,\ 0 \le t \le T,$$

and the nonlocal contact condition

$$u(0.5,t) = 0.25u^-(0.25,t) + 0.25u^+(0.75,t) + \frac{169}{256}(1 - e^{-t}),\ 0 \le t \le T,$$

where

$$f^-(x,t) = e^{-t}(-120 + 417x + 1611x^2 - 3040x^3 - 100x^4) + 8(15 - 51x - 210x^2 + 400x^3),$$

$$f^+(x,t) = \frac{16}{3}e^{-t}(2 + 4x - 11x^2 - 16x^3 - x^4) + \frac{32}{3}(-1 - 2x + 6x^2 + 8x^3).$$

The exact solution of this problem is

$$u(x,t) = \begin{cases} (100x(1-x)(0.3-x)^2(1-e^{-t}),\ 0 \le x \le 0.5,\ 0 \le t \le T, \\ \frac{16}{3}x^2(1-x^2)(1-e^{-t}),\ 0.5 \le x \le 1,\ 0 \le t \le T. \end{cases}$$

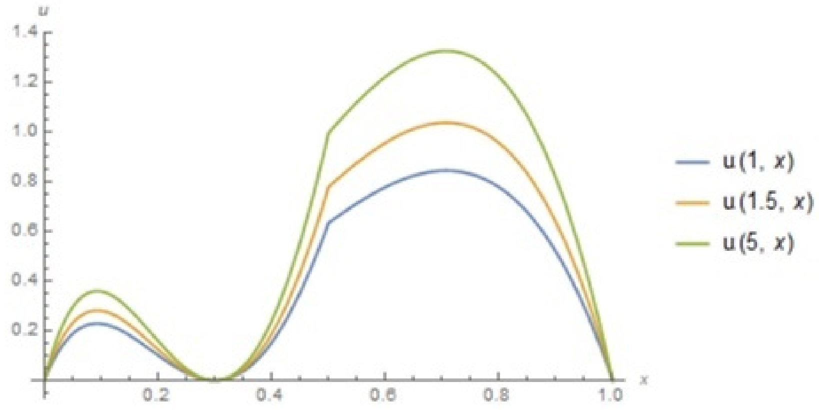

Figure 1. Exact solution for different values of t.

To solve this problem, we consider the following iteration process:

$$\frac{\partial[u^-]^{(k)}}{\partial t} = \frac{\partial}{\partial x}\left((1+2x)\frac{\partial[u^-]^{(k)}}{\partial x}\right) + f^-(x,t),\ 0 < x < 0.5,\ 0 < t \leq T,$$

$$\frac{\partial[u^+]^{(k)}}{\partial t} = \frac{\partial}{\partial x}\left((1+x)\frac{\partial[u^+]^{(k)}}{\partial x}\right) + f^+(x,t),\ 0.5 < x < 1,\ 0 < t \leq T,$$

$$[u^-(x,0)]^{(k)} = 0,\ 0 \leq x \leq 0.5,\ \ [u^+(x,0)]^{(k)} = 0,\ 0.5 \leq x \leq 1,$$

$$[u^-(0,t)]^{(k)} = 0,\ \ [u^+(l,t)]^{(k)} = 0,\ 0 \leq t \leq 3,$$

and the nonlocal contact condition

$$[u(0.5,t)]^{(k)} = 0.25[u^-(0.25,t)]^{(k-1)} + \\ + 0.25[u^+(0.75,t)]^{(k-1)} + \frac{169}{256}(1-e^{-t}),\ 0 \leq t \leq T,$$

where $k = 1, 2, \ldots$ and the initial values for $u^{(k)}(x,t)$ are equal to 0.

Below one can see the figures of approximate solution and respective absolute error for $k = 1$ and $k = 7$.

The absolute error decreases as $O(Q^k)$, where $Q = q^+ \sum_{i=1}^{m} \beta_i^+ + q^- \sum_{j=1}^{n} \beta_j^- < 1/4$ for this example, and $0 < q^-$, $q^+ < 1$.

The figure below compares the absolute error (C-norm) with its theoretical value $(1/4)^k$

We considered the case when $\gamma_i^- > 0$, $\gamma_i^+ > 0$, $\sum_{i=1}^{m} \gamma_i^- + \sum_{j=1}^{n} \gamma_j i^+ < 1$.

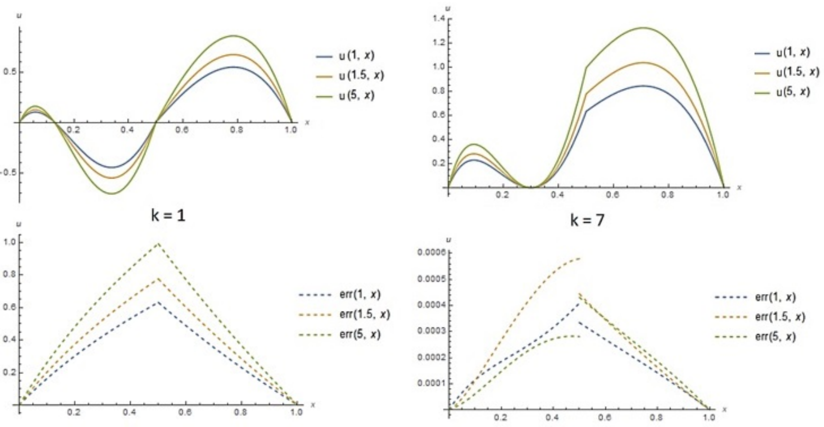

Figure 2. Approximate solution and absolute errors for $k = 1$, $k = 7$, for different t.

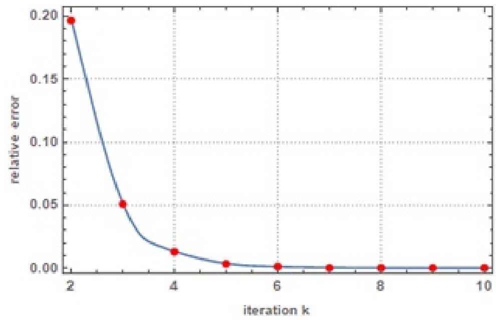

Figure 3. The relative error, $\frac{\|u_{exact}-u_{appr}\|_C}{\|u_{exact}\|_C}$ versus iteration k.

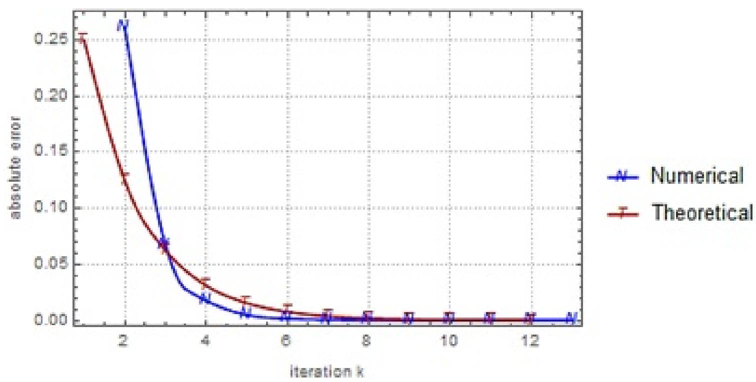

Figure 4. Absolute error (numerical) and Q^k (theoretical) versus iteration k.

Let us consider the example with nonlocal condition, where $\sum_{i=1}^{m}\gamma_i^- + \sum_{j=1}^{n}\gamma_j^+ =$

1, $T = 3$ and the nonlocal condition has the form:

$$[u(0.5,t)]^{(k)} = 0.5[u^-(0.25,t)]^{(k-1)} + \\ + 0.5[u^+(0.75,t)]^{(k-1)} + \frac{41}{128}(1-e^{-t}),\ 0 \le t \le T,$$

where $k = 1, 2, \ldots$ and the initial values for $u^{(k)}(x,t)$ are equal to 0. In this case the convergence is achieved as well:

Table 1. Relative error		
Iteration k	Relative error	Relative error,%
2	0.39321	39.32
4	0.10602	10.6
6	0.02982	2.98
8	0.00805	0.81
10	0.002460	0.25
12	0.00093	0.09

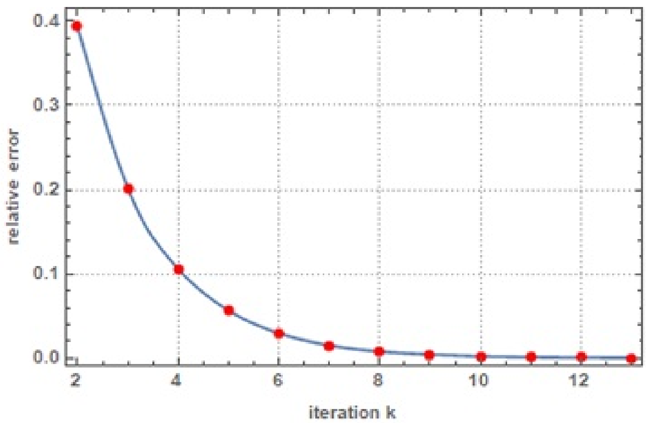

Figure 5. The relative error, $\frac{\|u_{exact} - u_{appr}\|_C}{\|u_{exact}\|_C}$ versus iteration k.

8. Conclusion

The present article investigates a new type of the nonlocal contact problem for parabolic equations. In section 2, the method of separation of variables is used for one dimensional parabolic equation with constant coefficients to construct the analytic solution of a given nonlocal contact problem and provide the existence and uniqueness of a regular solution.

In section 3, for an equation with variable coefficients the existence and uniqueness of the solution of a nonlocal contact problem is proved. A convergent iterative procedure is constructed to find the numerical solution of the considered problem.

In sections 3-6, for an equation with variable coefficients the existence and uniqueness of the solution of a nonlocal contact problem is proved and the convergent iterative procedure is constructed to find the numerical solution of the considered problem. For the equation with variable coefficients convergence is achieved under the more general conditions $\sum\limits_{i=1}^{m} \gamma_i^- + \sum\limits_{j=1}^{n} \gamma_j^+ \leq 1$. The technique used in the present article can also be applied for the problems with elliptic type equations.

References

[1] V.V. Shelukin, *A non-local in time model for radionuclide propagation in Stokes fluid*, Dinamika Splosh. Sredy, **107** (1993), 180-193

[2] A.M. Nakhushev, *Equations of Mathematical Biology* (Russian), Moscow, "Visshaia shkola", 1995, p. 302

[3] C.V. Pao, *Reaction diffusion equations, with nonlocal boundary and nonlocal initial conditions*, J. Math. Anal. Appl., **195** (1995), 702-718

[4] O. Diaz, Jesus Ildefonso, Rakotoson, Jean-Michell, *On a non-local stationary free-boundary problem arising in the confinement of a plasma in a stellarator geometry*, Arch. Rational. Mech. Anal., **134**, 1 (1996), 53-95

[5] E. Hilb, *Zur Theorie der Enwicklungen willkurlicher Funktionen nach Eigenfunctionen*, Mathematcishe Zeitscherift 1, Band, 1918

[6] T. Carleman, Sur la Tehorie des Equatuibs Integrals et ses Applications, Verh. Internat. Math. Kongr., Zurich, 1932, 1, (Orell Fussli, Zurich) (1933), 138-151

[7] R. Beals, *Nonlocal elliptic boundary value problems*, Bull. Amer. Math. Soc., **70**, 5 (1964), 693-696

[8] F.E. Browder, *Non-local elliptic boundary value problems*, Amer. J. Math, **86** (1964), 735-750

[9] J.R. Canon, *The solution of heat equation subject to the specification of energy*, Quart. Appl. Math., **21** (1963), 155-160

[10] A.V. Bitsadze, A.A. Samarskii, *On some simpliest generalised linear elliptic problems* (Russian), Dokl. AN SSSR, **185**, 2 (1969), 739-740

[11] M.P. Sapagovas, R.U. Chechis, *On some boundary problems with nonlocal conditions* (Russian), Diff. Uravnenia, **23**, 7 (1987), 1268

[12] A. Bouziani, *On a class of parabolic equations with a non-local boundary conditions*, Acad. Roy. Belg. Bull. CI, Sci. **(6) 10**, 1-6 (1999), 61-67

[13] D.G. Gordeziani, G.A. Avalishvili, *On the constructing of solutions of the nonlocal initial boundary value problems for one dimensional medium oscillation equations*, Mathem. Mod., **12**, 1 (2000), 93-103

[14] S. Mesloub, S. Messaoudi, *A nonlocal mixed semilinear problem for second order hyperbolic equations*, Electr. Journal of Diff. Equat., **30** (2003), 1-17

[15] F. Shakeris, M. Dehghan, *The method of lines for solution of the one-dimensional wave equation subject to an integral consideration*, Comput. and Math. with Appl., **56** (2008), 2175-2188.

[16] A. Ashyralyev, Gercek, Okan, *Nonlocal Boundary Value Problem for Elliptic-Parabolic Differential and Difference Equations*, Discrete Dyn. Nat. Soc. (2008), Art. ID 904824, 16 p.

[17] M.P. Sapagovas, *A difference method of increased order of accuracy for the poisson equation with nonlocal conditions*, Diff. Uravn., **44**, 7 (2008), 988-998

[18] G. Gordeziani, N. Gordeziani, G. Avalishvili, *Non-local boundary value problem for some partial differential equations*, Bulletin of the Georgian Academy of Sciences, **157**, 1 (1998), 365-369

[19] F. Criado, H. Meladze, N. Odishelidze, *An optimal control problem for helmholtz equation with nonlocal boundary conditions and quadratic functional*, Rev. R. Acad. Scienc. Exact. Fis. Mat. (Esp.), **91**, 1 (1997), 65-69

[20] A.V. Gulin, V.A. Morozova, *A family of selfjoint difference schemes*, Diff. Urav., **44**, 9 (2008), 1249-1254

[21] N.I. Ionkin, *Solution of boundary-value problem in heat-conduction theory with non-classical boundary conditions*, Diff, Urav., **13** (1977), 1177-1182

[22] D. Gordeziani, G. Avalishvili, *Time nonlocal problems for schrodinger type equations*, I and II parts Diff, Urav., **41**, 5,6, (2005), 703-711 and 852-859

[23] G. Gordeziani, H. Meladze, G, Avalishvili, *On One class of nonlocal in time problems for first order evolution equations*,Jurn. Vich. I Prikl. Mat., **1**, 88 (2003), 66-78

[24] I.P. Gavrilyuk, V.L. Makarov, D.O. Sytnyk, V.B. Basylyk, *Exponentially Convergent Method for m-point Nonlocal Problem for a First Order Differential Equation in Banach Space*, Numerical functional analysis and optimization **31**, 1 (2010), 1-21

[25] T. Davitashvili, K. Meladze, N. Skhirtladze, *"About one parallel algorithm of solving non-local contact problem for parabolic equations"*, Computer Science and Information Technologies, CSIT, (2017),145-149, doi: 10.1109/CSITechnol.2017.8312159

[26] D. Gordeziani, I. Meladze, *On a nonlocal contact problem*, Bulletin of the Georgian Academy of Sciences, **8**, 1 (2014), 40-46

[27] A.N. Tikhonov, A.A. Samarskii, *Equations of Mathematical Physics*, Dover Publications Inc., New York, (transl. from Russian), 1990

[28] A. Bitsadze, *Equations of Mathematical Physics*. M.: Nauka, 1982

[29] A. Friedman, *Partial Differential Equation of Parabolic Type*, Dover Publications Inc., 2008