The present Lecture Notes contains extended material based on the lectures presented at the Workshop on Mathematical Methods for Elastic Cusped Plates and Bars (Tbilisi, September 27–28, 2001).

It consists of two parts. The first one is devoted to cusped plates, while the second one deals with cusped beams.

For convenience of the readers the work is organized so that each part is self–contained and can be read independently.

We apply I.Vekua's dimension reduction method to construct variational hierarchical models for elastic cusped plates and beams. The corresponding two– and one–dimensional models are studied in appropriately chosen weighted Sobolev spaces. Moreover, we show the convergence of the sequence of approximate solutions (corresponding to the hierarchical models) to the exact solution of the original three-dimensional problem.

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## Preface

The present Lecture Notes contains extended material based on the lectures presented at the Workshop on Mathematical Methods for Elastic Cusped Plates and Bars (Tbilisi, September 27–28, 2001).

It consists of two parts. The first one is devoted to cusped plates, while the second one deals with cusped beams.

For the readers convenience the work is organized so that each part is self–contained and can be read independently.

In Part 1 we construct variational hierarchical two–dimensional models for cusped elastic plates. With the help of variational methods, existence and uniqueness theorems for the corresponding two–dimensional boundary value problems are proved in appropriate weighted function spaces. By means of the solutions of these two–dimensional boundary value problems, a sequence of approximate solutions in the corresponding three-dimensional region is constructed. We establish that this sequence converges in the Sobolev space $H^1$ to the solution of the original three-dimensional boundary value problem. The systems of differential equations corresponding to the two-dimensional variational hierarchical models are explicitly given for a general function system and for Legendre polynomials, in particular.

In Part 2 variational hierarchical one–dimensional models are constructed for cusped elastic beams. With the help of the variational methods the existence and uniqueness theorems for the corresponding one–dimensional boundary value problems are proved in appropriate weighted function spaces. By means of the solutions of these one–dimensional boundary value problems the sequence of approximate solutions in the corresponding three-dimensional region is constructed. It is established that this sequence converges (in the sense of the Sobolev space $H^1$) to the solution of the original three-dimensional boundary value problem.

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Authors

# Contents

**PART 2**

# PART 1

## HIERARCHICAL MODELS FOR CUSPED PLATES

In Part 1 we construct variational hierarchical two–dimensional models for cusped elastic plates. With the help of variational methods, existence and uniqueness theorems for the corresponding two–dimensional boundary value problems are proved in appropriate weighted function spaces. By means of the solutions of these two–dimensional boundary value problems, a sequence of approximate solutions in the corresponding three-dimensional region is constructed. We establish that this sequence converges in the Sobolev space $H^1$ to the solution of the original three-dimensional boundary value problem. The systems of differential equations corresponding to the two-dimensional variational hierarchical models are explicitly given for a general system and for Legendre polynomials, in particular.

# List of Notation

$\mathbb{N} := \{1, 2, \cdots\}, \qquad \mathbb{N}_0 := \{0, 1, 2, \cdots\}$

$\mathbb{R}^m$ $\quad m-$dimensional Euclidean space $(m \in \mathbb{N})$

$$\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \quad x := (x_1, x_2) \in \omega \subset \mathbb{R}^2, \quad \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x) \right\}$$

$\overline{\Omega} = \Omega \cup \partial\Omega$ $\quad$ prismatic shell of variable thickness

$\overline{\omega} \subset R^2$ projection of a prismatic shell $\overline{\Omega}$

$$\Gamma := \left\{ (x, x_3) \in \mathbb{R}^3 : \quad x \in \partial\omega, \ \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x) \right\}$$

$\gamma$ $\quad$ projection of $\Gamma$ on $\partial\omega$, $\gamma_0 := \partial\omega \backslash \bar{\gamma}$

$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x) \begin{cases} > 0, & x \in \omega \\ \geq 0, & x \in \partial\omega \end{cases}$ thickness of a prismatic shell at the point $x \in \bar{\omega}$

$$S^{\pm} := \left\{ \left( x, \overset{(\pm)}{h}(x) \right) \in \mathbb{R}^3 : \quad x \in \omega \right\}$$

$$2\widetilde{h}(x) := \overset{(+)}{h}(x) + \overset{(-)}{h}(x)$$

$$a(x) := \frac{1}{h(x)}$$

$$b(x) := \frac{\widetilde{h}(x)}{h(x)}$$

$$\Delta_m := \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_m^2}$$

$P_n$ $\quad$ Legendre polynomial of order $n$

$C^\infty(\Omega, \Gamma) := \{\varphi \in C^\infty(\Omega) : \varphi|_\Gamma = 0, \ \Gamma \subset \partial\Omega\}$

$H^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n) = W^s(\mathbb{R}^n)$ $\quad$ Bessel potential and Sobolev-Slobodetski spaces on $\mathbb{R}^n (s \in \mathbb{R})$

$H^s(\Omega) = W^s(\Omega)$ space of restrictions to $\Omega \subset \mathbb{R}^n$ of distributions from $H^s(\mathbb{R}^n)$

$\widetilde{H}^s(\Omega) := \{\varphi \in H^s(\mathbb{R}^n) : \text{supp } \varphi \subset \bar{\Omega}\}$ $\ (\text{s} \in \mathbb{R})$

$H^s(\partial\Omega)$ Sobolev-Slobodetski space on $\partial\Omega$ $(s \in \mathbb{R})$

$H^s(S^{\pm})$ space of restrictions to $S^{\pm}$ of distributions from $H^s(\partial\Omega)$ $(s \in \mathbb{R})$

$H^s(\Omega, \Gamma) := \{\varphi \in H^s(\Omega) : \ \varphi = 0 \ \text{on} \ \Gamma\}$ $(\text{s} \in \mathbb{R})$

$u := (u_1, u_2, u_3)^\top$ displacement vector

$e_{ij}(u) := \frac{1}{2}(u_{j,i} + u_{i,j}), \quad i, j = 1, 2, 3,$ strain tensor

$\sigma_{ij}(u) := \lambda \delta_{ij} e_{kk}(u) + 2\mu e_{ij}(u), \quad i, j = 1, 2, 3,$ stress tensor

$\lambda, \mu$ Lamé constants

$T(\partial, n)u$ surface stress vector

$[T(\partial, n)u]_j := \sigma_{ji}(u) n_i$ the $j-$th component of the vector $T(\partial, n)u$

$(\cdot)^\top$ transposition operation

$X_1 \times X_2 \times \cdots \times X_m$ direct product of spaces $X_j, \ j = 1, ..., m$

$X^m := \underbrace{X \times \cdots \times X}_{m \ times}$

$\partial := (\partial_1, \partial_2, \cdots, \partial_n)$

$\partial_j := \frac{\partial}{\partial x_j}, \ j = 1, \cdots, n$

$u_{i,j}(u) := \frac{\partial u_i}{\partial x_j}, \quad i, j = 1, 2, 3$

$u_{i,jk}(u) := \frac{\partial^2 u_i}{\partial x_j \partial x_k}, \quad i, j, k = 1, 2, 3$

$C^m(\Omega)$ $(C^m(\bar{\Omega}))$ $m$ times continuously differentiable functions in $\Omega$ $(\bar{\Omega})$

$C(\Omega) := C^0(\Omega), \quad C(\bar{\Omega})) := C^0(\bar{\Omega})$

$C^{m,\kappa}(\Omega)$ $(C^{m,\kappa}(\bar{\Omega}))$ $m$ times continuously differentiable functions

whose $m$–th order derivatives are Hölder continuous in $\Omega$ $(\bar{\Omega})$ with the exponent $\kappa \in (0, 1]$

$C^{0,1}(\Omega)$ $(C^{0,1}(\bar{\Omega}))$ space of Lipschitz continuous functions in $\Omega$ $(\bar{\Omega})$

# 1 Introduction

In the fifties of the twentieth century, I.Vekua [33] introduced a new mathematical model for elastic prismatic shells (i.e., of plates of variable thickness) which was based on expansions of the three–dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier-Legendre series with respect to the variable of plate thickness. By taking only the first $N+1$ terms of the expansions, he introduced the so–called $N$–th approximation. Each of these approximations for $N = 0, 1, ...$ can be considered as an independent mathematical model of plates. In particular, the approximation for $N = 1$ corresponds to the classical Kirchhoff–Love plate model. In the sixties, I. Vekua [34] developed the analogous mathematical model for thin shallow shells. All his results concerning plates and shells are collected in his monograph [71]. Works of I. Babuška, D. Gordeziani, V. Guliaev, I. Khoma, A. Khvoles, T. Meunargia, C. Schwab, T. Vashakmadze, V. Zhgenti, and others (see [5], [16], [17], [43], [44], [50], [60], [68], [72] and the references therein) are devoted to further analysis of I.Vekua's models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (to non-shallow shells, to the anisotropic case, etc.). At the same time, I. Vekua recommended to investigate also cusped plates, i.e., plates whose thickness vanishes on some part or on the whole boundary of the plate projection (for corresponding investigations see the survey [12] and also I. Vekua's comments in [71, p.86]).

In 1957 E. Makhover [47], [48], by using the results of S. Mikhlin [51], was the first who considered such a cusped plate with the stiffness $D(x_1, x_2)$ satisfying

$$D_1 x_2^{\kappa_1} \leq D(x_1, x_2) \leq D_2 x_2^{\kappa_1}, \ \ D_1, D_2, \ \kappa_1 = const > 0, \tag{1.1}$$

within the framework of classical bending theory. She particularly studied in which cases the deflection ($\kappa_1 < 2$) or its normal derivative ($\kappa_1 < 1$) on the cusped edge of the plate can be given. In 1971, A. Khvoles [44] represented the forth order Airy stress function operator as the product of two second order operators in the case when the plate thickness $2h$ is given by

$$2h = h_0 x_2^{\kappa_2}, \ h_0, \kappa_2 = const > 0, \ x_2 \geq 0, \tag{1.2}$$

and investigated the general representation of corresponding solutions. Since 1972 the works of G. Jaiani [13]–[35] are also devoted to these problems. By using more natural spaces than E. Makhover, G. Jaiani in [32] has analyzed in which cases the cusped edge can be freed ($\kappa_1 > 0$) or freely supported ($\kappa_1 < 2$). Moreover, he established well–posedness and the correct formulation of all admissible principal boundary value problems (BVPs). In [25], [26], [31] he also investigated the tension–compression problem of cusped plates, based on I. Vekua's model of shallow prismatic shells ($N = 0$). G. Jaiani's results can be summarized as follows.

Let $\nu$ be the inward normal of the plate's boundary. In the case of the tension-compression ($N = 0$) problem on the cusped edge, where

$$0 \leq \frac{\partial h}{\partial \nu} < +\infty \quad \text{(in the case (1.2) this means } \kappa_2 \geq 1) \,,$$

which will be called a sharp cusped edge, one can not prescribe the displacement vector; while on the cusped edge, where

$$\frac{\partial h}{\partial \nu} = +\infty \quad \text{(in the case (1.2) this means } \kappa_2 < 1) \,,$$

called a blunt cusped edge, the displacement vector can be prescribed. In the case of the classical bending problem with a cusped edge, where

$$\frac{\partial h}{\partial \nu} = O(d^{\kappa-1}) \ \text{ as } d \to 0, \ \kappa = const > 0 \tag{1.3}$$

and where $d$ is the distance between an interior reference point of the plate projection and the cusped edge, the edge can not be fixed if $\kappa \geq \frac{1}{3}$, but it can be fixed if $0 < \kappa < \frac{1}{3}$; it can not be freely supported if $\kappa \geq \frac{2}{3}$, and it can be freely supported if $0 < \kappa < \frac{2}{3}$; it can be free or arbitrarily loaded by a shear force and a bending moment if $\kappa > 0$. Note that in the case (1.2), the condition (1.3) implies that $d = x_2$ and $\kappa = \kappa_2 = \frac{\kappa_1}{3}$.

For the specific cases of cusped cylindrical and conical shell bending, the above results remain valid as it has been shown by G. Tsiskarishvili and N. Khomasuridse [63]-[66]. These results also remain valid in the case of classical bending of orthotropic cusped plates (see [35]). However, for general cusped shells and also for general anisotropic cusped plates, corresponding analysis is yet to be done.

The problems involving cusped plates lead to correct mathematical formulations of BVPs for even order elliptic equations and systems whose orders degenerate at the boundary (see [31], [36]-[39]).

Applying the functional–analytic method developed by G. Fichera in [12], [13] (see also [10], [11]), in [31] the particular case ($\lambda = \mu$) of Vekua's system for general cusped plates has been investigated.

The classical bending of plates with the stiffness (1.1) in energetic and in weighted Sobolev spaces has been studied by G. Jaiani in [32], [34]. In the energetic space some restrictions on the lateral load has been relaxed by G. Devdariani in [9]. G. Tsiskarishvili [64] characterized completely the classical axial symmetric bending of specific circular cusped plates without or with a hole.

In the case (1.2), the basic BVPs have been explicitly solved in [38] and [39] with the help of singular solutions depending only on the polar angle.

If we consider the cylindrical bending of a plate, in particular of a cusped one, with rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq \ell$, then we actually get the corresponding results also for cusped beams (see [33], [27], [30], [22]-[26], [5], [6], [40], [41], [4]).

This part deals with the existence, uniqueness, and regularity properties of the hierarchical models of cusped plates. In practice, such plates and beams are often encountered in spatial structures with partly fixed edges, e.g., stadium ceilings, aircraft wings etc., in machine–tool design, as in cutting–machines, planning–machines, in astronautics and in many other areas of engineering.

This paper is organized as follows.

In Section 2 we collect well–known auxiliary material from the three-dimensional theory of elasticity and the theory of Fourier–Legendre series.

In Section 3 we construct hierarchical models which reduce the original three–dimensional boundary value problem for cusped and prismatic shell type elastic bodies to two–dimensional problems. We recall that in the "regular" case (i.e., when the plate thickness does not vanish anywhere), the Fourier–Legendre coefficients of the displacement vector $u$, which solves the original three-dimensional problem in the space $H^1(\Omega)$, automatically belong to the space $H^1(\omega)$. Moreover, all the moments $\overset{\mathbf{N}}{w}_{i0}, \cdots, \overset{\mathbf{N}}{w}_{\underline{i}N_i}$ for $i = 1, 2, 3$ and $\mathbf{N} := (N_1, N_2, N_3)$, determined by the corresponding two-dimensional hierarchical models belong to the space $H^1(\omega)$, while the approximations of the displacement vector $\overset{\mathbf{N}}{w}$ represented by means of these moments belong to the space $H^1(\Omega)$. In the case of a cusped plate, the Fourier-Legendre coefficients of the displacement vector $u \in H^1(\Omega)$ do not belong to the space $H^1(\omega)$ any more, in general. Also, the space of approximate vectors $\overset{\mathbf{N}}{w}$ represented by the moments $\overset{\mathbf{N}}{w}_{i0}, \cdots, \overset{\mathbf{N}}{w}_{\underline{i}N_i}$ for $i = 1, 2, 3$, of the class $H^1(\omega)$, do not belong to the space $H^1(\Omega)$ either. Therefore, it is necessary to choose a space for the moment functions $\overset{\mathbf{N}}{w}_{i0}, \cdots, \overset{\mathbf{N}}{w}_{\underline{i}N_i}$ defined on $\omega$ such that the corresponding linear combinations of these moments with the Legendre polynomials as coefficients, belong to the space $H^1(\Omega)$. This is done in Subsection 3.2. In subsection 3.3, the expressions for the strain and stress tensors corresponding to a vector–function $\overset{\mathbf{N}}{w}$ are given. In Subsection 3.4, we establish uniqueness and existence results for two–dimensional variational hierarchical models. We remark here that the well–known approach of previous authors [3] needs modifications which are related to the peculiarities of the appropriate function spaces for the unknown moments. Subsection 3.5 is devoted to the convergence of the approximate solution $\overset{\mathbf{N}}{w}$ to the exact solutions $u$ of the three–dimensional original problem in the space $H^1(\Omega)$. There we give abstract error estimates with respect to the approximation order $N$ and to the maximal thickness of the plate.

Finally, in Section 4 we formulate explicitly the systems of differential equations corresponding to the two–dimensional variational hierarchical models for a general orthogonal system in Subsection 4.1, and for the Legendre polynomials in Subsection 4.2. It is shown that in the latter case our system is equivalent to the system of I. Vekua who obtained it with the help of quite different arguments (see [33]).

The approach presented in this part we also apply to cusped beams with rectangular cross–sections. These results will be considered in Part 2 of the present lecture notes.

# 2 Preliminary material

## 2.1 Cusped prismatic shell

Let a three–dimensional body in the form of an *elastic prismatic shell* (later on called "shell") occupy a bounded region $\bar{\Omega}$ with boundary $\partial\Omega$,

$$\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x = (x_1, x_2) \in \omega,\ \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x)\}, \tag{2.1}$$

where $\bar{\omega} = \omega \cup \partial\omega$ is the so-called *projection* of the shell $\bar{\Omega} = \Omega \cup \partial\Omega$.

In what follows we assume that $\overset{(\pm)}{h}(x) \in C^2(\omega) \cap C(\bar{\omega})$, and

$$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x) \begin{cases} > 0 \text{ for } x \in \omega \\ \geq 0 \text{ for } x \in \partial\omega \end{cases} \tag{2.2}$$

is the *thickness* of the shell $\bar{\Omega}$ at the point $x \in \bar{\omega}$.

Further, let $\partial\omega$ be a Lipschitz curve and

$$\Gamma := \Big\{(x, x_3) \in \mathbb{R}^3 :\ x \in \partial\omega,\ \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x)\Big\}, \tag{2.3}$$

$$S^{\pm} := \Big\{(x, \overset{(\pm)}{h}(x)) \in \mathbb{R}^3 :\ x \in \omega\Big\}; \tag{2.4}$$

denote by $\gamma$ the projection of $\Gamma$ onto $\partial\omega$ and let $\gamma_0 := \partial\omega \backslash \bar{\gamma}$.

Obviously,

$$\partial\Omega = \bar{\Gamma} \cup \bar{S}^+ \cup \bar{S}^-, \tag{2.5}$$

where $\bar{\Gamma}$ is a cylindrical *lateral surface*, while $S^+$ and $S^-$ are *upper* and *lower face surfaces* of the shell. Note that, in general, $\partial\Omega$ is *not* a Lipschitz surface.

If $\bar{S}^+ \cap \bar{S}^- \neq \emptyset$, then a shell is called *a cusped shell*; and the set

$$\Gamma_0 := \bar{S}^+ \cap \bar{S}^- = \{(x, x_3) \in R^3 :\ x \in \partial\omega,\ x_3 = \overset{(+)}{h}(x) = \overset{(-)}{h}(x)\} \tag{2.6}$$

will be referred to as a *cusped edge* of a cusped shell.

In Figures 1-3 we depict some examples of cusped shells (see Appendix A).

In Figures 4-9 we present all the possible profiles of cusped shells (see Appendix B).

## 2.2 Variational formulation of the basic three–dimensional problem for prismatic shell type bodies

The system of equations in the three-dimensional linear theory of isotropic elasticity in terms of the displacement vector $u := (u_1, u_2, u_3)^T$ reads as follows:

$$A(\partial)u := \mu\Delta_3 u + (\lambda + \mu) \text{ grad div } u = f, \tag{2.7}$$

where $A(\partial)$ is a strongly elliptic differential operator

$$A(\partial) = [A_{kj}(\partial)]_{3\times3} := \Big[\mu\delta_{kj}\Delta_3 + (\lambda+\mu)\frac{\partial^2}{\partial x_k \partial x_j}\Big]_{3\times3}, \tag{2.8}$$

where $\lambda$ and $\mu$ are the Lamé constants, $\delta_{kj}$ is Kronecker's symbol; and the vector $-f$ corresponds to some given volume force.

By

$$e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = \frac{1}{2}(u_{j,i} + u_{i,j}) \tag{2.9}$$

and $\sigma_{ij}(u)$ we denote the strain and stress tensors, respectively. They are related by Hooke's law

$$\sigma_{ij}(u) = \lambda\delta_{ij}e_{kk}(u) + 2\mu e_{ij}(u) = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}),\ i,j = 1,2,3. \tag{2.10}$$

Here and in what follows, for brevity, we employ abridged notations:

i) repeated indices imply summation if they are not underlined. Greek letters run from 1 to 2, and Latin letters from 1 to 3, unless stated otherwise;

ii) subscripts preceded by a comma will mean partial derivatives with respect to the corresponding coordinates (see the list of notations).

By $T(\partial, n)u$ we denote the stress vector calculated on the surface element with the unit normal vector $n = (n_1, n_2, n_3)$:

$$[T(\partial, n)u]_k := \sigma_{kj}(u)n_j \ \text{ for } k = 1,2,3. \tag{2.11}$$

Recall that (6.6) can also be written in the form

$$[A(\partial)u]_k = \sigma_{kj,j}(u) = f_k \ \text{ for } k = 1,2,3. \tag{2.12}$$

Let us consider the boundary value problem (BVP):

$$A(\partial)u = f \quad \text{in } \Omega, \tag{2.13}$$
$$Tu = g^+ \ \text{ on } S^+, \tag{2.14}$$
$$Tu = g^- \ \text{ on } S^-, \tag{2.15}$$
$$u = 0 \quad \text{on } \Gamma. \tag{2.16}$$

We look for a solution of the BVP (6.12)–(6.15) in the Sobolev space $[H^1(\Omega)]^3$.[1] Assuming $\Omega$ to be a Lipschitz domain, we require the given data to belong to the corresponding natural spaces (cf., e.g., [49, Chapt.4])

$$f_k \in \widetilde{H}^{-1}(\Omega) \text{ and } g_k^{\pm} \in H^{-\frac{1}{2}}(S^{\pm}) \text{ for } k = 1, 2, 3, \tag{2.17}$$

which, in the case $\bar{S}^+ \cap \bar{S}^- \neq \emptyset$ means that there exists a functional $g \in H^{-\frac{1}{2}}(S)$ on $S := \partial\Omega \backslash \bar{\Gamma}$ and $g|_{S^\pm} = g^\pm$ on $S^\pm$.

Equation (6.12) is understood in the distributional sense and the Dirichlet condition (6.15) in the trace sense ([46], [49]). The conditions (6.13) and (6.14) are understood in the sense of the space $H^{-\frac{1}{2}}(S^\pm)$ since for $u \in H^1(\Omega)$ with $Au \in \widetilde{H}^{-1}(\Omega)$, the functional $Tu \in H^{-\frac{1}{2}}(\Gamma)$ is defined by Green's identity,

$$\langle Tu, u^* \rangle_{\partial\Omega} := \int\limits_{\Omega} \sigma_{ij}(u) e_{ij}(u^*) dx + \langle f, u^* \rangle_{\Omega} \qquad \text{for all } u^* \in H^1(\Omega)\,, \tag{2.18}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the $L_2$–duality between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, while $\langle \cdot, \cdot \rangle_{\Omega}$ denotes the $L_2$–duality between $\widetilde{H}^{-1}(\Omega)$ and $H^1(\Omega)$ (cf. [49, Ch.4], [8]).

We further denote

$$H^1(\Omega, \Gamma) := \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma\}. \tag{2.19}$$

As is well known, the BVP (6.12)-(6.15) is equivalent to the following variational formulation.

*Problem* (I): *Find* $u \in H^1(\Omega; \Gamma)$ *such that*

$$B(u, u^*) = \mathcal{F}(u^*) \quad \textit{for all } u^* \in H^1(\Omega, \Gamma)\,, \tag{2.20}$$

*where*

$$B(u, u^*) := \int\limits_{\Omega} \sigma_{ij}(u) e_{ij}(u^*) dx, \tag{2.21}$$

$$\mathcal{F}(u^*) := \begin{cases} -\langle f, u^* \rangle_{\Omega} + \langle g^+, u^* \rangle_{S^+} + \langle g^-, u^* \rangle_{S^-} & \text{if } S^+ \cap S^- = \emptyset, \\ \\ -\langle f, u^* \rangle_{\Omega} + \langle g, u^* \rangle_{S} & \text{if } \bar{S}^+ \cap \bar{S}^- \neq \emptyset; \end{cases} \tag{2.22}$$

*here* $\langle \cdot, \cdot \rangle_M$ *is the duality pairing between the spaces* $H^r(M)$ *and* $\widetilde{H}^{-r}(M)$, *where* $r = 1$ *for* $M = \Omega$ *and* $r = 1/2$ *for* $M = S^+, S^-, S$.

Both formulations are equivalent to the *minimization problem:*

*Find* $u \in H^1(\Omega, \Gamma)$ *such that*

$$E(u^*) \geq E(u) \quad \textit{for all } u^* \in H^1(\Omega, \Gamma)\,, \tag{2.23}$$

*where*

[1] If all elements of a vector field $u = (u_1, u_2, \cdots, u_N)$ belong to the same space $X$, as a rule we shall write from now on $u \in X$ instead of $u \in [X]^N$.

$$E(u^*) := \frac{1}{2}B(u^*, u^*) - \mathcal{F}(u^*)\,. \tag{2.24}$$

The following existence and uniqueness results are well-known (see, e.g., [14], [58], [49]).

**THEOREM 2.1** *Let* $\Omega$ *be a Lipschitz domain,* $\Gamma \neq \emptyset$*, and the conditions* (6.16) *are fulfilled. Then the BVP* (6.12)-(6.15) *(i.e., the equation* (2.20) *and the minimization problem* (2.23)*,* (2.24)*) has a unique solution* $u \in H^1(\Omega, \Gamma)$ *and*

$$||u||_{H^1(\Omega)} \leq \begin{cases} C\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|g^+\|_{H^{-\frac{1}{2}}(S^+)} + \|g^-\|_{H^{-\frac{1}{2}}(S^-)}\,, \\ C\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(S)} \quad for \quad \bar{S}^+ \cap \bar{S}^- = \emptyset, \end{cases} \tag{2.25}$$

*where* $C$ *is a positive constant independent of* $u, f, g^{\pm}$*.*

The proof of the theorem is based on the Lax-Milgram lemma since :
i) $\mathcal{F}$ is a bounded linear functional;
ii) the bilinear form $B(\cdot, \cdot)$ is bounded

$$B(u, u^*) \leq \delta_1 ||u||_{[H^1(\Omega)]^3} \;\; ||u^*||_{[H^1(\Omega)]^3}, \;\; \delta_1 = const > 0; \tag{2.26}$$

iii) $B(\cdot, \cdot)$ is coercive (due to the Korn's inequality)

$$B(u, u) \geq \delta_2 ||u||^2_{[H^1(\Omega)]^3} \;\; \forall u \in [H^1(\Omega, \Gamma)]^3, \;\; \delta_2 = const > 0, \tag{2.27}$$

(see e.g., [49], Theorems 10.1 and 10.2, [59], Theorem 2.5).

Due to the regularity properties of solutions to the BVP (6.12)–(6.15), on some subsets of $\bar{\Omega}$ we get higher smoothness of the solution wherever the right–hand sides $f$ and $g^{\pm}$ are smoother.

More precisely, let $g^{\pm} \in H^{r+\frac{1}{2}}(S^{\pm}),\; f \in H^r(\Omega),\; S^{\pm} \in C^{r+1,1}$, where $r \geq 0$ is an integer. Then

$$u \in H^{r+2}(\Omega^*)\,,$$

where $\Omega^*$ is an arbitrary subdomain of $\bar{\Omega}$ such that $\bar{\Omega}^* \cap (\bar{\Gamma} \cup \gamma_0) = \emptyset$. Moreover, there exists a constant $C = C(\Omega^*) > 0$ such that

$$||u||_{H^{r+2}(\Omega^*)} \leq C\Big( ||f||_{H^r(\Omega)} + ||g^+||_{H^{r+\frac{1}{2}}(S^+)} + ||g^-||_{H^{r+\frac{1}{2}}(S^-)} \Big)\,. \tag{2.28}$$

In addition, if $g^{\pm} \in C^{1,\kappa}(S^{\pm}),\; f \in C^{0,\kappa}(\Omega),\; S^{\pm} \in C^{2,\kappa}$, then $u \in C^{2,\kappa}(\bar{\Omega}^*)$ with $0 < \kappa < 1$ (cf., e.g., [49], [1], [15], [62]).

**REMARK 2.2** *The results of Subsection* 2.2 *also hold true for the BVP with the conditions* (6.12)–(6.14) *and*

$$u|_{\Gamma_D} = 0, \quad Tu|_{\Gamma_T} = 0, \tag{2.29}$$

*where* $\Gamma_D, \Gamma_T \subset \Gamma,\; \Gamma_D \cap \Gamma_T = \emptyset,\; \Gamma \subset \bar{\Gamma}_D \cup \bar{\Gamma}_T,\; \Gamma_D \neq \emptyset$*.*

*In particular, Theorem* 2.1 *holds true for the BVP* (6.12)–(6.14)*,* (2.29)*.*

## 2.3 Fourier–Legendre series

Let

$$\begin{aligned} &\varphi(x,\cdot) \in L_2([\overset{(-)}{h}(x), \overset{(+)}{h}(x)]), \\ &2h(x) = \overset{(+)}{h}(x) - \overset{(-)}{h}(x) > 0 \quad \text{for} \quad x \in \omega. \end{aligned} \tag{2.30}$$

The function $\varphi(x,\cdot)$ can be then represented in the form of a Fourier–Legendre series (see, e.g., I. Vekua [33]), i.e.,

$$\varphi(x, x_3) = \sum_{k=0}^{\infty} (k + \frac{1}{2})\, a(x)\, \varphi_k(x)\, P_k(ax_3 - b), \tag{2.31}$$

which converges in $L_2$, where

$$a := a(x) = \frac{1}{h(x)},\ b := b(x) = \frac{\widetilde{h}(x)}{h(x)},\ 2\widetilde{h}(x) := \overset{(+)}{h}(x) + \overset{(-)}{h}(x)\,, \tag{2.32}$$

$$\varphi_k(x) = \int_{\overset{(-)}{h}(x)}^{\overset{(+)}{h}(x)} \varphi(x, x_3)\, P_k(ax_3 - b)\, dx_3 \quad \text{for} \ \ k = \overline{0, \infty}\,. \tag{2.33}$$

Note that

$$\int_{\overset{(-)}{h}}^{\overset{(+)}{h}} P_k(ax_3 - b)P_l(ax_3 - b)a(x)dx_3 = \begin{cases} 0 & \text{for} \quad k \neq l, \\ \dfrac{2}{2k+1} & \text{for} \quad k = l, \end{cases} \tag{2.34}$$

and

$$t := ax_3 - b = \begin{cases} 1 & \text{for} \ \ x_3 = \overset{(+)}{h}, \\ -1 & \text{for} \ \ x_3 = \overset{(-)}{h}. \end{cases} \tag{2.35}$$

Recall that if $\varphi(x,\cdot)$, $\varphi'_{,3}(x,\cdot)$, $\varphi''_{,33}(x,\cdot) \in C^0([\overset{(-)}{h}, \overset{(+)}{h}])$, then the corresponding Fourier–Legendre series converges to $\varphi(x,\cdot)$ uniformly on $[\overset{(-)}{h}, \overset{(+)}{h}]$ (see, e.g., [2, Ch. 15, §2]). For further convergence properties see, e.g., [18, Ch.7] and [42].

# 3 Hierarchical method for elastic cusped prismatic shells: reduction to two–dimensional models

## 3.1 Legendre moments

Let $f \in C^{0,\alpha}(\bar{\Omega})$, and $u \in H^1(\Omega) \cap C^{2,\alpha}(\Omega) \cap C^2(\Omega \cup S^+ \cup S^-)$ be the unique solution of the BVP (6.12)–(6.15). Then $u_i$ can be expanded into the Fourier–Legendre series in $\Omega$:

$$\begin{gathered} u_i(x, x_3) = \sum_{k=0}^{\infty} (k + \tfrac{1}{2})\, a\, u_{ik}(x)\, P_k(ax_3 - b) \\ \text{for } x \in \omega\,,\ \overset{(-)}{h}(x) < x_3 < \overset{(+)}{h}(x), \quad \text{and } \ i = 1, 2, 3, \end{gathered} \tag{3.1}$$

where

$$\begin{aligned} &\overset{(+)}{h}(x) - \overset{(-)}{h}(x) > 0 \quad \text{for} \quad x \in \omega, \\ &\overset{(+)}{h}(x) - \overset{(-)}{h}(x) \geq 0 \quad \text{for} \quad x \in \partial\omega\,, \end{aligned} \tag{3.2}$$

$$u_{ik}(x) = \int\limits_{\overset{(-)}{h}(x)}^{\overset{(+)}{h}(x)} u_i(x, x_3) P_k(ax_3 - b) dx_3 \ \text{ for } k = \overline{0, \infty}\,,\ i = 1, 2, 3. \tag{3.3}$$

Evidently, (6.15) implies

$$u_{ik}(x)|_\gamma = 0\,. \tag{3.4}$$

We recall that $\gamma$ is the projection of $\Gamma$ onto $\partial\omega$. Note that with the help of (2.35), the Fourier coefficient (7.4) can be rewritten as

$$\begin{aligned} u_{ik}(x) &= \frac{1}{a} \int\limits_{-1}^{+1} u_i(x, \frac{t + b}{a})\, P_k(t) dt \\ &= h(x) \int\limits_{-1}^{+1} u_i(x, h(x)t + \tilde{h}(x))\, P_k(t)\, dt, \end{aligned} \tag{3.5}$$

which shows that if $u_i$ is bounded in the spatial vicinity of $\gamma_0$, then

$$u_{ik}(x)|_{\gamma_0} = 0 \ \text{ for } k = \overline{0, \infty} \ \text{ and } i = 1, 2, 3, \tag{3.6}$$

since

$$\frac{1}{a(x)} = h(x) = \frac{1}{2}\left[\overset{(+)}{h}(x) - \overset{(-)}{h}(x)\right]$$

vanishes on $\gamma_0$.

Clearly, in general, $u_i \in H^1(\Omega)$ is not bounded and the condition (7.7) does not hold.

## 3.2 Approximating function spaces

Let us fix $\mathbf{N} := (N_1, N_2, N_3) \in \mathbb{N}_0^3$ and consider the linear combinations

$$w_i(x_1,x_2,x_3) \equiv \overset{\mathbf{N}}{w}_i(x_1,x_2,x_3) := \sum_{r_{\underline{i}}=0}^{N_i} \left(r_i + \tfrac{1}{2}\right) a \overset{\mathbf{N}}{w}_{\underline{i}r_i}(x_1,x_2) \;\; P_{r_i}(ax_3 - b) \quad \text{for } i = 1,2,3, \tag{3.7}$$

where $\overset{\mathbf{N}}{w}_{\underline{i}r_i} \equiv w_{\underline{i}r_i} \in H^1_{loc}(\omega)$ and where underlining an index means that these repeated indices do not imply summation. The functions $w_{\underline{i}r_i}$ are called the *moments of the function* $w_i$.

Denote by $\widetilde{V}_{\mathbf{N}}(\Omega) := \widetilde{V}_{N_1}(\Omega) \times \widetilde{V}_{N_2}(\Omega) \times \widetilde{V}_{N_3}(\Omega)$ the set of vector–functions with components of the form (7.8) which belong to $H^1(\Omega)$.

Let

$$w_i \in \widetilde{V}_{N_i}(\Omega,\Gamma) \subset H^1(\Omega), \tag{3.8}$$

where

$$\widetilde{V}_{N_i}(\Omega,\Gamma) := \{w_i \in \widetilde{V}_{N_i}(\Omega) \;:\; w_i = 0 \;\text{ on }\; \Gamma\}. \tag{3.9}$$

Our aim is now to choose the corresponding function spaces for the moments $w_{\underline{i}r_i}$.

Taking into account (7.9), (6.15), (2.35), and using the standard limiting procedure, we easily get

$$\int_\Omega |w_i(x_1,x_2,x_3)|^2 d\Omega = \int_\omega \sum_{r_{\underline{i}}=0}^{N_i} (r_i + \frac{1}{2})\; a\; |w_{\underline{i}r_i}(x_1,x_2)|^2 d\omega < +\infty, \tag{3.10}$$

which implies

$$a^{\frac{1}{2}} w_{\underline{i}r_i} \in L_2(\omega) \;\text{ for } r_i = \overline{0, N_i}. \tag{3.11}$$

Analogously, applying the formula [2]

$$P_r'(t) = \frac{1}{2}\sum_{s=0}^{r-1}(2s+1)[1-(-1)^{r+s}]P_s(t) \;\text{ with }\; t = ax_3 - b, \tag{3.12}$$

we obtain

$$\int_\Omega |w_{i,3}(x_1,x_2,x_3)|^2 d\Omega$$

$$= \int_\omega \int_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big\{ \sum_{r_{\underline{i}}=0}^{N_i} (r_i + \frac{1}{2}) a^2 \sum_{s_{\underline{i}}=0}^{r_i} (s_i + \frac{1}{2}) \left[1 - (-1)^{r_i+s_i}\right]$$

[2]Here and in what follows we assume that $\sum_{s=k}^{m}(\,\cdot\,) = 0$ for $m < k$.

$$\times P_{s_i}(ax_3-b)w_{\underline{i}r_i}(x_1,x_2)\Big\}^2 dx_3 d\omega$$

$$= \int\limits_{\omega} \sum_{s_{\underline{i}}=0}^{N_i} \Big\{ \sum_{r_{\underline{i}}=s_i}^{N_i} (r_i+\frac{1}{2})\Big[1-(-1)^{r_i+s_i}\Big] w_{\underline{i}r_i}(x_1,x_2)\Big\}^2$$

$$a^3(s_i+\frac{1}{2})d\omega < +\infty\,, \tag{3.13}$$

which implies

$$a^{3/2}\sum_{r_{\underline{i}}=s_i}^{N_i}(r_i+\frac{1}{2})[1-(-1)^{r_i+s_i}]w_{\underline{i}r_i} \in L_2(\omega) \ \text{ for } s_i=\overline{0,N_i}. \tag{3.14}$$

In turn these inclusions yield

$$a^{3/2}w_{\underline{i}r_i} \in L_2(\omega) \ \text{ for } r_i=\overline{1,N_i}. \tag{3.15}$$

Similarly, applying the formula

$$\begin{aligned} P_{r,\alpha}(ax_3-b) &= (a_{,\alpha}x_3-b_{,\alpha})P'_r(ax_3-b) \\ &= A_{\alpha 0}rP_r(ax_3-b)+\sum_{q=1}^{r}A_{\alpha q}(2r-2q+1)P_{r-q}(ax_3-b) \end{aligned} \tag{3.16}$$

with

$$A_{\alpha q} = -\frac{\overset{(+)}{h}_{,\alpha}-(-1)^q\overset{(-)}{h}_{,\alpha}}{2h}, \tag{3.17}$$

we arrive at the equation

$$\int\limits_{\Omega}|\frac{\partial}{\partial x_\alpha}w_i(x_1,x_2,x_3)|^2 d\Omega$$

$$= \int\limits_{\omega}\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big(\sum_{r_{\underline{i}}=0}^{N_i}(r_i+\frac{1}{2})\Big\{a_{,\alpha}P_{r_i}(ax_3-b)\;w_{\underline{i}r_i}$$

$$+a[A_{\alpha 0}r_{\underline{i}}P_{r_i}(ax_3-b)+\sum_{q_{\underline{i}}=1}^{r_i}A_{\alpha q_{\underline{i}}}(2r_i-2q_i+1)P_{r_i-q_i}(ax_3-b)]\;w_{\underline{i}r_i}$$

$$+aP_{r_i}(ax_3-b)w_{\underline{i}r_i,\alpha}\Big\}\Big)^2 dx_3 d\omega$$

$$= \int\limits_{\omega}\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big(\sum_{r_{\underline{i}}=0}^{N_i}a(r_i+\frac{1}{2})\{[(\frac{a_{,\alpha}}{a}+A_{\alpha 0}r_i)w_{\underline{i}r_i}+w_{\underline{i}r_i,\alpha}]P_{r_i}(ax_3-b)$$

$$+\sum_{s_{\underline{i}}=0}^{r_i-1} 2A_{\alpha r_i - s_i} w_{\underline{i}r_i}(s_i+\frac{1}{2})P_{s_i}(ax_3-b)\}\Big)^2 dx_3 d\omega. \tag{3.18}$$

Introduce the notation

$$\Psi^{(\alpha, r_i - s_i, r_{\underline{i}})} := 2A_{\alpha\, r_i - s_i} w_{\underline{i}r_i} \quad \text{for} \;\; r_i - s_i > 0\,, \tag{3.19}$$

$$(r_{\underline{i}}+\frac{1}{2})\Psi^{(\alpha,0,r_i)} := (\frac{a_{,\alpha}}{a}+A_{\alpha 0} r_i) w_{\underline{i}r_i} + w_{\underline{i}r_i,\alpha}$$

$$= -(r_i+1)\,\frac{h_{,\alpha}}{h}\, w_{\underline{i}r_i} + w_{\underline{i}r_i,\alpha}\,. \tag{3.20}$$

Then from (7.19) one obtains

$$\int_\Omega |w_{i,\alpha}(x_1,x_2,x_3)|^2 d\Omega$$

$$= \int_\omega \int_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big\{ \sum_{r_{\underline{i}}=0}^{N_i} (r_i+\frac{1}{2}) a \Big( \sum_{s_{\underline{i}}=0}^{r_i} \Psi^{(\alpha, r_i - s_i, r_i)}(s_i+\frac{1}{2})P_{s_i}(ax_3-b)\Big)\Big\}^2 dx_3 d\omega$$

$$= \int_\omega \int_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big\{ \sum_{s_{\underline{i}}=0}^{N_i} \Big( \sum_{r_{\underline{i}}=s_i}^{N_i} (r_i+\frac{1}{2})\Psi^{(\alpha, r_i - s_i, r_i)}\Big) a(s_i+\frac{1}{2})P_{s_i}(ax_3-b)\Big\}^2 dx_3 d\omega$$

$$= \int_\omega \sum_{s_{\underline{i}}=0}^{N_i} \Big( \sum_{r_{\underline{i}}=s_i}^{N_i} (r_i+\frac{1}{2})\Psi^{(\alpha, r_i - s_i, r_{\underline{i}})}\Big)^2 a(s_i+\frac{1}{2})\, d\omega. \tag{3.21}$$

Consequently,

$$\Phi^{(\alpha, s_{\underline{i}}, N_i)} := a^{1/2} \sum_{r_{\underline{i}}=s_i}^{N_i} (r_i+\frac{1}{2})\Psi^{(\alpha, r_i - s_i, r_{\underline{i}})} \in L_2(\omega) \;\; \text{for} \; s_i = \overline{0, N_i}. \tag{3.22}$$

Denote

$$v_{\underline{i}r_i} := h^{-r_i-1} w_{\underline{i}r_i}. \tag{3.23}$$

The functions $v_{\underline{i}r_i}$ are called *weighted moments of the function* $w_i$. From (7.11), (7.14), and (7.20)–(7.22) it follows that $w \equiv \overset{\mathbf{N}}{w} := (w_1, w_2, w_3)$ satisfies

$$||w||^2_{H^1(\Omega)} = \sum_{i=1}^{3} \Big[ \sum_{r_{\underline{i}}=0}^{N_i} (r_i+\frac{1}{2})||h^{r_i+\frac{1}{2}} v_{\underline{i}r_i}||^2_{L_2(\omega)} \tag{3.24}$$

$$+\sum_{s_{\underline{i}}=0}^{N_i} (s_i+\frac{1}{2})|| \sum_{r_{\underline{i}}=s_i}^{N_i} (r_i+\frac{1}{2})[1-(-1)^{r_i+s_i}] h^{r_i-\frac{1}{2}} v_{\underline{i}r_i}||^2_{L_2(\omega)}$$

$$+\sum_{\alpha=1}^{2}\sum_{s_{\underline{i}}=0}^{N_i}(s_i+\frac{1}{2})||h^{s_i+1/2}v_{\underline{i}s_i,\alpha}+\sum_{r_{\underline{i}}=s_i+1}^{N_i}(r_i+\frac{1}{2})\,2A_{\alpha\, r_i-s_i}h^{r_i+\frac{1}{2}}v_{\underline{i}r_i}||^2_{L_2(\omega)}\Big].$$

Let

$$v:=(v_{10},...,v_{1N_1},v_{20},...,v_{2N_2},v_{30},...,v_{3N_3})^T, \tag{3.25}$$

where $v_{\underline{i}r_i}$ are given by (7.24).

**DEFINITION 3.1** *By $H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$ we denote the subspace of vector–functions belonging to $[H^1_{loc}(\omega)]^{N_1+N_2+N_3+3}$ for which the norm $||.||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)}$ defined by the right–hand side of* (7.25) *is finite.*

**LEMMA 3.2** *The space $H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$ is complete.*

*Proof.* Denote for any $\varepsilon>0$:

$$\omega_\varepsilon:=\{x\in\omega\ :\ |x-y|\geq\varepsilon\ \text{ for all } y\in\gamma_0\}.$$

The norm $||\cdot||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)}$ is equivalent to the norm $||\cdot||_{[H^1(\omega_\varepsilon)]^{N_1+N_2+N_3+3}}$ since $h(x)>0$ for $x\in\overline{\omega_\varepsilon}$, i.e., $h(x)\geq const>0$.

Let $\{v_n\}\subset H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$ be a fundamental sequence. We will show that it converges to some vector $v\in H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$. Due to the above equivalence, the sequence $\{v_n\}$ is then also fundamental in the space $[H^1(\omega_\varepsilon)]^{N_1+N_2+N_3+3}$ for arbitrary $\varepsilon>0$. Since $H^1_{\mathbf{N}}(\omega_\varepsilon)$ is complete, there exists a vector $v^{(\varepsilon)}\in[H^1(\omega_\varepsilon)]^{N_1+N_2+N_3+3}$ such that

$$||v_n-v^{(\varepsilon)}||_{[H^1(\omega_\varepsilon)]^{N_1+N_2+N_3+3}}\longrightarrow 0\ \text{ as } n\longrightarrow+\infty.$$

Note that for $\varepsilon_1<\varepsilon_2$ (i.e., $\omega_{\varepsilon_1}\supset\omega_{\varepsilon_2}$) there holds for the restriction onto $\omega_{\varepsilon_2}$:

$$v^{(\varepsilon_1)}|_{\omega_{\varepsilon_2}}=v^{(\varepsilon_2)}.$$

Therefore, there exists a vector-function $v_0\in[H^1_{loc}(\omega)]^{N_1+N_2+N_3+3}$ such that for every $\varepsilon>0$

$$v_0|_{\omega_\varepsilon}=v^{(\varepsilon)}.$$

It is evident that

$$||v_n||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)}\leq M\ \text{ for all } n\in N,$$

where $M$ is some positive constant. Obviously,

$$||v_n||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)}\leq||v_n||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)}\leq M\quad\text{for every } \varepsilon>0,$$

and

$$||v_0||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} = ||v^{(\varepsilon)}||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} = \lim_{n\to+\infty} ||v_n||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} \le M.$$

Hence

$$||v_0||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)} := \lim_{\varepsilon\to 0} ||v_0||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} \le M,$$

which implies that $v_0 \in H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$.

It remains to show that $v_n \to v_0$ in the space $H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$. Indeed, for any arbitrary $\delta > 0$ there exists a number $N_0(\delta)$, such that

$$||v_n - v_m||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} \le ||v_n - v_m||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)} < \delta$$

for $n, m > N_0(\delta)$. Passing to the limit as $m \to \infty$ for fixed $n$ we get

$$||v_n - v^{(\varepsilon)}||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} = ||v_n - v_0||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega_\varepsilon)} \le \delta \ \text{ for all } n > N_0(\delta)\,.$$

Sending $\varepsilon$ to zero we conclude that

$$||v_n - v_0||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)} \le \delta \quad \text{for all} \ \ n > N_0(\delta),$$

which completes the proof. □

**COROLLARY 3.3** $H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$ *is a Hilbert space and a subspace of* $H^1_{loc}(\omega)$*, that is, for any* $\varepsilon > 0$ *and* $v \in H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)$ *there holds*

$$v \in [H^1(\omega_\varepsilon)]^{N_1+N_2+N_3+3}.$$

**COROLLARY 3.4** $\widetilde{V}_{\mathbf{N}}(\Omega)$ *is a closed subspace of* $H^1(\Omega)$*.*

*Proof.* It readily follows from the above isometry (see (7.25) and Definition 3.1 ) that

$$||w||_{\widetilde{V}_{\mathbf{N}}(\Omega)} = ||v||_{H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega)} = ||w||_{H^1(\Omega)}$$

for every $w \in \widetilde{V}_{\mathbf{N}}(\Omega)$, where $v$ corresponds to $w$ according to the relations (7.9), (7.24), and (7.26). □

We now introduce the spaces (cf. (3.9))

$$\widetilde{V}_{\mathbf{N}}(\Omega,\Gamma) \ := \ \{w \in \widetilde{V}_{\mathbf{N}}(\Omega): \ \ w|_\Gamma = 0\}, \tag{3.26}$$

$$H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega,\gamma) \ := \ \{v \in H^1_{\mathbf{N}}(\overset{(+)}{h},\overset{(-)}{h},\omega): \ \ v|_\gamma = 0\}. \tag{3.27}$$

**LEMMA 3.5** *If $w \in \widetilde{V}_{\mathbf{N}}(\Omega, \Gamma)$ then the corresponding vector of weighted moments $v \in H^1_{\mathbf{N}}(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma)$.*

*Proof.* Due to the trace theorem we have (see also (7.8) and (7.24))

$$0 = \tau_\Gamma(w_i) = \sum_{r_{\underline{i}}=0}^{N_i} (r + \frac{1}{2})[P_{r_i}(ax_3 - b)]_\Gamma \tau_\gamma(h^{r_i} v_{\underline{i}r_i}),$$

where $\tau_\Gamma$ and $\tau_\gamma$ are the trace operators on $\Gamma$ and $\gamma$, respectively.

The equality $||\tau_\Gamma(w_i)||^2_{L_2(\Gamma)} = 0$ along with the Fubini theorem and (6.15) gives

$$\tau_\gamma(h^{r_i} v_{\underline{i}r_i}) = 0 \ \text{ for } r_i = \overline{0, N_i} \ \text{ and } i = 1, 2, 3,$$

which implies $\tau_\gamma v_{\underline{i}r_i} = 0$ since $h|_\gamma > 0$. $\square$

**REMARK 3.6** *It is easy to see that $\widetilde{V}_{\mathbf{N}}(\Omega, \Gamma)$ and $H^1_{\mathbf{N}}(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma)$ are closed subspaces of $H^1(\Omega, \Gamma)$ and of $H^1_{\mathbf{N}}(\overset{(+)}{h}, \overset{(-)}{h}, \omega)$, respectively, and represent Hilbert spaces with respect to the respective natural scalar products induced by $H^1(\Omega)$ and $H^1_{\mathbf{N}}(\overset{(+)}{h}, \overset{(-)}{h}, \omega)$.*

## 3.3 Expressions for the strain and stress tensors

In what follows, for simplicity, we assume that

$$v_{\underline{i}r_i} = 0 \quad \text{for} \ \ N_i < r_i \leq N \tag{3.28}$$

with $N := \max\{N_1, N_2, N_3\}$.

In order to obtain expressions for the strain tensor in terms of the vector–function

$$w \in \widetilde{V}_{\mathbf{N}}(\Omega)\,,$$

we substitute (7.8) in (6.8) and distinguish the following three cases: $\{i = \alpha,\ j = \beta\}$, $\{i = \alpha,\ j = 3\}$, and $\{i = j = 3\}$. Due to (7.24) and (7.27) we assume that

$$w_{\underline{i}r_i} = 0 \quad \text{for} \quad N_i < r_i \leq N, \quad \text{and} \quad i = 1, 2, 3. \tag{3.29}$$

We begin with the first case $i = \alpha\,,\ j = \beta$. Applying (7.8), (3.29), and (7.17), we get

$$\begin{aligned} e_{\alpha\beta}(w) &= \frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha}) \\ &= \frac{1}{2}\sum_{r=0}^{N} \Big(r + \frac{1}{2}\Big)\Big\{a_{,\beta}\ w_{\alpha\underline{r}}\ P_r(ax_3 - b) + a\ w_{\alpha\underline{r},\beta}\ P_r(ax_3 - b) \end{aligned}$$

$$+a\ w_{\alpha\underline{r}}\ P_r'(ax_3-b)\ (a_{,\beta}x_3-b_{,\beta})+a_{,\alpha}\ w_{\beta r}P_r\ (ax_3-b)$$
$$+a\ w_{\beta\underline{r},\alpha}\ P_r(ax_3-b)+aw_{\beta\underline{r}}\ P_r'(ax_3-b)\ (a_{,\alpha}x_3-b_{,\alpha})\Big]$$
$$=\frac{1}{2}\sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)\Big\{\ [a(w_{\alpha\underline{r},\beta}+w_{\beta\underline{r},\alpha})+a_{,\beta}\ w_{\alpha\underline{r}}+a_{,\alpha}\ w_{\beta\underline{r}}]\,P_r(ax_3-b)$$
$$+aw_{\alpha\underline{r}}\Big[A_{\beta 0}\ rP_r(ax_3-b)+\sum_{q=1}^{r}A_{\beta\underline{q}}\ (2r-2q+1)\ P_{r-q}(ax_3-b)\Big]$$
$$+aw_{\beta\underline{r}}\Big[A_{\alpha 0}\ r\ P_r(ax_3-b)$$
$$+\sum_{q=1}^{r}A_{\alpha\underline{q}}(2r-2q+1)P_{r-q}(ax_3-b)\Big]\Big\}\,. \qquad (3.30)$$

By introducing

$$\overset{r}{b}_{\alpha s}:=\begin{cases}-(r+1)\ h_{,\alpha}\ h^{-1} & \text{if}\ \ s=r\,,\\ (2s+1)\ A_{\alpha\, r+\underline{s}} & \text{if}\ \ s\neq r\,,\end{cases} \qquad (3.31)$$

we find

$$a_{,\alpha}+aA_{\alpha 0}r=a\overset{r}{b}_{\alpha\underline{r}}\ \ \text{and}\ \ A_{\alpha r+\underline{s}}(2s+1)=\overset{r}{b}_{\alpha s},\quad \text{for}\quad s\neq r. \qquad (3.32)$$

Taking into account (7.32) and the obvious relation

$$\sum_{r=0}^{N}\ \sum_{s=0}^{r}c_{rs}=\sum_{s=0}^{N}\ \sum_{r=s}^{N}c_{rs}, \qquad (3.33)$$

after substitution $r-q=s$, from (7.30) we have

$$e_{\alpha\beta}(w)=\frac{1}{2}\sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)a(w_{\alpha\underline{r},\beta}+w_{\beta\underline{r},\alpha})P_r(ax_3-b)$$
$$+\frac{1}{2}\sum_{r=0}^{N}\ \sum_{s=0}^{r}\Big(r+\frac{1}{2}\Big)a\left(\overset{r}{b}_{\beta s}w_{\alpha\underline{r}}+\overset{r}{b}_{\alpha s}w_{\beta\underline{r}}\right)\ P_{\underline{s}}(ax_3-b)$$
$$=\frac{1}{2}\sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)a(w_{\alpha\underline{r},\beta}+w_{\beta\underline{r},\alpha})P_r(ax_3-b)$$
$$+\frac{1}{2}\sum_{s=0}^{N}\sum_{r=s}^{N}\Big(r+\frac{1}{2}\Big)a\Big(\overset{r}{b}_{\beta s}w_{\alpha\underline{r}}+\overset{r}{b}_{\alpha s}w_{\beta\underline{r}}\Big)P_{\underline{s}}(ax_3-b). \qquad (3.34)$$

Denote by $\Sigma_1$ the second sum and interchange there $r$ and $s$. By virtue of (7.31) and the identity

$$\Big(s+\frac{1}{2}\Big)(2r+1)=\Big(r+\frac{1}{2}\Big)(2s+1), \qquad (3.35)$$

which with (7.32) implies

$$\Big(\underline{r}+\frac{1}{2}\Big)\overset{r}{b}_{\alpha s}=\Big(\underline{s}+\frac{1}{2}\Big)\overset{s}{b}_{\alpha r}, \tag{3.36}$$

we obtain

$$\Sigma_1=\frac{1}{2}\sum_{r=0}^{N}\sum_{s=r}^{N}\left(s+\frac{1}{2}\right)a\left(\overset{s}{b}_{\beta r}w_{\alpha\underline{s}}+\overset{s}{b}_{\alpha r}w_{\beta\underline{s}}\right)P_{\underline{r}}(ax_3-b)$$
$$=\frac{1}{2}\sum_{r=0}^{N}\sum_{s=r}^{N}\left(r+\frac{1}{2}\right)a\left(\overset{r}{b}_{\beta s}w_{\alpha\underline{s}}+\overset{r}{b}_{\alpha s}w_{\beta\underline{s}}\right)P_{\underline{r}}(ax_3-b).$$

Finally, substituting the last expression into (7.34), we get

$$e_{\alpha\beta}(w)=\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)a(x)e_{\alpha\beta r}(x)P_{\underline{r}}(ax_3-b)\ \text{ for }\ \alpha,\beta=1,2, \tag{3.37}$$

where

$$e_{\alpha\beta r}=\frac{1}{2}(w_{\alpha r,\beta}+w_{\beta r,\alpha})+\frac{1}{2}\sum_{s=r}^{N}\left(\overset{r}{b}_{\beta s}w_{\alpha\underline{s}}+\overset{r}{b}_{\alpha s}w_{\beta\underline{s}}\right) \tag{3.38}$$

for $\alpha,\beta=1,2$ and $r=\overline{0,N}$.

In the second case, $i=\alpha$ and $j=3$, denoting

$$\overset{r}{b}_{3s}:=(2s+1)\frac{1-(-1)^{s+r}}{2h} \tag{3.39}$$

and applying formulae (7.13), (7.17), (7.32), and (7.33), we have

$$e_{\alpha 3}(w)=\frac{1}{2}(w_{\alpha,3}+w_{3,\alpha})$$
$$=\frac{1}{2}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\Big[a^2w_{\alpha\underline{r}}P_r'(ax_3-b)+a_{,\alpha}w_{3\underline{r}}P_r(ax_3-b)$$
$$+aw_{3\underline{r},\alpha}P_r(ax_3-b)+aw_{3\underline{r}}P_r'(ax_3-b)(a_{,\alpha}x_3-b_{,\alpha})\Big]$$
$$=\frac{1}{2}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\Big\{aw_{\alpha\underline{r}}\sum_{s=0}^{r}\overset{r}{b}_{3\underline{s}}P_s(ax_3-b)+(a_{,\alpha}w_{3r}+aw_{3r,\alpha})P_{\underline{r}}(ax_3-b)$$
$$+aw_{3\underline{r}}\Big[A_{\alpha 0}rP_r(ax_3-b)+\sum_{q=1}^{r}A_{\alpha\underline{q}}(2r-2q+1)P_{r-q}(ax_3-b)\Big]\Big\}$$
$$=\frac{1}{2}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\Big[aw_{3\underline{r},\alpha}P_r(ax_3-b)+aw_{\alpha r}\sum_{s=0}^{r}\overset{r}{b}_{3\underline{s}}P_s(ax_3-b)$$

$$+a\overset{r}{b}_{\alpha r} w_{3\underline{r}} P_r(ax_3-b) + a\sum_{s=0}^{r-1} A_{\alpha r+\underline{s}}(2s+1)P_s(ax_3-b)w_{3\underline{r}}\Big]$$

$$= \frac{1}{2}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right) a w_{3\underline{r},\alpha} P_r(ax_3-b)$$

$$+\frac{1}{2}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right) a \sum_{s=0}^{r}\left(\overset{r}{b}_{3s} w_{\alpha\underline{r}} + \overset{r}{b}_{\alpha s} w_{3\underline{r}}\right) P_{\underline{s}}(ax_3-b)$$

$$= \frac{1}{2}\sum_{r=0}^{N}\left(r+\frac{1}{2}\right) a w_{3\underline{r},\alpha} P_r(ax_3-b)$$

$$+\frac{1}{2}\sum_{s=0}^{N}\sum_{r=s}^{N}\left(r+\frac{1}{2}\right) a \left(\overset{r}{b}_{3s} w_{\alpha\underline{r}} + \overset{r}{b}_{\alpha s} w_{3\underline{r}}\right) P_{\underline{s}}(ax_3-b)\,. \tag{3.40}$$

Denote by $\sum_2$ the second sum and interchange there $r$ and $s$. By virtue of (7.36) and (7.39) which with (7.35) implies

$$\left(\underline{r}+\frac{1}{2}\right)\overset{r}{b}_{3s} = \left(\underline{s}+\frac{1}{2}\right)\overset{s}{b}_{3r}, \tag{3.41}$$

we obtain

$$\Sigma_2 = \frac{1}{2}\sum_{r=0}^{N}\sum_{s=r}^{N}\left(s+\frac{1}{2}\right) a \left(\overset{s}{b}_{3r} w_{\alpha\underline{s}} + \overset{s}{b}_{\alpha r} w_{3\underline{s}}\right) P_{\underline{r}}(ax_3-b)$$

$$= \frac{1}{2}\sum_{r=0}^{N}\sum_{s=r}^{N}\left(r+\frac{1}{2}\right) a \left(\overset{r}{b}_{3s} w_{\alpha\underline{s}} + \overset{r}{b}_{\alpha s} w_{3\underline{s}}\right) P_{\underline{r}}(ax_3-b)\,.$$

Substituting the last expression into (7.40), we get

$$e_{\alpha 3}(w) = \sum_{r=0}^{N}\left(r+\frac{1}{2}\right) a(x) e_{\alpha 3\underline{r}}(x) P_r(ax_3-b) \quad \text{for} \quad \alpha = 1,2, \tag{3.42}$$

where

$$e_{\alpha 3r} = \frac{1}{2} w_{3r,\alpha} + \frac{1}{2}\sum_{s=r}^{N}\left(\overset{r}{b}_{3s} w_{\alpha\underline{s}} + \overset{r}{b}_{\alpha s} w_{3\underline{s}}\right) \quad \text{for } \alpha = 1,2 \text{ and } r = \overline{0,N}\,. \tag{3.43}$$

Finally, in the third case $i=3$, $j=3$, by applying formulae (7.13), (7.39), (7.33), and (7.41), we have

$$e_{33}(w) = w_{3,3} = \sum_{r=0}^{N}\left(r+\frac{1}{2}\right) a^2 w_{3\underline{r}} P'_r(ax_3-b)$$

$$= \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a w_{3\underline{r}} \sum_{s=0}^{r} \overset{r}{b}_{3\underline{s}} P_s(ax_3 - b)$$
$$= \sum_{s=0}^{N} \sum_{r=s}^{N} \left( r + \frac{1}{2} \right) a \overset{r}{b}_{3\underline{s}} w_{3\underline{r}} P_s(ax_3 - b)$$
$$= \sum_{r=0}^{N} \sum_{s=r}^{N} \left( s + \frac{1}{2} \right) a \overset{s}{b}_{3\underline{r}} w_{3\underline{s}} P_r(ax_3 - b)$$
$$= \sum_{r=0}^{N} \sum_{s=r}^{N} \left( r + \frac{1}{2} \right) a \overset{r}{b}_{3s} w_{3\underline{s}} P_{\underline{r}}(ax_3 - b)$$
$$= \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a(x) e_{33\underline{r}}(x) w_{3\underline{s}} P_r(ax_3 - b), \tag{3.44}$$

where

$$e_{33r} = \sum_{s=r}^{N} \overset{r}{b}_{3s} w_{3\underline{s}} \quad \text{for } r = \overline{0, N}. \tag{3.45}$$

Now, we can rewrite all the formulae (7.38), (7.43), and (7.45) in an unified form, i.e.,

$$e_{ijr} = \tfrac{1}{2}(w_{ir,j} + w_{jr,i}) + \tfrac{1}{2} \sum_{s=r}^{N} \left( \overset{r}{b}_{js} w_{i\underline{s}} + \overset{r}{b}_{is} w_{j\underline{s}} \right) \quad \text{for } i, j = 1, 2, 3 \ \text{ and } r = \overline{0, N}. \tag{3.46}$$

The last formula has been derived by I. Vekua [33] who was using, however, different arguments.

In view of (7.24), (7.31), and (7.39) we conclude

$$e_{ijr}(v) = \frac{1}{2} \left[ \left( h^{r+1} v_{ir} \right)_{,j} + \left( h^{r+1} v_{jr} \right)_{,i} \right]$$
$$+ \frac{1}{2} \sum_{s=r}^{N} h^{s+1} \left( \overset{r}{b}_{js} v_{i\underline{s}} + \overset{r}{b}_{is} v_{j\underline{s}} \right)$$
$$= \frac{1}{2} h^{r+1} \left( v_{ir,j} + v_{jr,i} \right) + \frac{1}{2} (r+1) h^r \left( h_{,j} v_{ir} + h_{,i} v_{jr} \right)$$
$$- \frac{1}{2} (r+1) h^r \left( h_{,j} v_{ir} + h_{,i} v_{jr} \right) + \frac{1}{2} \sum_{s=r+1}^{N} h^{s+1} \left( \overset{r}{b}_{js} v_{i\underline{s}} + \overset{r}{b}_{is} v_{j\underline{s}} \right),$$

which provides us with the relations

$$e_{ijr}(v) = \frac{1}{2} h^{r+1} \left( v_{ir,j} + v_{jr,i} \right) + \frac{1}{2} \sum_{s=r+1}^{N} h^{s+1} \left( \overset{r}{b}_{js} v_{i\underline{s}} + \overset{r}{b}_{is} v_{j\underline{s}} \right) \quad \text{for } i, j = 1, 2, 3 \ \text{ and } r = \overline{0, N}\,. \tag{3.47}$$

Thus, by virtue of (7.37), (7.42), and (7.44),

$$e_{ij}(w) = \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a e_{ij\underline{r}}(v)\, P_r(ax_3 - b) \ \text{ for } \ i, j = 1, 2, 3\,, \tag{3.48}$$

where $e_{ijr}$ is given by (7.47).

Now, according to Hooke's law (6.9), we get the following expressions for the stress tensor:

$$\begin{aligned}
\sigma_{ij}(w) &= \lambda \delta_{ij} e_{kk}(w) + 2\mu e_{ij}(w) \\
&= \lambda \delta_{ij} \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a e_{kk\underline{r}} P_r(ax_3 - b) + 2\mu \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a e_{ij\underline{r}} P_r(ax_3 - b) \\
&= \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a \left[ \lambda \delta_{ij} e_{kkr} + 2\mu e_{ijr} \right] \, P_{\underline{r}}(ax_3 - b)\,,
\end{aligned} \tag{3.49}$$

i.e.,

$$\sigma_{ij}(w) = \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a \sigma_{ij\underline{r}}(v) P_r(ax_3 - b) \ \text{ for } i, j = 1, 2, 3, \tag{3.50}$$

where

$$\sigma_{ijr}(v) = \lambda \delta_{ij} e_{kkr}(v) + 2\mu e_{ijr}(v) \ \text{ for } i, j = 1, 2, 3, \quad r = \overline{0, N}\,. \tag{3.51}$$

From (7.49), by virtue of (7.47), we have

$$\begin{aligned}
\sigma_{ij}(w) &= \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) a \Big\{ \lambda \delta_{ij} h^{r+1} \Big( v_{\alpha r, \alpha} + \sum_{s=r+1}^{N} \overset{r}{b}_{ks} v_{k\underline{s}} \Big) \\
&\quad + \mu \Big[ h^{r+1} \left( v_{ir,j} + v_{jr,i} \right) + \sum_{s=r+1}^{N} h^{s+1} \left( \overset{r}{b}_{js} v_{i\underline{s}} + \overset{r}{b}_{is} v_{j\underline{s}} \right) \Big] \Big\} P_{\underline{r}}(ax_3 - b) \\
&= \sum_{r=0}^{N} \left( r + \frac{1}{2} \right) h^r \Big\{ \lambda \delta_{ij} \Big( v_{\alpha r, \alpha} + \sum_{s=r+1}^{N} \overset{r}{b}_{ks} v_{k\underline{s}} \Big) \\
&\quad + \mu \Big[ v_{ir,j} + v_{jr,i} + \sum_{s=r+1}^{N} h^{s-r} \left( \overset{r}{b}_{js} v_{i\underline{s}} + \overset{r}{b}_{is} v_{j\underline{s}} \right) \Big] \Big\} P_{\underline{r}}(ax_3 - b)
\end{aligned} \tag{3.52}$$

for $i, j = 1, 2, 3$. So, as a consequence, the moments of the stress tensor have the representation

$$\sigma_{ijr}(v) = \lambda \delta_{ij} h^{r+1} v_{\alpha r, \alpha} + \mu h^{r+1} \left( v_{ir,j} + v_{jr,i} \right) + \sum_{s=r+1}^{N} \overset{r}{B}_{ijks} h^{s+1} v_{k\underline{s}} \tag{3.53}$$

for $i, j = 1, 2, 3$, $r = \overline{0, N}$, where

$$\begin{gathered}
\overset{r}{B}_{ijks} := \lambda \delta_{ij} \overset{r}{b}_{ks} + \mu \delta_{ik} \overset{r}{b}_{js} + \mu \delta_{kj} \overset{r}{b}_{is} \\
\text{for } i, j, k, s = 1, 2, 3, \quad r = \overline{0, N}\,.
\end{gathered} \tag{3.54}$$

## 3.4 Variational formulation in particular spaces. Existence results

For any pair of elements

$$v, v^* \in H^1_{\mathbf{N}}\left(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma\right) \tag{3.55}$$

we construct $w,\ w^* \in \overset{\sim}{V}_{\mathbf{N}}(\Omega, \Gamma)$ according to the formulae (7.24) and (7.8):

$$w_i(x, x_3) \;=\; \sum_{r_i=0}^{N_i} \left(r_i + \frac{1}{2}\right) h^{r_i} v_{\underline{i} r_i}(x) P_{r_i}(ax_3 - b)\,, \tag{3.56}$$

$$w_i^*(x, x_3) \;=\; \sum_{r_i=0}^{N_i} \left(r_i + \frac{1}{2}\right) h^{r_i} v^*_{\underline{i} r_i}(x) P_{r_i}(ax_3 - b) \tag{3.57}$$

for $i = 1, 2, 3$. Then, consider the following variational problem.

*Problem* $(I^{\Omega}_{\mathbf{N}})$. *Find* $w \in \overset{\sim}{V}_{\mathbf{N}}(\Omega, \Gamma)$ *such that*

$$B(w, w^*) = \mathcal{F}(w^*) \;\; \textit{for all}\; w^* \in \overset{\sim}{V}_{\mathbf{N}}(\Omega, \Gamma), \tag{3.58}$$

*where the bilinear form* $B(\cdot, \cdot)$ *and the linear functional* $\mathcal{F}(\cdot)$ *are given by* (2.21) *and* (2.22), *respectively, with* $f$ *and* $g^{\pm}$ *satisfying the same conditions as in Subsection 2.2.*

Due to the coerciveness (2.27) and the Lax–Milgram lemma along with Corollary 3.4 we obtain the following statement.

**LEMMA 3.7** *The variational problem* $(I^{\Omega}_{\mathbf{N}})$ *has a unique solution.*

Further, we now reduce the three–dimensional variational Problem $(I^{\Omega}_{\mathbf{N}})$ (see (7.58)) to the two-dimensional variational problem for the vector-function of weighted moments. To this end we have to substitute (7.56) and (7.57) into (7.58), apply formulae (2.21), (2.22), (7.48), (7.50), and (7.51) along with the orthogonality property of the Legendre polynomials (6.15). We get

$$B(w, w^*) = \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sum_{r=0}^{N} \sum_{s=0}^{N} \left(r + \frac{1}{2}\right) \left(s + \frac{1}{2}\right) a^2 \sigma_{ij\underline{r}}(v) e_{ij\underline{s}}(v^*)$$

$$\times P_r(ax_3 - b) P_s(ax_3 - b) dx_3$$

$$= \int\limits_{\omega} \sum_{r=0}^{N} \left(r + \frac{1}{2}\right) a \sigma_{ij\underline{r}}(v) e_{ijr}(v^*) d\omega$$

$$= \int\limits_{\omega} \sum_{r=0}^{N} \left(r + \frac{1}{2}\right) a \left[\lambda \delta_{ij} e_{kk\underline{r}}(v) e_{ijr}(v^*) + 2\mu e_{ij\underline{r}}(v) e_{ijr}(v^*)\right] \, d\omega$$

$$= \sum_{r=0}^{N} \left(r + \frac{1}{2}\right) \int\limits_{\omega} a\Big[\lambda e_{kk\underline{r}}(v) e_{iir}(v^*) + 2\mu e_{ij\underline{r}}(v) e_{ijr}(v^*)\Big] d\omega$$

$$=: B^{\omega}_{\mathbf{N}}(v, v^*) \,, \tag{3.59}$$

$$\mathcal{F}(w^*) = \sum_{i=1}^{3} \sum_{r_i=0}^{N} \left(r + \frac{1}{2}\right) \int\limits_{\omega} a\Big[ - f_{\underline{i}r_i} + g_i^+ \sqrt{1 + (\nabla \overset{(+)}{h})^2}$$

$$+(-1)^{r_i} g_{\underline{i}}^- \sqrt{1 + (\nabla \overset{(-)}{h})^2}\Big] h^{r_i+1} v^*_{\underline{i}r_i} d\omega \;=:\; \mathcal{F}^{\omega}_{\mathbf{N}}(v^*) \,, \tag{3.60}$$

where

$$f_{\underline{i}r_i}(x) = \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} f_{\underline{i}}(x, x_3) P_{r_i}(a x_3 - b) dx_3 \;\; \text{for } i, j = 1, 2, 3, \;\; r_i = \overline{0, N_i} \,, \tag{3.61}$$

and we assume

$$f_{\underline{i}r_i}(x) = 0 \quad \text{for} \quad N_i < r_i \leq N \,. \tag{3.62}$$

Thus, (7.58) is equivalent to the following two–dimensional variational Problem $(I^{\omega}_{\mathbf{N}})$:

*Find* $v \in H^1_{\mathbf{N}}\left(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma\right)$ *such that*

$$B^{\omega}_{\mathbf{N}}(v, v^*) = \mathcal{F}^{\omega}_{\mathbf{N}}(v^*) \;\; \textit{for all} \; v^* \in H^1_{\mathbf{N}}\left(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma\right) . \tag{3.63}$$

For this problem, we show the following existence result.

**THEOREM 3.8** *If* $\omega$ *is a Lipschitz domain and*

$$\psi_{\underline{i}r_i} := h^{-\frac{1}{2}} \Bigg[ -f_{\underline{i}r_i} + g_i^+ \sqrt{1 + (\nabla \overset{(+)}{h})^2} \\ +(-1)^{r_i} g_{\underline{i}}^- \sqrt{1 + (\nabla \overset{(-)}{h})^2} \Bigg] \in L_2(\omega) \,, \tag{3.64}$$

*then Problem* $(I^{\omega}_{\mathbf{N}})$ *has a unique solution* $v$ *which satisfies the estimate*

$$||v||_{H^1_{\mathbf{N}}\left(\overset{(+)}{h}, \overset{(-)}{h}, \omega\right)} \leq \frac{||\mathcal{F}^{\omega}_{\mathbf{N}}||}{\delta_2}, \tag{3.65}$$

*where* $\delta_2$ *is the constant involved in* (2.27).

*Proof.* Due to the equality (3.59), the coerciveness and boundedness of $B(\cdot,\cdot)$ (see inequalities (2.26) and (2.27)) and isometry (7.25) (see Definition 3.1) it follows that the bilinear form $B^{\omega}_{\mathbf{N}}(\cdot,\cdot)$ is coercive and bounded:

$$B^{\omega}_{\mathbf{N}}(v,v) = B(w,w) \geq \delta_2 ||w||^2_{H^1(\Omega)} = \delta_2 ||v||^2_{H^1_{\mathbf{N}}\left(\overset{(+)}{h},\overset{(-)}{h},\omega\right)}, \tag{3.66}$$

$$\begin{aligned} B^{\omega}_{\mathbf{N}}(v,v^*) &= B(w,w^*) \leq \delta_1 ||w||_{H^1(\Omega)} ||w^*||_{H^1(\Omega)} \\ &= \delta_1 ||v||_{H^1_{\mathbf{N}}\left(\overset{(+)}{h}\overset{(-)}{h},\omega\right)} ||v^*||_{H^1_{\mathbf{N}}\left(\overset{(+)}{h},\overset{(-)}{h},\omega\right)}. \end{aligned} \tag{3.67}$$

The condition (3.64) implies that $\mathcal{F}^{\omega}_{\mathbf{N}}(\cdot)$ is a bounded linear functional on the space $H^1_{\mathbf{N}}\left(\overset{(+)}{h},\overset{(-)}{h},\omega,\gamma\right)$ as can be seen as follows:

$$\mathcal{F}^{\omega}_{\mathbf{N}}(v^*) = \sum_{i=1}^{3}\sum_{r_i=0}^{N}\left(r_i+\frac{1}{2}\right)\int_{\omega} \psi_{\underline{i}r_i} h^{r_i+\frac{1}{2}} v^*_{\underline{i}r_i}\, d\omega\,, \tag{3.68}$$

$$\begin{aligned} |\mathcal{F}^{\omega}_{\mathbf{N}}(v^*)| &\leq \sum_{i=1}^{3}\sum_{r_i=0}^{N}\left(r_i+\frac{1}{2}\right)^{1/2} ||\psi_{\underline{i}r_i}||_{L_2(\omega)}\, \Big|\Big| \left(r_i+\frac{1}{2}\right)^{1/2} h^{r_i+\frac{1}{2}} v^*_{\underline{i}r_i}\Big|\Big|_{L_2(\omega)} \\ &\leq \Big(\sum_{i=1}^{3}\sum_{r_i=0}^{N}\left(r_i+\frac{1}{2}\right)^{1/2} ||\psi_{\underline{i}r_i}||_{L_2(\omega)}\Big) ||v^*||_{H^1_{\mathbf{N}}\left(\overset{(+)}{h},\overset{(-)}{h},\omega\right)}, \end{aligned}$$

since

$$\Big\|\left(r_i+\frac{1}{2}\right)^{\frac{1}{2}} h^{r_i+\frac{1}{2}} v^*_{\underline{i}r_i}\Big\|_{L_2(\omega)} \leq ||v^*||_{H^1_{\mathbf{N}}\left(\overset{(+)}{h},\overset{(-)}{h},\omega\right)}$$

$$\text{for } r_i=\overline{0,N} \text{ and } i=1,2,3\,.$$

So, taking into account

$$\left(r_i+\frac{1}{2}\right)^{\frac{1}{2}} \leq \left(N+\frac{1}{2}\right)^{\frac{1}{2}},$$

we get

$$|\mathcal{F}^{\omega}_{\mathbf{N}}(v^*)| \leq \Big[\left(N+\frac{1}{2}\right)^{\frac{1}{2}} \sum_{i=1}^{3}\sum_{r_i=0}^{N} ||\psi_{\underline{i}r_i}||_{L_2(\omega)}\Big] ||v^*||_{H^1_{\mathbf{N}}\left(\overset{(+)}{h},\overset{(-)}{h},\omega\right)}. \tag{3.69}$$

Now, Remark 3.6 along with the Lax–Milgram lemma completes the proof. □

**REMARK 3.9** *If $f_i \in L_2(\Omega)$ then for almost every $x \in \omega$:*

$$f_i(x_1, x_2, x_3) = \sum_{r_i=0}^{\infty} \left( r_i + \frac{1}{2} \right) a f_{\underline{i} r_i}(x) P_{r_i}(a x_3 - b)$$

*and*

$$h^{-\frac{1}{2}} f_{\underline{i} r_i} \in L_2(\omega) \ \ for \ \ i = 1, 2, 3 \ \ and \ \ r_i = \overline{0, \infty}\,, \tag{3.70}$$

*due to the inequalities*

$$\sum_{r_i=0}^{N} \left( r_i + \frac{1}{2} \right) \int\limits_{\omega} a \ |f_{\underline{i} r_i}|^2 d\omega \leq \int\limits_{\Omega} |f_i|^2 d\Omega \ \ for \ \ i = 1, 2, 3.$$

*In this case,* (3.64) *is equivalent to the conditions*

$$h^{-\frac{1}{2}} \sqrt{1 + \left( \nabla \overset{(\pm)}{h} \right)^2} g_i^{\pm} \in L_2(\omega) \qquad\qquad for \ \ N \geq 1\,, \tag{3.71}$$

$$h^{-\frac{1}{2}} \left[ \sqrt{1 + (\nabla \overset{(+)}{h})^2} g_i^{+} + \sqrt{1 + (\nabla \overset{(-)}{h})^2} g_i^{-} \right] \in L_2(\omega) \quad for \ \ N = 0. \tag{3.72}$$

*Clearly,* (3.71) *implies*

$$g_i^{\pm} \in L_2(\omega). \tag{3.73}$$

For the reader's convenience here we present the following technical lemma.

**LEMMA 3.10** *Let $\Omega$ be a Lipschitz domain described in (6.1). The union $\bigcup\limits_{N_1=0}^{\infty} \widetilde{V}_{N_1}(\Omega, \Gamma)$ is dense in $H^1(\Omega, \Gamma)$.*

*Proof.* We shall prove the lemma in three steps.

*Step 1.* For $f \in C([0,1])$ and any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(\xi'') - f(\xi')| \leq \varepsilon \ \text{ if } \ |\xi' - \xi''| < \delta,$$

due to the uniform continuity of the function $f$. Denote by $B_n(\xi)$ the Bernstein polynomial corresponding to the function $f$:

$$B_n(f; \xi) = \sum_{k=0}^{n} \binom{n}{k} f(\frac{k}{n})\, \xi^k (1 - \xi)^{n-k} \ \text{ for } \xi \in [0,1] \ \text{ and } n \in \mathbb{N}\,. \tag{3.74}$$

Then (see the proof of Bernstein's theorem in [57, Ch.4, Section 5]) we have the following estimate:

$$|B_n(f; \xi) - f(\xi)| \leq \varepsilon + \frac{M}{2n\delta^2} \ \text{ for } \xi \in [0,1], \tag{3.75}$$

where $M := \max_{[0,1]}|f(\xi)|$.

In addition, we assume that $f$ is not constant, $f \in C^1([0,1])$, and

$$M' := \max_{[0,1]}|f'(\xi)| > 0. \tag{3.76}$$

The case $M' = 0$ is trivial as will be shown below at the end of Step 1.

It is evident that we can choose $\delta$ as

$$\delta = \frac{\varepsilon}{M'}\,. \tag{3.77}$$

The relations (3.75) and (3.77) give

$$|B_n(f;\xi) - f(\xi)| \le \varepsilon + \frac{M(M')^2}{2n\varepsilon^2} \quad \text{for } \xi \in [0,1] \text{ and } \varepsilon > 0\,. \tag{3.78}$$

Substitution $\varepsilon = \dfrac{M'}{\sqrt[3]{n}}$ leads to the inequality

$$|B_n(f;\xi) - f(\xi)| \le \frac{1}{\sqrt[3]{n}}(M' + \frac{1}{2}M) \tag{3.79}$$

for $\xi \in [0,1]$ and every $n \in \mathbb{N}$.

Further, if $f \in C^1([d_1,d_2])$, then (3.79) implies

$$|\widetilde{B}_n(f;\xi) - f(\xi)| \le \frac{1}{\sqrt[3]{n}}\Big(\frac{1}{2}\widetilde{M} + (d_2 - d_1)\widetilde{M'}\Big)\,, \tag{3.80}$$

where

$$\widetilde{B}_n(f;\xi) = \sum_{k=0}^{n} \binom{n}{k} f\Big(d_1 + (d_2 - d_1)\frac{k}{n}\Big)$$

$$\times\Big(\frac{\xi - d_1}{d_2 - d_1}\Big)^k \Big(\frac{d_2 - \xi}{d_2 - d_1}\Big)^{n-k}, \tag{3.81}$$

$$\widetilde{M} = \max_{[d_1,d_2]}|f(\xi)|, \quad \widetilde{M'} = \max_{[d_1,d_2]}|f'(\xi)|\,. \tag{3.82}$$

Put

$$\max\left\{\frac{1}{2}\,,\ (d_2 - d_1)\right\} = C^*. \tag{3.83}$$

Then (3.80) can be rewritten as

$$|\widetilde{B}_n(f;\xi) - f(\xi)| \le \frac{C^*}{\sqrt[3]{n}}\ \ ||f||_{C^1([d_1,d_2])}\,. \tag{3.84}$$

Remark that if $f(\xi) = const =: C$, i.e., $M' = 0$ (cf. (3.76)), it is evident that (3.84) holds, since

$$\widetilde{B}_n(\xi, C) = C \qquad \text{for all} \quad n \geq 0 \ \text{ and } \ \xi \in [d_1, d_2]\,.$$

If $f \in C^{0,\alpha}([0,1])$, $0 < \alpha < 1$, we can show, analogously, that

$$|B_n(f;\xi) - f(\xi)| \leq \frac{1}{n^{\frac{\alpha}{\alpha+2}}} \left( \overline{M}' + \frac{1}{2} M \right),$$

where

$$M := \max_{[0,1]} |f(\xi)|, \quad \overline{M}' := \sup_{0 \leq \xi, \eta \leq 1} \frac{|f(\xi) - f(\eta)|}{|\xi - \eta|^\alpha}.$$

*Step 2.* Let $\varphi$ be an arbitrary element of $H^1(\Omega, \Gamma)$.

Due to Lemma 1.10 in [45], for any $\varepsilon > 0$ there exists a function $\widetilde{\varphi} \in H^1(\Omega)$ such that the projection of supp $\widetilde{\varphi}$ onto the plane $Ox_1x_2$ is a subset of $\omega$, i.e., $\widetilde{\varphi}$ vanishes in some three–dimensional neighbourhood of the set $\bar{\Gamma} \cup \bar{\gamma}_0$, and

$$\|\varphi - \widetilde{\varphi}\|_{H^1(\Omega)} \leq \frac{\varepsilon}{3}\,. \tag{3.85}$$

Now, choose constants $d_1$ and $d_2$ such that

$$\Omega \subset \Omega_0 := \omega \times (d_1, d_2).$$

Denote

$$\tilde{\Gamma}_0 = \gamma \times (d_1, d_2).$$

Further, let $\Phi \in H^1(\Omega_0, \tilde{\Gamma}_0)$ be some extension of $\widetilde{\varphi}$ onto $\Omega_0$ (see, e.g., [49, Theorem A1]). Due to the density property of $C^\infty(\overline{\Omega}_0, \tilde{\Gamma}_0)$ in $H^1(\Omega_0, \tilde{\Gamma}_0)$ there exists a function $\psi \in C^3(\bar{\Omega}_0)$ such that

$$\|\Phi - \psi\|_{H^1(\Omega_0)} \leq \frac{\varepsilon}{3}, \tag{3.86}$$

and $\psi$ vanishes in some three-dimensional neighbouhood of $\tilde{\Gamma}_0$.

Our goal is to construct a polynomial in the variable $x_3$,

$$\psi_n(x, x_3) = \sum_{k=0}^{n} q_k(x) x_3^k \in C^1(\bar{\Omega}_0), \tag{3.87}$$

with

$$q_k \in C^1(\bar{\omega}) \quad \text{and} \quad q_k|_{\omega_\delta} = 0, \tag{3.88}$$

such that

$$\|\psi - \psi_n\|_{C^1(\bar{\Omega}_0)} \leq \frac{\varepsilon}{3(\text{mes}\,\Omega_0)^{1/2}}, \tag{3.89}$$

where $\omega_\delta$ is some two–dimensional $\delta$–neighbourhood of $\partial\omega$ (with sufficiently small $\delta > 0$).

It is easy to see that this leads to the proof of the lemma. Indeed, from (3.85), (3.86), and (3.82) it follows that

$$\begin{aligned}\|\varphi-\psi_n\|_{H^1(\Omega)} &\leq \|\varphi-\widetilde{\varphi}\|_{H^1(\Omega)}+\|\Phi-\psi_n\|_{H^1(\Omega_0)}\\ &\leq \|\varphi-\widetilde{\varphi}\|_{H^1(\Omega)}+\|\Phi-\psi\|_{H^1(\Omega_0)}+\|\psi-\psi_n\|_{H^1(\Omega_0)}\\ &\leq \frac{2\varepsilon}{3}+(\mathrm{mes}\,\Omega_0)^{1/2}\|\psi-\psi_n\|_{C^1(\bar{\Omega}_0)}\leq\frac{2\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\end{aligned}$$

since

$$\|\chi\|_{H^1(\Omega_0)}\leq(\mathrm{mes}\,\Omega_0)^{1/2}\|\chi\|_{C^1(\bar{\Omega}_0)}\ \text{ for } \chi\in C^1(\bar{\Omega}_0)\,.$$

Taking into consideration that the Legendre polynomials $\{P_k\}_{k=0}^n$ generate a basis in the set of polynomials of degree $m$ with $m \leq n$, we see that $\psi_n \in \widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)$ due to (3.88), and the lemma would be shown provided the polynomial (3.87) were constructed.

*Step 3.* To this end let us consider the polynomial

$$\begin{aligned}\widetilde{\psi}_{n-1}(x,x_3)=\sum_{k=0}^{n-1}\binom{n-1}{k}f\Big(x,d_1+(d_2-d_1)\tfrac{k}{n-1}\Big)&\\ \times\Big(\frac{x_3-d_1}{d_2-d_1}\Big)^k\Big(\frac{d_2-x_3}{d_2-d_1}\Big)^{n-k-1}&,\end{aligned}\tag{3.90}$$

where $n \geq 2$ and

$$f(x,x_3)=\frac{\partial\psi(x,x_3)}{\partial x_3}.\tag{3.91}$$

Evidently,

$$f\in C^2(\overline{\Omega_0}).\tag{3.92}$$

By (3.84) we obtain

$$\Big\|\frac{\partial\psi}{\partial x_3}-\widetilde{\psi}_{n-1}\Big\|_{C(\overline{\Omega_0})}\leq\frac{C^*}{\sqrt[3]{n-1}}\ \|\psi\|_{C^2(\overline{\Omega_0})}.\tag{3.93}$$

Introduce

$$\psi_n(x,x_3):=\psi(x,d_1)+\int_{d_1}^{x_3}\widetilde{\psi}_{n-1}(x,t)dt.\tag{3.94}$$

From (3.93),(3.94) there follows

$$\Big\|\frac{\partial\psi}{\partial x_3}-\frac{\partial\psi_n}{\partial x_3}\Big\|_{C(\overline{\Omega_0})}\leq\frac{C^*}{\sqrt[3]{n-1}}\|\psi\|_{C^2(\overline{\Omega_0})}.\tag{3.95}$$

In what follows we prove

$$\|\psi - \psi_n\|_{C(\overline{\Omega_0})} \leq \frac{C^*(d_2 - d_1)}{\sqrt[3]{n-1}} \ \|\psi\|_{C^2(\overline{\Omega_0})}\,, \tag{3.96}$$

$$\left\|\frac{\partial \psi}{\partial x_\alpha} - \frac{\partial \psi_n}{\partial x_\alpha}\right\|_{C(\overline{\Omega_0})} \leq \frac{C^*(d_2 - d_1)}{\sqrt[3]{n-1}} \|\psi\|_{C^3(\overline{\Omega_0})} \ \text{ for } \alpha = 1,2\,. \tag{3.97}$$

First let us note that, by virtue of (3.94) and (3.90), we have

$$\frac{\partial}{\partial x_\alpha}\Big(\frac{\partial \psi_n(x,x_3)}{\partial x_3}\Big) = \sum_{k=0}^{n-1} \binom{n-1}{k} \times f_{,\alpha}\Big(x, d_1 + (d_2 - d_1)\frac{k}{n-1}\Big)\Big(\frac{x_3 - d_1}{d_2 - d_1}\Big)^k \Big(\frac{d_2 - x_3}{d_2 - d_1}\Big)^{n-k-1},$$

$$\tilde{B}_{n-1}(\psi_{,3\alpha}(x,.);\ x_3) = \psi_{n,3\alpha}(x,x_3)\,, \tag{3.98}$$

due to (3.91). In view of (3.84) and (3.98) we get

$$\|\psi_{,3\alpha} - \psi_{n,3\alpha}\|_{C(\overline{\Omega_0})} \leq \frac{C^*}{\sqrt[3]{n-1}} \|\psi\|_{C^3(\overline{\Omega_0})}\,. \tag{3.99}$$

Applying the relations

$$\psi_{n,\alpha}(x,d_1) = \psi_{,\alpha}(x,d_1)\,,$$

$$\psi(x,x_3) - \psi(x,d_1) = \int_{d_1}^{x_3} \psi_{,3}(x,t)dt\,,$$

$$\psi_{,\alpha}(x,x_3) - \psi_{n,\alpha}(x,x_3) = \int_{d_1}^{x_3} [\psi_{,3\alpha}(x,t) - \psi_{n,3\alpha}(x,t)]dt$$

along with (3.93), (3.94), and (3.99) we conclude

$$\begin{aligned} |\psi(x,x_3) - \psi_n(x,x_3)| &= \Big|\int_{d_1}^{x_3} [\psi_{,3}(x,t) - \widetilde{\psi}_{n-1}(x,t)]dt\Big| \\ &\leq \frac{C^*(d_2 - d_1)}{\sqrt[3]{n-1}} \|\psi\|_{C^2(\overline{\Omega_0})}, \end{aligned} \tag{3.100}$$

$$\begin{aligned} |\psi_{,\alpha}(x,x_3) - \psi_{n,\alpha}(x,x_3)| &\leq \int_{d_1}^{x_3} \frac{C^*}{\sqrt[3]{n-1}} \|\psi\|_{C^3(\overline{\Omega_0})} dt \\ &\leq \frac{C^*(d_2 - d_1)}{\sqrt[3]{n-1}} \|\psi\|_{C^3(\overline{\Omega_0})}\,, \end{aligned} \tag{3.101}$$

which coincide with (3.96) and (3.97).

Combining the estimates (3.95), (3.96), and (3.97) we obtain

$$\|\psi - \psi_n\|_{C^1(\overline{\Omega_0})} \le \frac{3C^*(d_2 - d_1)}{\sqrt[3]{n-1}} \|\psi\|_{C^3(\overline{\Omega_0})}. \tag{3.102}$$

Taking $n$ sufficiently large,

$$n \ge 1 + \left( \frac{9(\text{mes } \Omega_0)^{1/2} C^* \|\psi\|_{C^3(\overline{\Omega_0})}}{\varepsilon} \right)^3 ,$$

we get the inequality (3.89).

It is evident that $\psi_n$ given by (3.94) can be represented in the form (3.87) with $q_k$ satisfying conditions (3.88), because of $f(x, x_3) = 0$ for $x \in \omega_\delta$ since $\psi$ vanishes in some three–dimensional neighbourhood of $\tilde{\Gamma}_0$. The proof is completed. □

## 3.5 Convergence results

**THEOREM 3.11** *Assume that*

$$f \in L_2(\Omega) \ \ \textit{and} \ \ h^{-1/2}[1 + (\nabla \overset{(\pm)}{h})^2]^{1/2} g^{\pm} \in L_2(\omega)\,. \tag{3.103}$$

*For* $\mathbf{N} \in \mathbb{N}_0^3$ *let* $\overset{\mathbf{N}}{v} \in H^1_{\mathbf{N}}(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma)$ *be the unique solution to the problem* $(I^\omega_{\mathbf{N}})$ *(see* (3.63)*) and let* $\overset{\mathbf{N}}{w} \in \widetilde{V}_{\mathbf{N}}(\Omega, \Gamma) \subset H^1(\Omega, \Gamma)$ *be the corresponding vector constructed by* (7.56)*, which represents a solution to the variational problem* (7.58).

*Finally, let* $u \in H^1(\Omega, \Gamma)$ *be the unique solution of the three-dimensional variational problem* (2.20).

*Then*

$$\|\overset{\mathbf{N}}{w} - u\|_{H^1(\Omega)} \to 0 \ \text{ as } \ N_{min} := \min\{N_1, N_2, N_3\} \to +\infty. \tag{3.104}$$

*Proof.* By standard arguments it can be shown that $\overset{\mathbf{N}}{v}$ minimizes the functional

$$J_{\mathbf{N}}(v) := \frac{1}{2} B^\omega_{\mathbf{N}}(v, v) - \mathcal{F}^\omega_{\mathbf{N}}(v) \ \text{ where } v \in H^1_{\mathbf{N}} \left( \overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma \right) , \tag{3.105}$$

$$\text{i.e.,} \qquad J_{\mathbf{N}}(\overset{\mathbf{N}}{v}) \le J_{\mathbf{N}}(v) \ \text{ for all } v \in H^1_{\mathbf{N}} \left( \overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma \right). \tag{3.106}$$

Note that (cf. (7.58) and (3.63))

$$B^\omega_{\mathbf{N}}(\overset{\mathbf{N}}{v}, v) = \mathcal{F}^\omega_{\mathbf{N}}(v) = \mathcal{F}(w) = B(\overset{\mathbf{N}}{w}, w) \ \text{ for } v \in H^1_{\mathbf{N}} \left( \overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma \right) ; \tag{3.107}$$

here and in what follows $w$ corresponds to $v$ via the formula (7.56) and is an arbitrary element of the space $\widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)$.

Further, with the help of (3.105)–(3.107) and (2.20) we find

$$\begin{aligned} B(u-\overset{\mathbf{N}}{w},u-\overset{\mathbf{N}}{w}) &= B(u,u)-2B(u,\overset{\mathbf{N}}{w})+B(\overset{\mathbf{N}}{w},\overset{\mathbf{N}}{w}) \\ &= B(u,u)-2\mathcal{F}(\overset{\mathbf{N}}{w})+B^{\omega}_{\mathbf{N}}(\overset{\mathbf{N}}{v},\overset{\mathbf{N}}{v})-2\mathcal{F}^{\omega}_{\mathbf{N}}(\overset{\mathbf{N}}{v})+2\mathcal{F}^{\omega}_{\mathbf{N}}(\overset{\mathbf{N}}{v}) \\ &\le B(u,u)-2\mathcal{F}(\overset{\mathbf{N}}{w})+B^{\omega}_{\mathbf{N}}(v,v)-2\mathcal{F}^{\omega}_{\mathbf{N}}(v)+2\mathcal{F}^{\omega}_{\mathbf{N}}(\overset{\mathbf{N}}{v}) \\ &= B(u,u)+B(w,w)-2B(u,w) \\ &= B(u-w,u-w) \qquad \text{for all } w\in \widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)\,, \end{aligned}$$

that is,

$$B(u-\overset{\mathbf{N}}{w},u-\overset{\mathbf{N}}{w}) \le B(u-w,u-w) \ \text{ for all } w\in \widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)\,. \tag{3.108}$$

From (3.108) and the coerciveness of the bilinear form $B$ (see (2.27)) it follows that

$$\delta_2\|u-\overset{\mathbf{N}}{w}\|^2_{H^1(\Omega,\Gamma)} \le B(u-\overset{\mathbf{N}}{w},u-\overset{\mathbf{N}}{w}) \le \varepsilon_{\mathbf{N}} \tag{3.109}$$

with

$$\varepsilon_{\mathbf{N}} := \inf\ B(u-w,u-w)\ge 0 \ \text{ for all } w\in \widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)\,. \tag{3.110}$$

Since $\widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)\subset \widetilde{V}_{\mathbf{N}'}(\Omega,\Gamma)$ for $N_i\le N_i'$ $(i=1,2,3)$, we conclude that $\varepsilon_{\mathbf{N}}\ge\varepsilon_{\mathbf{N}'}$, and, therefore, there exists the limit

$$\lim_{N_{min}\to+\infty}\varepsilon_{\mathbf{N}}=\varepsilon\ge 0.$$

Due to (3.109) it remains to show that $\varepsilon=0$. We prove it by contradiction. Assume that $\varepsilon>0$.

Note that, under our assumption

$$\varepsilon_{\mathbf{N}}\ge\varepsilon>0 \ \text{ for all } \mathbf{N}\in\mathbb{N}_0^3. \tag{3.111}$$

By Lemma 3.10, the union

$$\bigcup_{\mathbf{N}}\widetilde{V}_{\mathbf{N}}(\Omega,\Gamma) \quad \text{is dense in } \ H^1(\Omega,\Gamma)\,. \tag{3.112}$$

Because of (3.112), there exists a vector–function $w'$ and a vector $\mathbf{N}'$, such that

$$w'\in\widetilde{V}_{\mathbf{N}'}(\Omega,\Gamma) \ \text{ and } \ \|w'-u\|^2_{H^1(\Omega)}<\frac{\varepsilon}{2\delta_1}$$

with the same $\delta_1>0$ as in (2.26). Therefore,

$$\varepsilon_{\mathbf{N}'}\le B(u-w',\ u-w')\le\delta_1\|u-w'\|^2_{H^1(\Omega)}\le\frac{\varepsilon}{2},$$

which contradicts (3.111). Thus, $\varepsilon = 0$ and the result follows. □

Note that if $h(x) \neq 0$ for $x \in \bar{\omega}$, then there holds the following known result (see [3])

$$\|u - \overset{\mathbf{N}}{w}\|^2_{H^1(\Omega)} \leq N^{-(2s-3)} q(N), \quad \text{with } N_1 = N_2 = N_3 = N, \tag{3.113}$$

where $u$ and $\overset{\mathbf{N}}{w}$ are as in Theorem 3.11 and in addition $u \in H^s(\Omega)$ with $s \geq 2$; here $q(N) \to 0$ as $N \to +\infty$. In the case $\overset{(+)}{h} = -\overset{(-)}{h} = h = const > 0$ :

$$\|u - \overset{\mathbf{N}}{w}\|^2_{H^1(\Omega)} \leq \frac{h^{2(s-1)}}{N^{2s-3}} q(N).$$

The approach developed in [3] is essentially based on the fact that in the case of non-cusped shells the Fourier-Legendre coefficients of the solution $u \in H^1(\Omega)$ automatically belong to the space $H^1(\omega)$, which implies that the partial sums of the corresponding Fourier–Legendre series belong to the space $H^1(\Omega)$. In the case of cusped prismatic shells, in general, as has been shown above, the Fourier–Legendre coefficients and the corresponding partial sums do not belong to the spaces $H^1(\omega)$ and $H^1(\Omega)$, respectively. Therefore, the above approach needs some modifications.

**THEOREM 3.12** *Let $u$ and $\overset{\mathbf{N}}{w}$ be as in Theorem 3.11 and, in addition, let the conditions*

$$h^{-1}u \ , \ h^{-1}\overset{(\pm)}{h}_{,\alpha} u \ , \ \overset{(\pm)}{h}_{,\alpha} u_{,3} \ , \ h_*^{1/2} u \in L_2(\Omega) \ \ \textit{for } \alpha = 1, 2 \, , \tag{3.114}$$

*be fulfilled with*

$$h_* := h^{-4}\Big[\frac{1}{3} h^2_{,\alpha}\Big(\overset{(+)}{h}{}^2 + \overset{(+)}{h}\,\overset{(-)}{h} + \overset{(-)}{h}{}^2\Big) + \Big(\overset{(+)}{h}_{,\alpha} h - h_{,\alpha} \overset{(+)}{h}\Big)^2\Big]. \tag{3.115}$$

*Moreover, let*

$$\begin{aligned} &\partial_3^k u_i \in L_2(\Omega) \ \ \textit{for } k = \overline{0, s}, \\ &\partial_3^p \partial_\alpha u_i \in L_2(\Omega) \ \ \textit{for } p = \overline{0, s-1}\,, \quad \textit{and } s \geq 2, \end{aligned} \tag{3.116}$$

$$h_{,\alpha}\partial_3^m u_i \ , \ \widetilde{h}_{,\alpha}\partial_3^m u_i \in L_2(\Omega) \ \ \textit{for } m = \overline{1, s} \tag{3.117}$$

*with $h$ and $\widetilde{h}$ given by (6.2) and (2.32), respectively.*

*Then*

$$\Big\|u - \overset{\mathbf{N}}{w}\Big\|^2_{H^1(\Omega)} \leq \frac{h_0^{2s-2}}{N_{min}^{2s-3}}\, q(\mathbf{N}), \tag{3.118}$$

*where $N - N_{min} \leq C_0$ with some constant $C_0$ independent of $\mathbf{N}$ and*

$$h_0 := \max_{x \in \overline{\omega}} h(x) \quad \textit{ and } \ \ q(\mathbf{N}) \to 0 \ \ \textit{as} \ \ N_{min} \to \infty\,. \tag{3.119}$$

*Proof.* The conditions (3.114) (which are sufficient) imply that the partial Fourier-Legendre sums $S_{N_i}(u_{\underline{i}})$ of the components $u_i$, i.e.,

$$S_{N_i}(u_{\underline{i}})(x,x_3) := \sum_{r=0}^{N} a\left(r+\tfrac{1}{2}\right) u_{\underline{i}\,\underline{r}}(x) P_r(ax_3-b) \qquad \text{for } N_i=\overline{0,\infty},\quad i=1,2,3, \tag{3.120}$$

where

$$u_{ir}(x) := \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} u_i(x,x_3)P_r(ax_3-b)dx_3 \ \text{ for } r=\overline{0,\infty}\,, \tag{3.121}$$

belong to the space $\widetilde{V}_{N_i}(\Omega,\Gamma)\subset H^1(\Omega,\Gamma)$.

Introduce

$$\begin{aligned}\varepsilon_{N_i}(x,x_3) &:= u_i(x,x_3) - S_{N_i}(u_{\underline{i}})(x,x_3)\\ &= \sum_{r=N_i+1}^{\infty} a\left(r+\frac{1}{2}\right) u_{\underline{i}\,\underline{r}}(x)P_r(ax_3-b)\,.\end{aligned} \tag{3.122}$$

Our goal is to estimate the norm of $\varepsilon_{N_i}$ in the space $H^1(\Omega)$. Applying the recurrence formula

$$P_r(t)=\frac{1}{2r+1}\left[P'_{r+1}(t)-P'_{r-1}(t)\right] \ \text{ for } r\ge 1\,,$$

we get from (3.121)

$$u_{ir}(x)=\frac{1}{a(2r+1)}\Big[(\partial_3 u_i)_{r-1}-(\partial_3 u_i)_{r+1}\Big] \ \text{ for } r=\overline{1,\infty}\,. \tag{3.123}$$

With the help of (3.123) we obtain

$$\left|\left|h^{-1/2}\left(r+\frac{1}{2}\right)^{1/2} u_{ir}\right|\right|^2_{L_2(\omega)} \le \frac{c\,h_0^{2s}}{r^{2s}}\sum_{l=r-s}^{r+s}\left|\left|h^{-1/2}\left(l+\frac{1}{2}\right)^{1/2}(\partial_3^s u_i)_l\right|\right|^2_{L_2(\omega)} \tag{3.124}$$

with $c=2^{2s-1}(2s+1)[(s-1)!]^2$.

By the Parseval equality along with some additional calculations we arrive at the relations

$$\left|\left|\varepsilon_{N_i}\right|\right|^2_{L_2(\Omega)} = \sum_{r=N_i+1}^{\infty}\int\limits_{\omega} h^{-1}\Big(r+\frac{1}{2}\Big)(u_{\underline{i}r})^2 d\omega\,, \tag{3.125}$$

$$\left|\left|\partial_3\varepsilon_{N_i}\right|\right|^2_{L_2(\Omega)} = \sum_{r=N_{\underline{i}}}^{\infty}\int\limits_{\omega} h^{-1}\Big(r+\frac{1}{2}\Big)[(\partial_3 u_i)_r]^2 d\omega$$

$$+\frac{N_i(N_i-1)}{4}\int\limits_{\omega} h^{-1}[(\partial_3 u_{\underline{i}})_N]^2 d\omega$$

$$+\frac{N_i(N_i+1)}{4}\int\limits_{\omega} h^{-1}[(\partial_3 u_{\underline{i}})_{N_i+1}]^2 d\omega, \tag{3.126}$$

$$\Big|\Big|\partial_\alpha \varepsilon_{N_i}\Big|\Big|^2_{L_2(\Omega)} \leq 2\Big\{\sum_{r=N_i-1}^{\infty}\int\limits_{\omega} h^{-1}\Big(r+\frac{1}{2}\Big)[(\partial_\alpha u_{\underline{i}})_r]^2 d\omega$$
$$+N_i^2\Big[\int\limits_{\omega} h^{-1}[(h_{,\alpha}\partial_3 u_{\underline{i}})_{N_i}]^2 d\omega + \int\limits_{\omega} h^{-1}[(\widetilde{h}_{,\alpha}\partial_3 u_{\underline{i}})_{N_i}]^2 d\omega$$
$$+\int\limits_{\omega} h^{-1}[(h_{,\alpha}\partial_3 u_{\underline{i}})_{N_i+1}]^2 d\omega + \int\limits_{\omega} h^{-1}[(\widetilde{h}_{,\alpha}\partial_3 u_{\underline{i}})_{N_i+1}]^2 d\omega\Big]\Big\} \tag{3.127}$$

with $h$ and $\widetilde{h}$ given by (6.2) and (2.32), respectively.

From (3.125) and (3.124), there follows

$$\Big|\Big|\varepsilon_{N_i}\Big|\Big|^2_{L_2(\Omega)} \leq \frac{h_0^{2s}}{N_i^{2s}}\ \ q_{i1}(N_{\underline{i}}) \tag{3.128}$$

with $h_0$ given by (3.119); here and in what follows $q_{ij}(N_{\underline{i}}) \to 0$ as $N_i \to \infty$.

Further, by virtue of (3.126) and (3.123) we get

$$\Big|\Big|\partial_3 \varepsilon_{N_i}\Big|\Big|^2_{L_2(\Omega)} \leq \frac{h_0^{2s-2}}{N_i^{2s-3}}\ \ q_{i2}(N_{\underline{i}}). \tag{3.129}$$

In view of conditions (3.116) and (3.117) we analogously derive the estimate

$$\Big|\Big|\partial_\alpha \varepsilon_{N_i}\Big|\Big|^2_{L_2(\Omega)} \leq \frac{h_0^{2s-2}}{N_i^{2s-3}}\ \ q_{i3}(N_{\underline{i}})\ \ \text{for } \alpha=1,2\,. \tag{3.130}$$

Combining (3.128)–(3.130) we have

$$\sum_{i=1}^{3}\Big|\Big|\varepsilon_{N_i}\Big|\Big|^2_{H^1(\Omega)} \leq \frac{h_0^{2s-2}}{N_{min}^{2s-3}}\ \ q_4(\mathbf{N}) \tag{3.131}$$

with $q_4(\mathbf{N}) \to 0$ as $N_{min} \to \infty$.

Invoking the inequality (3.108) with

$$w = (S_{N_1}(u_1), S_{N_2}(u_2), S_{N_3}(u_3)) \in \widetilde{V}_{\mathbf{N}}(\Omega,\Gamma)$$

and taking into consideration the coerciveness and boundedness properties of the bilinear form $B(\cdot,\cdot)$ we obtain inequality (3.118). □

**REMARK 3.13** *In the symmetric case, i.e., when* $\overset{(+)}{h} = -\overset{(-)}{h} = h$, *conditions* (3.114) *are simplified and read as*

$$h^{-1}u\,,\ h^{-1}h_{,\alpha}u\,,\ h_{,\alpha}u_{,3} \in L_2(\Omega)\ \ \textit{for}\ \alpha = 1,2\,, \tag{3.132}$$

*since in the case under consideration* $h_* = \frac{1}{3}\left(\frac{h_{,\alpha}}{h}\right)^2$.

**REMARK 3.14** *In the particular case of power degeneration, i.e., when* $\overset{(+)}{h} = -\overset{(-)}{h} = h = c_0 x_2^\kappa$ *with constants* $c_0 > 0$ *and* $0 \le \kappa \le 1$, *the conditions* (3.132) *have the form*

$$x_2^{-\kappa}u\,,\ \kappa x_2^{-1}u\,,\ \kappa x_2^{\kappa-1}u_{,3} \in L_2(\Omega)\,. \tag{3.133}$$

# 4 Derivation of the basic system of two–dimensional models

## 4.1 The case of general systems

By virtue of (2.20) we obtain

$$B(\overset{\mathbf{N}}{u},\overset{\mathbf{N}}{v}) = \mathcal{F}(\overset{\mathbf{N}}{v})\ \text{ for all } \overset{\mathbf{N}}{v}_i = \sum_{m_{\underline{i}}=0}^{N_i} \overset{\mathbf{N}}{V}_{im_i}(x)P_{m_i}(ax_3-b), \tag{4.1}$$

where $P_{m_i}(\cdot)$ is a general system on $[-1,1]$ and

$$\overset{\mathbf{N}}{u}_i(x_1,x_2) = \sum_{k_{\underline{i}}=0}^{N_i} \overset{\mathbf{N}}{U}_{ik_i}(x)P_{k_i}(ax_3-b). \tag{4.2}$$

According to (2.21), (4.2), and (4.1), we have

$$\begin{aligned} B(\overset{\mathbf{N}}{u},\overset{\mathbf{N}}{v}) &= \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{ij}(\overset{\mathbf{N}}{u})e_{ij}(\overset{\mathbf{N}}{v})dx_3 = \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{ij}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i,j}dx_3 \\ &= \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i,\alpha}dx_3 + \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{i3}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i,3}dx_3 \end{aligned}$$

$$= \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{i3}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i,3} dx_3 + \int\limits_{S^{+}\cup S^{-}} \sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i} n_\alpha dS$$

$$- \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{i\alpha,\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i} dx_3 \, . \tag{4.3}$$

Let

$$I_1 := - \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{i\alpha,\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i} dx_3 \, ,$$

$$I_2 := \int\limits_{S^{+}\cup S^{-}} \sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i} n_\alpha dS \, , \quad I_3 := \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \sigma_{i3}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i,3} dx_3 \, .$$

Due to Hooke's law (6.9) we have

$$\sigma_{i\alpha}(\overset{\mathbf{N}}{u}) = \lambda\delta_{i\alpha} \sum_{k_l=0}^{N_l} \left(\overset{\mathbf{N}}{U}_{lk_l}(x) P_{k_l}(ax_3-b)\right)_{,l}$$

$$+\mu\Big\{ \sum_{k_{\underline{i}}=0}^{N_i} \left(\overset{\mathbf{N}}{U}_{ik_i}(x) P_{k_i}(ax_3-b)\right)_{,\alpha} + \sum_{k_{\underline{\alpha}}=0}^{N_\alpha} \left(\overset{\mathbf{N}}{U}_{\alpha k_\alpha}(x) P_{k_\alpha}(ax_3-b)\right)_{,i} \Big\} \, .$$

By direct calculations, we obtain

$$\sigma_{i\alpha,\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_{i} = \Big[\lambda\delta_{i\alpha} \sum_{k_l=0}^{N_l} \left(\overset{\mathbf{N}}{U}_{lk_l} P_{k_l}(ax_3-b)\right)_{,l\alpha} + \mu\Big\{ \sum_{k_i=0}^{N_i} \left(U_{ik_i} P_{k_i}(ax_3-b)\right)_{,\alpha\alpha}$$

$$+ \sum_{k_\alpha=0}^{N_\alpha} \left(\overset{\mathbf{N}}{U}_{\alpha k_\alpha} P_{k_\alpha}(ax_3-b)\right)_{,i\alpha} \Big\}\Big] \sum_{m_i=0}^{N_i} \overset{\mathbf{N}}{V}_{im_i} P_{m_i}(ax_3-b)$$

$$= \sum_{m_i=0}^{N_i} \Big[\lambda\delta_{i\alpha} \sum_{k_l=0}^{N_l} \Big\{ \overset{\mathbf{N}}{U}_{lk_l,l\alpha}(x) P_{k_l}(ax_3-b) + \overset{\mathbf{N}}{U}_{lk_l,l}(x) \left(P_{k_l}(ax_3-b)\right)_{,\alpha}$$

$$+\overset{\mathbf{N}}{U}_{lk_l,\alpha}(x) \left(P_{k_l}(ax_3-b)\right)_{,l} + \overset{\mathbf{N}}{U}_{lk_l}(x) \left(P_{k_l}(ax_3-b)\right)_{,l\alpha} \Big\}$$

$$+\mu \sum_{k_i=0}^{N_i} \Big\{ \overset{\mathbf{N}}{U}_{ik_i,\alpha\alpha}(x) P_{k_i}(ax_3-b) + 2\overset{\mathbf{N}}{U}_{ik_i,\alpha}(x) \left(P_{k_i}(ax_3-b)\right)_{,\alpha}$$

$$+\overset{\mathbf{N}}{U}_{ik_i}(x)\left(P_{k_i}(ax_3-b)\right)_{,\alpha\alpha}\Big\}$$

$$+\mu\sum_{k_\alpha=0}^{N_\alpha}\Big\{\overset{\mathbf{N}}{U}_{\alpha k_\alpha,i\alpha}(x)P_{k_\alpha}(ax_3-b)+\overset{\mathbf{N}}{U}_{\alpha k_\alpha,i}(x)\left(P_{k_\alpha}(ax_3-b)\right)_{,\alpha}$$

$$+\overset{\mathbf{N}}{U}_{\alpha k_\alpha,\alpha}(x)\left(P_{k_\alpha}(ax_3-b)\right)_{,i}+\overset{\mathbf{N}}{U}_{\alpha k_\alpha}(x)\left(P_{k_\alpha}(ax_3-b)\right)_{,i\alpha}\Big\}\Big]$$

$$\times P_{m_i}(ax_3-b)\overset{\mathbf{N}}{V}_{im_i}(x). \tag{4.4}$$

Let us introduce the notation

$$\mathcal{P}^{(p,ij)}_{(q,kl)}:=\int_{\overset{(-)}{h}}^{\overset{(+)}{h}}\left(P_p(ax_3-b)\right)_{,ij}\left(P_q(ax_3-b)\right)_{,kl}dx \tag{4.5}$$

$$\text{for } i,j,k,l,p,q=1,2,3\,,$$

$$\overset{(\pm)}{n}_3=\frac{\pm1}{\sqrt{1+(\nabla_x\overset{(\pm)}{h})^2}},\qquad \overset{(\pm)}{n}_\alpha=\frac{\mp\overset{(\pm)}{h}_{,\alpha}}{\sqrt{1+(\nabla_x\overset{(\pm)}{h})^2}}, \tag{4.6}$$

$$dS=\sqrt{1+(\overset{(\pm)}{h}_{,1})^2+(\overset{(\pm)}{h}_{,2})^2}dx=\sqrt{1+(\nabla_x\overset{(\pm)}{h})^2}dx\,.$$

Invoking (4.4) and (4.5), we obtain

$$I_1=-\sum_{m_i=0}^{N_i}\int_\omega\Big[\lambda\delta_{i\alpha}\sum_{k_l=0}^{N_l}\Big\{\overset{\mathbf{N}}{U}_{lk_l,l\alpha}\mathcal{P}^{(k_l,00)}_{(m_i,00)}+\overset{\mathbf{N}}{U}_{lk_l,l}\mathcal{P}^{(k_l,0\alpha)}_{(m_i,00)}$$

$$+\overset{\mathbf{N}}{U}_{lk_l,\alpha}\mathcal{P}^{(k_l,0l)}_{(m_i,00)}+\overset{\mathbf{N}}{U}_{lk_l}\mathcal{P}^{(k_l,l\alpha)}_{(m_i,00)}\Big\}$$

$$+\mu\sum_{k_i=0}^{N_i}\Big\{\overset{\mathbf{N}}{U}_{ik_i,\alpha\alpha}\mathcal{P}^{(k_i,00)}_{(m_i,00)}+2\overset{\mathbf{N}}{U}_{ik_i,\alpha}\mathcal{P}^{(k_i,0\alpha)}_{(m_i,00)}+\overset{\mathbf{N}}{U}_{ik_i}\mathcal{P}^{(k_i,\alpha\alpha)}_{(m_i,00)}\Big\}$$

$$+\mu\sum_{k_\alpha=0}^{N_\alpha}\Big\{\overset{\mathbf{N}}{U}_{\alpha k_\alpha,i\alpha}\mathcal{P}^{(k_\alpha,00)}_{(m_i,00)}+\overset{\mathbf{N}}{U}_{\alpha k_\alpha,i}\mathcal{P}^{(k_\alpha,0\alpha)}_{(m_i,00)}$$

$$+\overset{\mathbf{N}}{U}_{\alpha k_\alpha,\alpha}\mathcal{P}^{(k_\alpha,0i)}_{(m_i,00)}+\overset{\mathbf{N}}{U}_{\alpha k_\alpha}\mathcal{P}^{(k_\alpha,i\alpha)}_{(m_i,00)}\Big\}\Big]\overset{\mathbf{N}}{V}_{im_i}d\omega\,. \tag{4.7}$$

Further, taking into account (4.6), we get

$$I_2=\int_\omega\Big[\sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_i\overset{(+)}{n}_\alpha\Big]_{x_3=\overset{(+)}{h}(x)}\sqrt{1+(\nabla_x\overset{(+)}{h})^2}dx$$

$$+\int\limits_{\omega}\Big[\sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_i\overset{(-)}{n}_{\alpha}\Big]_{x_3=\overset{(-)}{h}(x)}\sqrt{1+(\nabla_x\overset{(-)}{h})^2}dx$$

$$=-\int\limits_{\omega}\Big[\sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_i\overset{(+)}{h}_{,\alpha}\Big]_{x_3=\overset{(+)}{h}(x)}dx+\int\limits_{\omega}\Big[\sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_i\overset{(-)}{h}_{,\alpha}\Big]_{x_3=\overset{(-)}{h}(x)}dx. \quad (4.8)$$

Note that

$$\sigma_{i\alpha}(\overset{\mathbf{N}}{u})\overset{\mathbf{N}}{v}_i=\Big[\lambda\delta_{i\alpha}\overset{\mathbf{N}}{u}_{l,l}+\mu(\overset{\mathbf{N}}{u}_{i,\alpha}+\overset{\mathbf{N}}{u}_{\alpha,i})\Big]\overset{\mathbf{N}}{v}_i$$

$$=\Big[\lambda\delta_{i\alpha}\sum_{k_l=0}^{N_l}\{\overset{\mathbf{N}}{U}_{lk_l}(x)P_{k_l}(ax_3-b)\}_{,l}+\mu\sum_{k_i=0}^{N_i}\{\overset{\mathbf{N}}{U}_{ik_i}(x)P_{k_i}(ax_3-b)\}_{,\alpha}$$

$$+\mu\sum_{k_{\alpha}=0}^{N_{\alpha}}\{\overset{\mathbf{N}}{U}_{\alpha k_{\alpha}}(x)P_{k_{\alpha}}(ax_3-b)\}_{,i}\Big]\sum_{m_i=0}^{N_i}\overset{\mathbf{N}}{V}_{im_i}(x)P_{m_i}(ax_3-b)\,. \quad (4.9)$$

In view of (4.9), from (4.8) we find

$$I_2=-\sum_{m_i=0}^{N_i}\int\limits_{\omega}\overset{(+)}{h}_{,\alpha}(x)\Big[\lambda\delta_{i\alpha}\sum_{k_l=0}^{N_l}\{\overset{\mathbf{N}}{U}_{lk_l,l}(x)P_{k_l}(1)$$

$$+\overset{\mathbf{N}}{U}_{lk_l}(x)(P_{k_l}(ax_3-b))_{,l}\Big|_{x_3=\overset{(+)}{h}(x)}\}$$

$$+\mu\sum_{k_i=0}^{N_i}\{\overset{\mathbf{N}}{U}_{ik_i,\alpha}(x)P_{k_i}(1)$$

$$+\overset{\mathbf{N}}{U}_{ik_i}(x)(P_{k_i}(ax_3-b))_{,\alpha}\Big|_{x_3=\overset{(+)}{h}(x)}\}$$

$$+\mu\sum_{k_{\alpha}=0}^{N_{\alpha}}\{\overset{\mathbf{N}}{U}_{\alpha k_{\alpha},i}(x)P_{k_{\alpha}}(1)$$

$$+\overset{\mathbf{N}}{U}_{\alpha k_{\alpha}}(x)(P_{k_{\alpha}}(ax_3-b))_{,i}\Big|_{x_3=\overset{(+)}{h}(x)}\}\Big]\overset{\mathbf{N}}{V}_{im_i}(x)P_{m_i}(1)dx$$

$$+\sum_{m_i=0}^{N_i}\int\limits_{\omega}\overset{(-)}{h}_{,\alpha}(x)\Big[\lambda\delta_{i\alpha}\sum_{k_l=0}^{N_l}\{\overset{\mathbf{N}}{U}_{lk_l,l}(x)P_{k_l}(-1)$$

$$+\overset{\mathbf{N}}{U}_{lk_l}(x)(P_{k_l}(ax_3-b))_{,l}\Big|_{x_3=\overset{(-)}{h}(x)}\}$$

$$+\mu\sum_{k_i=0}^{N_i}\{\overset{\mathbf{N}}{U}_{ik_i,\alpha}(x)P_{k_i}(-1)+\overset{\mathbf{N}}{U}_{ik_i}(x)(P_{k_i}(ax_3-b))_{,\alpha}\Big|_{x_3=\overset{(-)}{h}(x)}\}$$

$$+\mu\sum_{k_{\alpha}=0}^{N_{\alpha}}\{\overset{\mathbf{N}}{U}_{\alpha k_{\alpha},i}(x)P_{k_{\alpha}}(-1)+\overset{\mathbf{N}}{U}_{\alpha k_{\alpha}}(x)(P_{k_{\alpha}}(ax_3-b))_{,i}\Big|_{x_3=\overset{(-)}{h}(x)}\}\Big]$$

$$\times \overset{\mathbf{N}}{V}_{im_i}(x) P_{m_i}(-1) dx \,. \tag{4.10}$$

By virtue of Hooke's law (6.9), we have

$$I_3 = \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \left[\lambda \delta_{i3} \overset{\mathbf{N}}{u}_{l,l} + \mu \left(\overset{\mathbf{N}}{u}_{i,3} + \overset{\mathbf{N}}{u}_{3,i}\right)\right] \overset{\mathbf{N}}{v}_{i,3} dx_3$$

$$= \sum_{m_i=0}^{N_i} \int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big[\lambda \delta_{i3} \sum_{k_l=0}^{N_l} \{\overset{\mathbf{N}}{U}_{lk_l,l}(x) P_{k_l}(ax_3 - b)$$

$$+\overset{\mathbf{N}}{U}_{lk_l}(x) \left(P_{k_l}(ax_3 - b)\right)_{,l} \}$$

$$+\mu \sum_{k_i=0}^{N_i} \{\overset{\mathbf{N}}{U}_{ik_i}(x) \left(P_{k_i}(ax_3 - b)\right)_{,3} \}$$

$$+\mu \sum_{k_3=0}^{N_3} \{\overset{\mathbf{N}}{U}_{3k_3,i}(x)\, P_{k_3}(ax_3 - b) + \overset{\mathbf{N}}{U}_{3k_3}(x)\ \left(P_{k_3}(ax_3 - b)\right)_{,i} \}\Big]$$

$$\times \overset{\mathbf{N}}{V}_{im_i}(x) \left(P_{m_i}(ax_3 - b)\right)_{,3} dx_3 \,. \tag{4.11}$$

From (4.11) in view of (4.5) we get

$$I_3 = \sum_{m_i=0}^{N_i} \int\limits_{\omega} \Big[\lambda \delta_{i3} \sum_{k_l=0}^{N_l} \{\overset{\mathbf{N}}{U}_{lk_l,l}(x) \mathcal{P}^{(k_l,00)}_{(m_i,03)}(x) + \overset{\mathbf{N}}{U}_{lk_l}(x) \mathcal{P}^{(k_l,0l)}_{(m_i,03)}\}$$

$$+\mu \sum_{k_i=0}^{N_i} \{\overset{\mathbf{N}}{U}_{ik_i} \mathcal{P}^{(k_i,03)}_{(m_i,03)}\} + \mu \sum_{k_3=0}^{N_3} \{\overset{\mathbf{N}}{U}_{3k_3,i} \mathcal{P}^{(k_3,00)}_{(m_i,03)} + \overset{\mathbf{N}}{U}_{3k_3} \mathcal{P}^{(k_3,0i)}_{(m_i,03)}\}\Big] \overset{\mathbf{N}}{V}_{im_i} d\omega \,. \tag{4.12}$$

Taking into account (2.22), we have

$$\mathcal{F}(\overset{\mathbf{N}}{v}) = -\int\limits_{\Omega} f \cdot \overset{\mathbf{N}}{v}\, d\Omega + \int\limits_{S^+} g^+ \cdot \overset{\mathbf{N}}{v}\, dS + \int\limits_{S^-} g^- \cdot \overset{\mathbf{N}}{v}\, dS$$

$$= -\int\limits_{\omega} d\omega \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} f_i \ \sum_{m_i=0}^{N_i} \overset{\mathbf{N}}{V}_{im_i}(x) P_{m_i}(ax_3 - b) dx_3$$

$$+ \int\limits_{\omega} g_i^+(x) \sum_{m_i=0}^{N_i} \overset{\mathbf{N}}{V}_{im_i} P_{m_i}(1) \sqrt{1 + (\nabla_x \overset{(+)}{h})^2} d\omega$$

$$+\int\limits_{\omega} g_i^-(x) \sum_{m_i=0}^{N_i} \overset{\mathbf{N}}{V}_{im_i} P_{m_i}(-1)\sqrt{1+(\nabla_x \overset{(-)}{h})^2}d\omega$$

$$= \sum_{m_i=0}^{N_i} \int\limits_{\omega} \Big[ - \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} f_i(x,x_3) P_{m_i}(ax_3-b)dx_3$$

$$+P_{m_i}(1)\, g_i^+ \sqrt{1+(\nabla_x \overset{(+)}{h})^2}$$

$$+P_{m_i}(-1) g_i^- \sqrt{1+(\nabla_x \overset{(-)}{h})^2}\Big] \overset{\mathbf{N}}{V}_{im_i}(x)d\omega\,. \tag{4.13}$$

From (4.1) and (4.3) it follows that

$$I_1+I_2+I_3 = B(\overset{\mathbf{N}}{u},\overset{\mathbf{N}}{v}) = \mathcal{F}(\overset{\mathbf{N}}{v}) \text{ for all } \overset{\mathbf{N}}{v} \in \overset{\sim}{V}_{\mathbf{N}}(\Omega). \tag{4.14}$$

Because of the arbitrariness of $\overset{\mathbf{N}}{V}_{\underline{i}m_i}$, by substitution of (4.7), (4.10), (4.12), and (4.13) in (4.14), we get

$$-\Big[\lambda\delta_{\underline{i}\alpha} \sum_{k_l=0}^{N_l} \Big\{ \mathcal{P}^{(k_l,00)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{lk_l,l\alpha} + \mathcal{P}^{(k_l,0\alpha)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{lk_l,l} + \mathcal{P}^{(k_l,0l)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{lk_l,\alpha} + \mathcal{P}^{(k_l,l\alpha)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{lk_l} \Big\}$$

$$+\mu \sum_{k_{\underline{i}}=0}^{N_i} \Big\{ \mathcal{P}^{(k_i,00)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{ik_i,\alpha\alpha} + 2\mathcal{P}^{(k_i,0\alpha)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{ik_i,\alpha} + \mathcal{P}^{(k_i,\alpha\alpha)}_{(m_i,00)} \overset{\mathbf{N}}{U}_{ik_i} \Big\}$$

$$+\mu \sum_{k_\alpha=0}^{N_\alpha} \Big\{ \mathcal{P}^{(k_\alpha,00)}_{(m_{\underline{i}},00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha,i\alpha} + \mathcal{P}^{(k_\alpha,0\alpha)}_{(m_{\underline{i}},00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha,i} + \mathcal{P}^{(k_\alpha,0i)}_{(m_{\underline{i}},00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha,\alpha} + \mathcal{P}^{(k_\alpha,i\alpha)}_{(m_{\underline{i}},00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha} \Big\}\Big]$$

$$-\Big[P_{m_i}(1) \overset{(+)}{h}_{,\alpha} \Big( \lambda\delta_{i\alpha} \sum_{k_l=0}^{N_l} \Big\{ \overset{\mathbf{N}}{U}_{lk_l,l} P_{k_l}(1) + \overset{\mathbf{N}}{U}_{lk_l} (P_{k_l,l})^+ \Big\}$$

$$+\mu \sum_{k_{\underline{i}}=0}^{N_i} \Big\{ \overset{\mathbf{N}}{U}_{ik_i,\alpha} P_{k_i}(1) + \overset{\mathbf{N}}{U}_{ik_i} (P_{k_i,\alpha})^+ \Big\}$$

$$+\mu \sum_{k_\alpha=0}^{N_\alpha} \Big\{ \overset{\mathbf{N}}{U}_{\alpha k_\alpha,i} P_{k_\alpha}(1) + \overset{\mathbf{N}}{U}_{\alpha k_\alpha} (P_{k_\alpha,i})^+ \Big\}\Big)$$

$$-P_{m_{\underline{i}}}(-1) \overset{(-)}{h}_{,\alpha} \Big( \lambda\delta_{i\alpha} \sum_{k_l=0}^{N_l} \Big\{ \overset{\mathbf{N}}{U}_{lk_l,l} P_{k_l}(-1) + \overset{\mathbf{N}}{U}_{lk_l} (P_{k_l,l})^- \Big\}$$

$$+\mu \sum_{k_{\underline{i}}=0}^{N_i} \Big\{ \overset{\mathbf{N}}{U}_{ik_i,\alpha} P_{k_i}(-1) + \overset{\mathbf{N}}{U}_{ik_i} (P_{k_i,\alpha})^- \Big\}$$

$$+\mu\sum_{k_\alpha=0}^{N_\alpha}\Big\{\overset{\mathbf{N}}{U}_{\alpha k_\alpha,i}P_{k_\alpha}(-1)+\overset{\mathbf{N}}{U}_{\alpha k_\alpha}(P_{k_\alpha,i})^{-}\Big\}\Big)\Big]$$

$$+\lambda\delta_{\underline{i}3}\sum_{k_l=0}^{N_l}\Big\{\mathcal{P}^{(k_l,00)}_{(m_i,03)}\overset{\mathbf{N}}{U}_{lk_l,l}+\mathcal{P}^{(k_l,0l)}_{(m_i,03)}\overset{\mathbf{N}}{U}_{lk_l}\Big\}$$

$$+\mu\sum_{k_{\underline{i}}=0}^{N_i}\Big\{\mathcal{P}^{(k_i,03)}_{(m_i,03)}\overset{\mathbf{N}}{U}_{ik_i}\Big\}+\mu\sum_{k_3=0}^{N_3}\Big\{\mathcal{P}^{(k_3,00)}_{(m_{\underline{i}},03)}\overset{\mathbf{N}}{U}_{3k_3,i}+\mathcal{P}^{(k_3,0i)}_{(m_{\underline{i}},03)}\overset{\mathbf{N}}{U}_{3k_3}\Big\}$$

$$=-\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}}f_{\underline{i}}P_{m_i}dx_3+P_{m_i}(1)g^{+}_{\underline{i}}\sqrt{1+(\nabla_x\overset{(+)}{h})^2}$$

$$+P_{m_i}(-1)\,g^{-}_{\underline{i}}\sqrt{1+(\nabla_x\overset{(-)}{h})^2}, \tag{4.15}$$

where $i=1,2,3$, $m_i=\overline{0,N_i}$, and

$$(P_{k_l,i})^{\pm}:=(P_{k_l,i}(ax_3-b))\big|_{x_3=\overset{(\pm)}{h}(x)}. \tag{4.16}$$

Recall that in equations (4.15) the set of functions $\{P_k(ax_3-b)\}_{k=0}^{\infty}$ is an arbitrary system on $(\overset{(-)}{h},\overset{(+)}{h})$.

## 4.2 The case of the Legendre polynomials – Vekua's system

If $\{P_k(ax_3-b)\}_{k=0}^{\infty}$ is the system of Legendre polynomials, then

$$\mathcal{P}^{(k,00)}_{(\ell,00)}=\frac{\delta_{\ell k}}{a}\frac{2}{2\ell+1},\quad P_k(1)=1,\quad P_k(-1)=(-1)^k,$$

and from (4.15) we actually get I. Vekua's system (see [33]).

In what follows we rewrite this system in vector form.

If $i=\beta$, from (4.15) we have for $\beta=1,2$ and $m_\beta=\overline{0,N_\beta}$

$$-\lambda\delta_{\underline{\beta}\alpha}\sum_{k_\tau=0}^{N_\tau}\mathcal{P}^{(k_\tau,00)}_{(m_\beta,00)}\overset{\mathbf{N}}{U}_{\tau k_\tau,\tau\alpha}-\mu\Big[\sum_{k_{\underline{\beta}}=0}^{N_\beta}\mathcal{P}^{(k_\beta,00)}_{(m_\beta,00)}\overset{\mathbf{N}}{U}_{\beta k_\beta,\alpha\alpha}+\sum_{k_\alpha=0}^{N_\alpha}\mathcal{P}^{(k_\alpha,00)}_{(m_{\underline{\beta}},00)}\overset{\mathbf{N}}{U}_{\alpha k_\alpha,\beta\alpha}\Big]$$

$$-\sum_{k_\tau=0}^{N_\tau}\Big\{\lambda\delta_{\beta\alpha}\mathcal{P}^{(k_\tau,0\alpha)}_{(m_{\underline{\beta}},00)}+\lambda\delta_{\beta\alpha}\Big(\overset{(+)}{h}_{,\alpha}-(-1)^{m_{\underline{\beta}}+k_\tau}\overset{(-)}{h}_{,\alpha}\Big)\Big\}\overset{\mathbf{N}}{U}_{\tau k_\tau,\tau}$$

$$-\lambda\delta_{\beta\alpha}\Big[\sum_{k_\tau=0}^{N_\tau}\mathcal{P}^{(k_\tau,0\tau)}_{(m_{\underline{\beta}},00)}\overset{\mathbf{N}}{U}_{\tau k_\tau,\alpha}+\sum_{k_3=0}^{N_3}\mathcal{P}^{(k_3,03)}_{(m_{\underline{\beta}},00)}\overset{\mathbf{N}}{U}_{3k_3,\alpha}\Big]$$

$$
-\mu \sum_{k_{\underline{\beta}}=0}^{N_\beta} \Big\{ 2\mathcal{P}^{(k_\beta,0\alpha)}_{(m_\beta,00)} + \overset{(+)}{h}_{,\alpha} - (-1)^{m_\beta+k_\beta} \overset{(-)}{h}_{,\alpha} \Big\} \overset{\mathbf{N}}{U}_{\beta k_\beta,\alpha}
$$

$$
-\mu \sum_{k_\alpha=0}^{N_\alpha} \Big\{ \mathcal{P}^{(k_\alpha,0\alpha)}_{(m_{\underline{\beta}},00)} + \overset{(+)}{h}_{,\alpha} - (-1)^{m_\beta+k_\alpha} \overset{(-)}{h}_{,\alpha} \Big\} \overset{\mathbf{N}}{U}_{\alpha k_\alpha,\beta}
$$

$$
-\mu \sum_{k_\alpha=0}^{N_\alpha} \mathcal{P}^{(k_\alpha,0\beta)}_{(m_{\underline{\beta}},00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha,\alpha} + \mu \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,00)}_{(m_{\underline{\beta}},03)} \overset{\mathbf{N}}{U}_{3k_3,\beta}
$$

$$
-\lambda \delta_{\underline{\beta}\alpha} \Big[ \sum_{k_\tau=0}^{N_\tau} \Big\{ \mathcal{P}^{(k_\tau,\tau\alpha)}_{(m_\beta,00)} + \overset{(+)}{h}_{,\alpha} (P_{k_\tau,\tau})^+ - (-1)^{m_\beta} \overset{(-)}{h}_{,\alpha} (P_{k_\tau,\tau})^- \Big\} \overset{\mathbf{N}}{U}_{\tau k_\tau}
$$

$$
+ \sum_{k_3=0}^{N_3} \Big\{ \mathcal{P}^{(k_3,3\alpha)}_{(m_\beta,00)} + \overset{(+)}{h}_{,\alpha} (P_{k_3,3})^+ - (-1)^{m_\beta} \overset{(-)}{h}_{,\alpha} (P_{k_3,3})^- \Big\} \overset{\mathbf{N}}{U}_{3k_3} \Big]
$$

$$
-\mu \sum_{k_{\underline{\beta}}=0}^{N_\beta} \Big\{ \mathcal{P}^{(k_\beta,\alpha\alpha)}_{(m_\beta,00)} + \overset{(+)}{h}_{,\alpha} (P_{k_\beta,\alpha})^+ - (-1)^{m_\beta} \overset{(-)}{h}_{,\alpha} (P_{k_\beta,\alpha})^- - \mathcal{P}^{(k_\beta,03)}_{(m_\beta,03)} \Big\} \overset{\mathbf{N}}{U}_{\beta k_\beta}
$$

$$
-\mu \sum_{k_\alpha=0}^{N_\alpha} \Big\{ \mathcal{P}^{(k_\alpha,\beta\alpha)}_{(m_{\underline{\beta}},00)} + \overset{(+)}{h}_{,\alpha} (P_{k_\alpha,\beta})^+ - (-1)^{m_\beta} \overset{(-)}{h}_{,\alpha} (P_{k_\alpha,\beta})^- \Big\} \overset{\mathbf{N}}{U}_{\alpha k_\alpha}
$$

$$
+\mu \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,0\beta)}_{(m_\beta,03)} \overset{\mathbf{N}}{U}_{3k_3} = - \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} f_{\underline{\beta}} P_{m_\beta}(ax_3 - b) dx_3 + g^+_{\underline{\beta}} \sqrt{1 + \Big( \nabla_x \overset{(+)}{h} \Big)^2}
$$

$$
+(-1)^{m_\beta} g^-_{\underline{\beta}} \sqrt{1 + (\nabla_x \overset{(-)}{h})^2} \, . \tag{4.17}
$$

If $i = 3$, then for $m_3 = \overline{0, N_3}$,

$$
-\mu \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,00)}_{(m_3,00)} \overset{\mathbf{N}}{U}_{3k_3,\alpha\alpha} - \mu \sum_{k_3=0}^{N_3} \Big[ 2\mathcal{P}^{(k_3,0\alpha)}_{(m_3,00)} + \overset{(+)}{h}_{,\alpha}
$$

$$
-(-1)^{m_3+k_3} \overset{(-)}{h}_{,\alpha} \Big] \overset{N}{U}_{3k_3,\alpha} + \lambda \sum_{k_\tau=0}^{N_\tau} \mathcal{P}^{(k_\tau,00)}_{(m_3,03)} \overset{\mathbf{N}}{U}_{\tau k_\tau,\tau} - \mu \sum_{k_\alpha=0}^{N_\alpha} \mathcal{P}^{(k_\alpha,03)}_{(m_3,00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha,\alpha}
$$

$$
-\mu \sum_{k_3=0}^{N_3} \Big\{ \mathcal{P}^{(k_3,\alpha\alpha)}_{(m_3,00)} + \overset{(+)}{h}_{,\alpha} (P_{k_3,\alpha})^+ - (-1)^{m_3} \overset{(-)}{h}_{,\alpha} (P_{k_3,\alpha})^- - \mathcal{P}^{(k_3,03)}_{(m_3,03)} \Big\} \overset{\mathbf{N}}{U}_{3k_3}
$$

$$
-\mu \sum_{k_\alpha=0}^{N_\alpha} \Big\{ \mathcal{P}^{(k_\alpha,3\alpha)}_{(m_3,00)} + \overset{(+)}{h}_{,\alpha} (P_{k_\alpha,3})^+ - (-1)^{m_3} \overset{(-)}{h}_{,\alpha} (P_{k_\alpha,3})^- \Big\} \overset{\mathbf{N}}{U}_{\alpha k_\alpha}
$$

$$
+\lambda \Big[ \sum_{k_\tau=0}^{N_\tau} \mathcal{P}^{(k_\tau,0\tau)}_{(m_3,03)} \overset{\mathbf{N}}{U}_{\tau k_\tau} + \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,03)}_{(m_3,03)} \overset{\mathbf{N}}{U}_{3k_3} \Big] + \mu \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,03)}_{(m_3,03)} \overset{\mathbf{N}}{U}_{3k_3}
$$

$$= -\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} f_3 P_{m_3} dx_3 + g_3^+ \sqrt{1 + (\nabla_x \overset{(+)}{h})^2} + (-1)^{m_3} g_3^- \sqrt{1 + (\nabla_x \overset{(-)}{h})^2}\,. \qquad (4.18)$$

In order to write the whole set of equations in some compact form, let us introduce the unknown vector

$$\overset{\mathbf{N}}{U} := \left( \overset{\mathbf{N}}{U}_{10}, \ldots, \overset{\mathbf{N}}{U}_{1N_1}; \overset{\mathbf{N}}{U}_{20}, \ldots, \overset{\mathbf{N}}{U}_{2N_2}; \overset{\mathbf{N}}{U}_{30}, \ldots; \overset{\mathbf{N}}{U}_{3N_3} \right)^T. \qquad (4.19)$$

By $\overset{\mathbf{N}}{\mathcal{A}}$ we denote the operator corresponding to the system (4.17) and (4.18), and represent $\overset{\mathbf{N}}{\mathcal{A}}$ as the sum

$$\overset{\mathbf{N}}{\mathcal{A}} = \overset{\mathbf{N}}{\mathcal{A}}{}^{(2)} + \overset{\mathbf{N}}{\mathcal{A}}{}^{(1)} + \overset{\mathbf{N}}{\mathcal{A}}{}^{(0)}, \qquad (4.20)$$

where the operators $\overset{\mathbf{N}}{\mathcal{A}}{}^{(i)}$ for $i = 0, 1, 2$, contain only the $i-$th order derivatives and are $(N_1 + N_2 + N_3 + 3) \times (N_1 + N_2 + N_3 + 3)$ matrices. In the sequel we shall write the elements of these matrices explicitly.

The principal part of the operators in the left–hand side of the system (4.17), defining the entries of the matrix $\mathcal{A}^{(2)}$, has then the form

$$\begin{aligned} &-\mu \sum_{k_\beta=0}^{N_\beta} \mathcal{P}^{(k_\beta,00)}_{(m_\beta,00)} \Delta_2 \overset{\mathbf{N}}{U}_{\beta k_\beta} - (\lambda+\mu) \sum_{k_\alpha=0}^{N_\alpha} \mathcal{P}^{(k_\alpha,00)}_{(m_\beta,00)} \overset{\mathbf{N}}{U}_{\alpha k_\alpha, \beta\alpha} \\ &= -\mu \sum_{k_\beta=0}^{N_\beta} \delta_{k_\beta m_\beta} \kappa_{m_\beta} \Delta_2 \overset{\mathbf{N}}{U}_{\beta k_\beta} - (\lambda+\mu) \sum_{k_\alpha=0}^{N_\alpha} \delta_{k_\alpha m_\beta} \kappa_{m_\beta} \overset{\mathbf{N}}{U}_{\alpha k_\alpha, \beta\alpha}, \qquad (4.21) \\ &\quad \text{for } m_\beta = \overline{0, N_\beta} \ \text{ and } \beta = 1, 2, \end{aligned}$$

where

$$\kappa_{m_\beta} := \frac{2}{(2m_\beta + 1)a} \ \text{ and } \Delta_2 \text{ is the two–dimensional Laplacian}\ . \qquad (4.22)$$

Hence, for $\beta = 1$ and $m_1 = \overline{0, N_1}$:

$$\begin{aligned} -\kappa_{m_1} \Big[ \mu \sum_{k_1=0}^{N_1} \delta_{k_1 m_1} \Delta_2 \overset{\mathbf{N}}{U}_{1k_1} + (\lambda+\mu) \Big\{ \sum_{k_1=0}^{N_1} \delta_{k_1 m_1} \overset{\mathbf{N}}{U}_{1k_1,11} \\ + \sum_{k_2=0}^{N_2} \delta_{k_2 m_1} \overset{\mathbf{N}}{U}_{2k_2,12} \Big\} \Big], \qquad (4.23) \end{aligned}$$

while for $\beta = 2$ and $m_2 = \overline{0, N_2}$

$$-\kappa_{m_2}\Big[\mu\sum_{k_2=0}^{N_2}\delta_{k_2m_2}\Delta_2\overset{\mathbf{N}}{U}_{2k_2}+(\lambda+\mu)\Big\{\sum_{k_1=0}^{N_1}\delta_{k_1m_2}\overset{\mathbf{N}}{U}_{1k_1,21}$$

$$+\sum_{k_2=0}^{N_2}\delta_{k_2m_2}\overset{\mathbf{N}}{U}_{2k_2,22}\Big\}\Big]. \tag{4.24}$$

Without loss of generality we can take $N_1\leq N_2$, so that

$$m_1=0,1,\cdots,N_1 \ \text{ and } m_2=0,1,\cdots,N_1,N_1+1,\cdots,N_2. \tag{4.25}$$

For $\ell=\overline{0,N_1}$, the expressions (4.23) and (4.24) are simplified and can be rewritten as the two–component vector–function

$$\begin{pmatrix} -\kappa_\ell\left[\mu\Delta_2\overset{\mathbf{N}}{U}_{1\ell}+(\lambda+\mu)\left\{\overset{\mathbf{N}}{U}_{1\ell,11}+\overset{\mathbf{N}}{U}_{2l,12}\right\}\right] \\ -\kappa_\ell\left[\mu\Delta_2\overset{\mathbf{N}}{U}_{2\ell}+(\lambda+\mu)\left\{\overset{\mathbf{N}}{U}_{1\ell,21}+\overset{\mathbf{N}}{U}_{2\ell,22}\right\}\right]\end{pmatrix}$$
$$=:-\kappa_\ell A^{(2)}(\partial_x)\begin{pmatrix}\overset{\mathbf{N}}{U}_{1\ell}\\ \overset{\mathbf{N}}{U}_{2\ell}\end{pmatrix},$$

where $A^{(2)}(\partial_x):=[A^{(2)}_{kj}(\partial_x)]_{2\times 2}$ denotes the matrix of operators defined by the left–hand side expressions, i.e.,

$$\begin{aligned} &A^{(2)}_{11}(\partial_x):=\mu\Delta_2+(\lambda+\mu)\frac{\partial^2}{\partial x_1^2}, \quad A^{(2)}_{12}(\partial_x):=(\lambda+\mu)\frac{\partial^2}{\partial x_1\partial x_2},\\ &A^{(2)}_{12}(\partial_x):=(\lambda+\mu)\frac{\partial^2}{\partial x_1\partial x_2}, \qquad A^{(2)}_{22}(\partial_x):=\mu\Delta_2+(\lambda+\mu)\frac{\partial^2}{\partial x_2^2}.\end{aligned} \tag{4.26}$$

For $\ell=\overline{N_1+1,N_2}$ from (4.24), we have

$$-\kappa_\ell\Big[\mu\Delta_2\overset{\mathbf{N}}{U}_{2\ell}+(\lambda+\mu)\overset{\mathbf{N}}{U}_{2\ell,22}\Big]=-\kappa_\ell\,\Lambda_2(\partial_x)\overset{\mathbf{N}}{U}_{2\ell}, \tag{4.27}$$

where

$$\Lambda_2(\partial_x)\equiv A^{(2)}_{22}(\partial_x):=\mu\Delta_2+(\lambda+\mu)\frac{\partial^2}{\partial x_2^2}, \tag{4.28}$$

The principal part of the operator on the left–hand side of (4.18) has the form

$$-\kappa_{m_3}\mu\Delta_2\overset{\mathbf{N}}{U}_{3m_3} \ \text{ for } m_3=\overline{0,N_3}\,. \tag{4.29}$$

Now we are able to construct the $(N_1+N_2+N_3+3)\times(N_1+N_2+N_3+3)$ matrix operator $\overset{\mathbf{N}}{\mathcal{A}}{}^{(2)}$ in the following form,

$$\overset{\mathbf{N}}{\mathcal{A}}{}^{(2)}:=\begin{bmatrix}\Phi^{(11)} & \Phi^{(12)} & O^{(1)} & O^{(2)}\\ \Phi^{(21)} & \Phi^{(22)} & O^{(1)} & O^{(2)}\\ O^{(3)} & O^{(3)} & \Phi^{(1)} & O^{(4)}\\ O^{(5)} & O^{(5)} & O^{(6)} & \Phi^{(2)}\end{bmatrix}_{M\times M}, \tag{4.30}$$

where $M = N_1 + N_2 + N_3 + 3$, and

$$\begin{aligned}
\Phi^{(kj)} &= [\Phi^{(kj)}_{(pq)}]_{(N_1+1)\times(N_1+1)} := \mathrm{diag}[\kappa_1, \ldots, \kappa_{N_1+1}]\ A^{(2)}_{kj}(\partial_x) \\
& \qquad\qquad\qquad\qquad \text{for } k, j = 1, 2\,, \\
\Phi^{(1)} &= [\Phi^{(1)}_{(pq)}]_{(N_2-N_1)\times(N_2-N_1)} := \mathrm{diag}[\kappa_{N_1+1}, \ldots, \kappa_{N_2}]\ \Lambda_2(\partial_x)\,, \\
\Phi^{(2)} &= [\Phi^{(2)}_{(pq)}]_{(N_3+1)\times(N_3+1)} := \mathrm{diag}[\kappa_0, \ldots, \kappa_{N_3}]\ \mu\ \Delta_2(\partial_x)
\end{aligned}$$

with $A^{(2)}_{kj}(\partial_x)$, $\Lambda_2(\partial_x)$, and $\Delta_2(\partial_x)$ given by (4.26), (4.28), and (4.22), respectively. Here $O^{(i)}$ , $i = \overline{1, 6}$, are appropriate zero matrices.

Let us establish the explicit form of the operator $\mathcal{A}^1$.

If $\beta = 1$, the left–hand side in (4.17) becomes for $m_1 = \overline{0, N_1}$

$$\begin{aligned}
& -\sum_{k_1=0}^{N_1} \left\{ \lambda \mathcal{P}^{(k_1,01)}_{(m_1,00)} + \lambda \left( \overset{(+)}{h}_{,1} - (-1)^{m_1+k_1} \overset{(-)}{h}_{,1} \right) \right\} \overset{\mathbf{N}}{U}_{1k_1,1} \\
& -\sum_{k_2=0}^{N_2} \left\{ \lambda \mathcal{P}^{(k_2,01)}_{(m_1,00)} + \lambda \left( \overset{(+)}{h}_{,1} - (-1)^{m_1+k_2} \overset{(-)}{h}_{,1} \right) \right\} \overset{\mathbf{N}}{U}_{2k_2,2} \\
& -\lambda \sum_{k_1=0}^{N_1} \mathcal{P}^{(k_1,01)}_{(m_1,00)} \overset{\mathbf{N}}{U}_{1k_1,1} \\
& -\lambda \sum_{k_2=0}^{N_2} \mathcal{P}^{(k_2,02)}_{(m_1,00)} \overset{\mathbf{N}}{U}_{2k_2,1} - \lambda \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,03)}_{(m_1,00)} \overset{\mathbf{N}}{U}_{3k_3,1} \\
& -\mu \sum_{k_1=0}^{N_1} \left\{ 2\mathcal{P}^{(k_1,01)}_{(m_1,00)} + \overset{(+)}{h}_{,1} - (-1)^{m_1+k_1} \overset{(-)}{h}_{,1} \right\} \overset{\mathbf{N}}{U}_{1k_1,1} \\
& -\mu \sum_{k_1=0}^{N_1} \left\{ 2\mathcal{P}^{(k_1,02)}_{(m_1,00)} + \overset{(+)}{h}_{,2} - (-1)^{m_1+k_1} \overset{(-)}{h}_{,2} \right\} \overset{\mathbf{N}}{U}_{1k_1,2} \\
& -\mu \sum_{k_1=0}^{N_1} \left\{ \mathcal{P}^{(k_1,01)}_{(m_1,00)} + \overset{(+)}{h}_{,1} - (-1)^{m_1+k_1} \overset{(-)}{h}_{,1} \right\} \overset{\mathbf{N}}{U}_{1k_1,1} \\
& -\mu \sum_{k_2=0}^{N_2} \left\{ \mathcal{P}^{(k_2,02)}_{(m_1,00)} + \overset{(+)}{h}_{,2} - (-1)^{m_1+k_2} \overset{(-)}{h}_{,2} \right\} \overset{\mathbf{N}}{U}_{2k_2,1} \\
& -\mu \sum_{k_1=0}^{N_1} \mathcal{P}^{(k_1,01)}_{(m_1,00)} \overset{\mathbf{N}}{U}_{1k_1,1} - \mu \sum_{k_2=0}^{N_2} \mathcal{P}^{(k_2,01)}_{(m_1,00)} \overset{\mathbf{N}}{U}_{2k_2,2} \\
& +\mu \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,00)}_{(m_1,03)} \overset{\mathbf{N}}{U}_{3k_3,1}\,. \qquad (4.31)
\end{aligned}$$

If $\beta = 2$, then the left–hand side from (4.17) becomes for $m_2 = \overline{0, N_2}$:

$$
\begin{aligned}
- & \sum_{k_1=0}^{N_1} \left\{ \lambda \mathcal{P}^{(k_1,02)}_{(m_2,00)} + \lambda \left( \overset{(+)}{h}_{,2} - (-1)^{m_2+k_1} \overset{(-)}{h}_{,2} \right) \right\} \overset{\mathbf{N}}{U}_{1k_1,1} \\
- & \sum_{k_2=0}^{N_2} \left\{ \lambda \mathcal{P}^{(k_2,02)}_{(m_2,00)} + \lambda \left( \overset{(+)}{h}_{,2} - (-1)^{m_2+k_2} \overset{(-)}{h}_{,2} \right) \right\} \overset{\mathbf{N}}{U}_{2k_2,2} \\
- & \lambda \sum_{k_1=0}^{N_1} \mathcal{P}^{(k_1,01)}_{(m_2,00)} \overset{\mathbf{N}}{U}_{1k_1,2} - \lambda \sum_{k_2=0}^{N_2} \mathcal{P}^{(k_2,02)}_{(m_2,00)} \overset{\mathbf{N}}{U}_{2k_2,2} \\
& -\lambda \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,03)}_{(m_2,00)} \overset{\mathbf{N}}{U}_{3k_3,2} \\
- & \mu \sum_{k_2=0}^{N_2} \left\{ 2\mathcal{P}^{(k_2,01)}_{(m_2,00)} + \overset{(+)}{h}_{,1} - (-1)^{m_2+k_2} \overset{(-)}{h}_{,1} \right\} \overset{\mathbf{N}}{U}_{2k_2,1} \\
- & \mu \sum_{k_2=0}^{N_2} \left\{ 2\mathcal{P}^{(k_2,02)}_{(m_2,00)} + \overset{(+)}{h}_{,2} - (-1)^{m_2+k_2} \overset{(-)}{h}_{,2} \right\} \overset{\mathbf{N}}{U}_{2k_2,2} \\
- & \mu \sum_{k_1=0}^{N_1} \left\{ \mathcal{P}^{(k_1,01)}_{(m_2,00)} + \overset{(+)}{h}_{,1} - (-1)^{m_2+k_1} \overset{(-)}{h}_{,1} \right\} \overset{\mathbf{N}}{U}_{1k_1,2} \\
- & \mu \sum_{k_2=0}^{N_2} \left\{ \mathcal{P}^{(k_2,02)}_{(m_2,00)} + \overset{(+)}{h}_{,2} - (-1)^{m_2+k_2} \overset{(-)}{h}_{,2} \right\} \overset{\mathbf{N}}{U}_{2k_2,2} \\
- & \mu \sum_{k_1=0}^{N_1} \mathcal{P}^{(k_1,02)}_{(m_2,00)} \overset{\mathbf{N}}{U}_{1k_1,1} - \mu \sum_{k_2=0}^{N_2} \mathcal{P}^{(k_2,02)}_{(m_2,00)} \overset{\mathbf{N}}{U}_{2k_2,2} \\
& +\mu \sum_{k_3=0}^{N_3} \mathcal{P}^{(k_3,00)}_{(m_2,03)} \overset{\mathbf{N}}{U}_{3k_3,2} \, .
\end{aligned}
\tag{4.32}
$$

From (4.18) we have the following terms containing the first order derivatives for $m_3 = \overline{0, N_3}$,

$$
\begin{aligned}
- & \mu \sum_{k_3=0}^{N_3} \left\{ \left[ 2\mathcal{P}^{(k_3,01)}_{(m_3,00)} + \overset{(+)}{h}_{,1} - (-1)^{m_3+k_3} \overset{(-)}{h}_{,1} \right] \overset{\mathbf{N}}{U}_{3k_3,1} \right. \\
+ & \left. \left[ 2\mathcal{P}^{(k_3,02)}_{(m_3,00)} + \overset{(+)}{h}_{,2} - (-1)^{m_3+k_3} \overset{(-)}{h}_{,2} \right] \overset{\mathbf{N}}{U}_{3k_3,2} \right\} \\
+ & \lambda \sum_{k_1=0}^{N_1} \mathcal{P}^{(k_1,00)}_{(m_3,03)} \overset{\mathbf{N}}{U}_{1k_1,1} + \lambda \sum_{k_2=0}^{N_2} \mathcal{P}^{(k_2,00)}_{(m_3,03)} \overset{\mathbf{N}}{U}_{2k_2,2}
\end{aligned}
$$

$$- \quad \mu \sum_{k_1=0}^{N_1} \mathcal{P}_{(m_3,00)}^{(k_1,03)} \overset{\mathbf{N}}{U}_{1k_1,1} - \mu \sum_{k_2=0}^{N_2} \mathcal{P}_{(m_3,00)}^{(k_2,03)} \overset{\mathbf{N}}{U}_{2k_2,2}\,. \tag{4.33}$$

The expressions (4.31)–(4.33), for $i = 1, 2, 3$ and $m_i = \overline{0, N_i}$, can be written as

$$\begin{aligned} &\sum_{k_1=0}^{N_1} \left[ \overset{(i1)}{b}{}^{1}_{m_i k_1} \frac{\partial}{\partial x_1} + \overset{(i1)}{b}{}^{2}_{m_i k_1} \frac{\partial}{\partial x_2} \right] \overset{\mathbf{N}}{U}_{1k_1} \\ &+ \sum_{k_2=0}^{N_2} \left[ \overset{(i2)}{b}{}^{1}_{m_i k_2} \frac{\partial}{\partial x_1} + \overset{(i2)}{b}{}^{2}_{m_i k_2} \frac{\partial}{\partial x_2} \right] \overset{\mathbf{N}}{U}_{2k_2} \\ &+ \sum_{k_3=0}^{N_3} \left[ \overset{(i3)}{b}{}^{1}_{m_i k_3} \frac{\partial}{\partial x_1} + \overset{(i3)}{b}{}^{2}_{m_i k_3} \frac{\partial}{\partial x_2} \right] \overset{\mathbf{N}}{U}_{3k_3}\,, \end{aligned} \tag{4.34}$$

where, for $i = \beta = 1$ and $k_j = \overline{0, N_j}\,,\ j = 1, 2, 3\,,\ m_1 = \overline{0, N_1}$:

$$\begin{cases} \overset{(11)}{b}{}^{1}_{m_1 k_1} := -(\lambda + 2\mu) \left[ 2\mathcal{P}_{(m_1,00)}^{(k_1,01)} + \overset{(+)}{h}_{,1} - (-1)^{m_1+k_1} \overset{(-)}{h}_{,1} \right], \\ \overset{(11)}{b}{}^{2}_{m_1 k_1} := -\mu \left[ 2\mathcal{P}_{(m_1,00)}^{(k_1,02)} + \overset{(+)}{h}_{,2} - (-1)^{m_1+k_1} \overset{(-)}{h}_{,2} \right], \\ \overset{(12)}{b}{}^{1}_{m_1 k_2} := -(\lambda + \mu)\mathcal{P}_{(m_1,00)}^{(k_2,02)} - \mu \left[ \overset{(+)}{h}_{,2} - (-1)^{m_1+k_2} \overset{(-)}{h}_{,2} \right], \\ \overset{(12)}{b}{}^{2}_{m_1 k_2} := -(\lambda + \mu)\mathcal{P}_{(m_1,00)}^{(k_2,01)} - \lambda \left[ \overset{(+)}{h}_{,1} - (-1)^{m_1+k_2} \overset{(-)}{h}_{,1} \right], \\ \overset{(13)}{b}{}^{1}_{m_1 k_3} := -\lambda\mathcal{P}_{(m_1,00)}^{(k_3,03)} + \mu\mathcal{P}_{(m_1,03)}^{(k_3,00)}, \\ \overset{(13)}{b}{}^{2}_{m_1 k_3} = 0; \end{cases} \tag{4.35}$$

while for $i = \beta = 2$ and $k_j = \overline{0, N_j}\,,\ j = 1, 2, 3\,,\ m_2 = \overline{0, N_2}$:

$$\begin{cases} \overset{(21)}{b}{}^{1}_{m_2 k_1} := -(\lambda + \mu)\mathcal{P}_{(m_2,00)}^{(k_1,02)} - \lambda \left[ \overset{(+)}{h}_{,2} - (-1)^{m_2+k_1} \overset{(-)}{h}_{,2} \right], \\ \overset{(21)}{b}{}^{2}_{m_2 k_1} := -(\lambda + \mu)\mathcal{P}_{(m_2,00)}^{(k_1,01)} - \mu \left[ \overset{(+)}{h}_{,1} - (-1)^{m_2+k_1} \overset{(-)}{h}_{,1} \right], \\ \overset{(22)}{b}{}^{1}_{m_2 k_2} := -\mu \left[ 2\mathcal{P}_{(m_2,00)}^{(k_2,01)} + \overset{(+)}{h}_{,1} - (-1)^{m_2+k_2} \overset{(-)}{h}_{,1} \right], \\ \overset{(22)}{b}{}^{2}_{m_2 k_2} := -(\lambda + 2\mu) \left[ 2\mathcal{P}_{(m_2,00)}^{(k_2,02)} + \overset{(+)}{h}_{,2} - (-1)^{m_2+k_2} \overset{(-)}{h}_{,2} \right], \\ \overset{(23)}{b}{}^{1}_{m_2 k_3} := 0, \quad \overset{(23)}{b}{}^{2}_{m_2 k_3} := \mu\mathcal{P}_{(m_2,03)}^{(k_3,00)} - \lambda\mathcal{P}_{(m_2,00)}^{(k_3,03)}; \end{cases} \tag{4.36}$$

for $i=3$ and $k_j=\overline{0,N_j}$ , $j=1,2,3$ , $m_3=\overline{0,N_3}$:

$$
\left\{
\begin{array}{l}
\overset{(31)}{b}{}^{1}_{m_3k_1} := \lambda\mathcal{P}^{(k_1,00)}_{(m_3,03)} - \mu\mathcal{P}^{(k_1,03)}_{(m_3,00)}, \quad \overset{(31)}{b}{}^{2}_{m_3k_1} := 0, \\
\overset{(32)}{b}{}^{1}_{m_3k_2} := 0, \quad \overset{(32)}{b}{}^{2}_{m_3k_2} := \lambda\mathcal{P}^{(k_2,00)}_{(m_3,03)} - \mu\mathcal{P}^{(k_2,03)}_{(m_3,00)}, \\
\overset{(33)}{b}{}^{1}_{m_3k_3} := -\mu\left[2\mathcal{P}^{(k_3,01)}_{(m_3,00)} + \overset{(+)}{h}{}_{,1} - (-1)^{m_3+k_3}\overset{(-)}{h}{}_{,1}\right], \\
\overset{(33)}{b}{}^{2}_{m_3k_3} := -\mu\left[2\mathcal{P}^{(k_3,02)}_{(m_3,00)} + \overset{(+)}{h}{}_{,2} - (-1)^{m_3+k_3}\overset{(-)}{h}{}_{,2}\right].
\end{array}
\right.
\tag{4.37}
$$

Thus, the operator $\overset{\mathbf{N}}{\mathcal{A}}{}^{(1)}$ has the block form

$$
\overset{\mathbf{N}}{\mathcal{A}}{}^{(1)} := \begin{bmatrix}
\overset{\mathbf{N}}{A}{}^{(1)}_{11} & \overset{\mathbf{N}}{A}{}^{(1)}_{12} & \overset{\mathbf{N}}{A}{}^{(1)}_{13} \\
\overset{\mathbf{N}}{A}{}^{(1)}_{21} & \overset{\mathbf{N}}{A}{}^{(1)}_{22} & \overset{\mathbf{N}}{A}{}^{(1)}_{23} \\
\overset{\mathbf{N}}{A}{}^{(1)}_{31} & \overset{\mathbf{N}}{A}{}^{(1)}_{32} & \overset{\mathbf{N}}{A}{}^{(1)}_{33}
\end{bmatrix}_{M\times M}, \tag{4.38}
$$

where $M=N_1+N_2+N_3+3$ and the entries of the block matrices $\overset{\mathbf{N}}{A}{}^{(1)}_{ij}$ are given by the equalities

$$
\left(\overset{\mathbf{N}}{A}{}^{(1)}_{ij}\right)_{m_{\underline{i}}k_{\underline{j}}} = \overset{(ij)}{b}{}^{1}_{m_{\underline{i}}k_{\underline{j}}}\frac{\partial}{\partial x_1} + \overset{(ij)}{b}{}^{2}_{m_{\underline{i}}k_{\underline{j}}}\frac{\partial}{\partial x_2}
\quad \text{for } m_i=\overline{0,N_i}, \quad k_j=\overline{0,N_j}, \quad i,j=1,2,3. \tag{4.39}
$$

Let us now treat the matrix $\mathcal{A}^0$. From (4.17) for $\beta=1$, we get

$$
\begin{aligned}
&-\lambda\sum_{k_1=0}^{N_1}\left\{\mathcal{P}^{(k_1,11)}_{(m_1,00)} + \overset{(+)}{h}{}_{,1}(P_{k_1,1})^+ - (-1)^{m_1}\overset{(-)}{h}{}_{,1}(P_{k_1,1})^-\right\}\overset{\mathbf{N}}{U}_{1k_1} \\
&-\lambda\sum_{k_2=0}^{N_2}\left\{\mathcal{P}^{(k_2,21)}_{(m_1,00)} + \overset{(+)}{h}{}_{,1}(P_{k_2,2})^+ - (-1)^{m_1}\overset{(-)}{h}{}_{,1}(P_{k_2,2})^-\right\}\overset{\mathbf{N}}{U}_{2k_2} \\
&-\lambda\sum_{k_3=0}^{N_3}\left\{\mathcal{P}^{(k_3,31)}_{(m_1,00)} + \overset{(+)}{h}{}_{,1}(P_{k_3,3})^+ - (-1)^{m_1}\overset{(-)}{h}{}_{,1}(P_{k_3,3})^-\right\}\overset{\mathbf{N}}{U}_{3k_3} \\
&-\mu\sum_{k_1=0}^{N_1}\left\{\mathcal{P}^{(k_1,\alpha\alpha)}_{(m_1,00)} + \overset{(+)}{h}{}_{,\alpha}(P_{k_1,\alpha})^+ - (-1)^{m_1}\overset{(-)}{h}{}_{,\alpha}(P_{k_1,\alpha})^- - \mathcal{P}^{(k_1,03)}_{(m_1,03)}\right\}\overset{\mathbf{N}}{U}_{1k_1} \\
&-\mu\sum_{k_1=0}^{N_1}\left\{\mathcal{P}^{(k_1,11)}_{(m_1,00)} + \overset{(+)}{h}{}_{,1}(P_{k_1,1})^+ - (-1)^{m_1}\overset{(-)}{h}{}_{,1}(P_{k_1,1})^-\right\}\overset{\mathbf{N}}{U}_{1k_1} \\
&-\mu\sum_{k_2=0}^{N_2}\left\{\mathcal{P}^{(k_2,12)}_{(m_1,00)} + \overset{(+)}{h}{}_{,2}(P_{k_2,1})^+ - (-1)^{m_1}\overset{(-)}{h}{}_{,2}(P_{k_2,1})^-\right\}\overset{\mathbf{N}}{U}_{2k_2}
\end{aligned}
$$

$$
+ \mu \sum_{k_3=0}^{N_3} \mathcal{P}_{(m_1,03)}^{(k_3,01)} \overset{\mathbf{N}}{U}_{3k_3}
$$

$$
= \sum_{k_1=0}^{N_1} \overset{(11)}{c}_{m_1k_1} \overset{\mathbf{N}}{U}_{1k_1} + \sum_{k_2=0}^{N_2} \overset{(12)}{c}_{m_1k_2} \overset{\mathbf{N}}{U}_{2k_2} + \mu \sum_{k_3=0}^{N_3} \overset{(13)}{c}_{m_1k_3} \overset{\mathbf{N}}{U}_{3k_3}, \tag{4.40}
$$

where for $m_1 = \overline{0, N_1}\,,\ k_j = \overline{0, N_j}\,,\ j = 1, 2, 3$:

$$
\begin{cases}
\overset{(11)}{c}_{m_1k_1} := -\Big\{ (\lambda + 2\mu) \Big[ \mathcal{P}_{(m_1,00)}^{(k_1,11)} + \overset{(+)}{h}_{,1} (P_{k_1,1})^{+} - (-1)^{m_1} \overset{(-)}{h}_{,1} (P_{k_1,1})^{-} \Big] \\
\qquad + \mu \Big[ \mathcal{P}_{(m_1,00)}^{(k_1,22)} + \overset{(+)}{h}_{,2} (P_{k_1,2})^{+} - (-1)^{m_1} \overset{(-)}{h}_{,2} (P_{k_1,2})^{-} - \mathcal{P}_{(m_1,03)}^{(k_1,03)} \Big] \Big\}, \\
\overset{(12)}{c}_{m_1k_2} := -\Big[ \lambda \Big\{ \mathcal{P}_{(m_1,00)}^{(k_2,21)} + \overset{(+)}{h}_{,1} (P_{k_2,2})^{+} - (-1)^{m_1} \overset{(-)}{h}_{,1} (P_{k_2,2})^{-} \Big\} \\
\qquad + \mu \Big\{ \mathcal{P}_{(m_1,00)}^{(k_2,12)} + \overset{(+)}{h}_{,2} (P_{k_2,1})^{+} - (-1)^{m_1} \overset{(-)}{h}_{,2} (P_{k_2,1})^{-} \Big\} \Big], \\
\overset{(13)}{c}_{m_1k_3} := -\Big[ \lambda \Big\{ \mathcal{P}_{(m_1,00)}^{(k_3,31)} + \overset{(+)}{h}_{,1} (P_{k_3,3})^{+} - (-1)^{m_1} \overset{(-)}{h}_{,1} (P_{k_3,3})^{-} \Big\} \\
\qquad - \mu \mathcal{P}_{(m_1,03)}^{(k_3,01)} \Big].
\end{cases} \tag{4.41}
$$

From (4.17), for $\beta = 2$, we have

$$
- \lambda \sum_{k_1=0}^{N_1} \Big\{ \mathcal{P}_{(m_2,00)}^{(k_1,12)} + \overset{(+)}{h}_{,2} (P_{k_1,1})^{+} - (-1)^{m_2} \overset{(-)}{h}_{,2} (P_{k_1,1})^{-} \Big\} \overset{\mathbf{N}}{U}_{1k_1}
$$

$$
- \lambda \sum_{k_2=0}^{N_2} \Big\{ \mathcal{P}_{(m_2,00)}^{(k_2,22)} + \overset{(+)}{h}_{,2} (P_{k_2,2})^{+} - (-1)^{m_2} \overset{(-)}{h}_{,2} (P_{k_2,2})^{-} \Big\} \overset{\mathbf{N}}{U}_{2k_2}
$$

$$
- \lambda \sum_{k_3=0}^{N_3} \Big\{ \mathcal{P}_{(m_2,00)}^{(k_3,32)} + \overset{(+)}{h}_{,2} (P_{k_3,3})^{+} - (-1)^{m_2} \overset{(-)}{h}_{,2} (P_{k_3,3})^{-} \Big\} \overset{\mathbf{N}}{U}_{3k_3}
$$

$$
- \mu \sum_{k_2=0}^{N_2} \Big\{ \mathcal{P}_{(m_2,00)}^{(k_2,\alpha\alpha)} + \overset{(+)}{h}_{,\alpha} (P_{k_2,\alpha})^{+} - (-1)^{m_2} \overset{(-)}{h}_{,\alpha} (P_{k_2,\alpha})^{-} - \mathcal{P}_{(m_2,03)}^{(k_2,03)} \Big\} \overset{\mathbf{N}}{U}_{2k_2}
$$

$$
- \mu \sum_{k_1=0}^{N_1} \Big\{ \mathcal{P}_{(m_2,00)}^{(k_1,21)} + \overset{(+)}{h}_{,1} (P_{k_1,2})^{+} - (-1)^{m_2} \overset{(-)}{h}_{,1} (P_{k_1,2})^{-} \Big\} \overset{\mathbf{N}}{U}_{1k_1}
$$

$$
- \mu \sum_{k_2=0}^{N_2} \Big\{ \mathcal{P}_{(m_2,00)}^{(k_2,22)} + \overset{(+)}{h}_{,2} (P_{k_2,2})^{+} - (-1)^{m_2} \overset{(-)}{h}_{,2} (P_{k_2,2})^{-} \Big\} \overset{\mathbf{N}}{U}_{2k_2}
$$

$$+\mu\sum_{k_3=0}^{N_3}\mathcal{P}^{(k_3,02)}_{(m_2,03)}\overset{\mathbf{N}}{U}_{3k_3}$$

$$=\sum_{k_1=0}^{N_1}\overset{(21)}{c}_{m_2k_1}\overset{\mathbf{N}}{U}_{1k_1}+\sum_{k_2=0}^{N_2}\overset{(22)}{c}_{m_2k_2}\overset{\mathbf{N}}{U}_{2k_2}+\mu\sum_{k_3=0}^{N_3}\overset{(23)}{c}_{m_2k_3}\overset{\mathbf{N}}{U}_{3k_3}, \tag{4.42}$$

where for $m_2=\overline{0,N_2}\,,\ k_j=\overline{0,N_j}\,,\ j=1,2,3$:

$$\left\{\begin{aligned}
\overset{(21)}{c}_{m_2k_1} :=& -\Bigg[\lambda\left\{\mathcal{P}^{(k_1,12)}_{(m_2,00)}+\overset{(+)}{h}_{,2}(P_{k_1,1})^+-(-1)^{m_2}\overset{(-)}{h}_{,2}(P_{k_1,1})^-\right\}\\
&+\mu\left\{\mathcal{P}^{(k_1,21)}_{(m_2,00)}+\overset{(+)}{h}_{,1}(P_{k_1,2})^+-(-1)^{m_2}\overset{(-)}{h}_{,1}(P_{k_1,2})^-\right\}\Bigg],\\
\overset{(22)}{c}_{m_2k_2} :=& -\Bigg[(\lambda+2\mu)\left\{\mathcal{P}^{(k_2,22)}_{(m_2,00)}+\overset{(+)}{h}_{,2}(P_{k_2,2})^+-(-1)^{m_2}\overset{(-)}{h}_{,2}(P_{k_2,2})^-\right\}\\
&+\mu\Bigg\{\mathcal{P}^{(k_2,11)}_{(m_2,00)}+\overset{(+)}{h}_{,1}(P_{k_2,1})^+-(-1)^{m_2}\overset{(-)}{h}_{,1}(P_{k_2,1})^-\\
&-\mathcal{P}^{(k_2,03)}_{(m_2,03)}\Bigg\}\Bigg],\\
\overset{(23)}{c}_{m_2k_3} :=& -\Bigg[\lambda\left\{\mathcal{P}^{(k_3,32)}_{(m_2,00)}+\overset{(+)}{h}_{,2}(P_{k_3,3})^+-(-1)^{m_2}\overset{(-)}{h}_{,2}(P_{k_3,3})^-\right\}\\
&-\mu\mathcal{P}^{(k_3,02)}_{(m_2,03)}\Bigg].
\end{aligned}\right. \tag{4.43}$$

From (4.18) we derive

$$\begin{aligned}
&-\mu\sum_{k_3=0}^{N_3}\left\{\mathcal{P}^{(k_3,\alpha\alpha)}_{(m_3,00)}+\overset{(+)}{h}_{,\alpha}(P_{k_3,\alpha})^+-(-1)^{m_3}\overset{(-)}{h}_{,\alpha}(P_{k_3,\alpha})^--\mathcal{P}^{(k_3,03)}_{(m_3,03)}\right\}\overset{\mathbf{N}}{U}_{3k_3}\\
&-\mu\sum_{k_\alpha=0}^{N_\alpha}\left\{\mathcal{P}^{(k_\alpha,3\alpha)}_{(m_3,00)}+\overset{(+)}{h}_{,\alpha}(P_{k_\alpha,3})^+-(-1)^{m_3}\overset{(-)}{h}_{,\alpha}(P_{k_\alpha,3})^-\right\}\overset{\mathbf{N}}{U}_{\alpha k_\alpha}\\
&+\lambda\left[\sum_{k_\alpha=0}^{N_\alpha}\mathcal{P}^{(k_\alpha,0\alpha)}_{(m_3,03)}\overset{N}{U}_{\alpha k_\alpha}+\sum_{k_3=0}^{N_3}\mathcal{P}^{(k_3,03)}_{(m_3,03)}\overset{\mathbf{N}}{U}_{3k_3}\right]\\
&+\mu\sum_{k_3=0}^{N_3}\mathcal{P}^{(k_3,03)}_{(m_3,03)}\overset{\mathbf{N}}{U}_{3k_3}\\
=&\sum_{k_1=0}^{N_1}\overset{(31)}{c}_{m_3k_1}\overset{\mathbf{N}}{U}_{1k_1}+\sum_{k_2=0}^{N_2}\overset{(32)}{c}_{m_3k_2}\overset{\mathbf{N}}{U}_{2k_2}+\sum_{k_3=0}^{N_3}\overset{(33)}{c}_{m_3k_3}\overset{\mathbf{N}}{U}_{3k_3},
\end{aligned} \tag{4.44}$$

where

$$
\left\{
\begin{aligned}
&\overset{(31)}{c}_{m_3k_1} := -\mu\left[\mathcal{P}^{(k_1,31)}_{(m_3,00)} + \overset{(+)}{h}_{,1}(P_{k_1,3})^+ - (-1)^{m_3}\overset{(-)}{h}_{,1}(P_{k_1,3})^-\right]\\
&\qquad +\lambda\mathcal{P}^{(k_1,01)}_{(m_3,03)},\\
&\overset{(32)}{c}_{m_3k_2} := -\mu\left[\mathcal{P}^{(k_2,32)}_{(m_3,00)} + \overset{(+)}{h}_{,2}(P_{k_2,3})^+ - (-1)^{m_3}\overset{(-)}{h}_{,2}(P_{k_2,3})^-\right]\\
&\qquad +\lambda\mathcal{P}^{(k_2,02)}_{(m_3,03)},\\
&\overset{(33)}{c}_{m_3k_3} := -\mu\left[\mathcal{P}^{(k_3,\alpha\alpha)}_{(m_3,00)} + \overset{(+)}{h}_{,\alpha}(P_{k_3,\alpha})^+ - (-1)^{m_3}\overset{(-)}{h}_{,\alpha}(P_{k_3,\alpha})^-\right]\\
&\qquad +(\lambda+2\mu)\mathcal{P}^{(k_3,03)}_{(m_3,03)}\,,\\
&\qquad \text{for } m_3=\overline{0,N_3}\,,\ k_j=\overline{0,N_j}\,,\ j=1,2,3\,.
\end{aligned}
\right.
\tag{4.45}
$$

Thus, for the matrix $\overset{N}{\mathcal{A}}{}^{(0)}$ we obtain the block form

$$
\overset{\mathbf{N}}{\mathcal{A}}{}^{(0)} := \begin{bmatrix} \overset{\mathbf{N}}{A}{}^{(0)}_{11} & \overset{\mathbf{N}}{A}{}^{(0)}_{12} & \overset{\mathbf{N}}{A}{}^{(0)}_{13}\\ \overset{\mathbf{N}}{A}{}^{(0)}_{21} & \overset{\mathbf{N}}{A}{}^{(0)}_{22} & \overset{\mathbf{N}}{A}{}^{(0)}_{23}\\ \overset{\mathbf{N}}{A}{}^{(0)}_{31} & \overset{\mathbf{N}}{A}{}^{(0)}_{32} & \overset{\mathbf{N}}{A}{}^{(0)}_{33} \end{bmatrix}_{M\times M} ;
\tag{4.46}
$$

here $M = N_1+N_2+N_3+3$ and the $\overset{\mathbf{N}}{A}{}^{(0)}_{ij}$ are $(N_i+1)\times(N_j+1)$ matrices with entries

$$
\left(\overset{\mathbf{N}}{A}{}^{(0)}_{ij}\right)_{m_{\underline{i}}k_{\underline{j}}} = \overset{(ij)}{c}_{m_{\underline{i}}k_{\underline{j}}} \ \text{ for } m_i=\overline{0,N_i},\quad k_j=\overline{0,N_j},\quad i,j=1,2,3.
\tag{4.47}
$$

Finally, from (4.20), (4.19), (4.30), (4.38), (4.46) and (4.17), (4.18) we have

$$
\overset{\mathbf{N}}{\mathcal{A}}(\partial)\overset{\mathbf{N}}{U} = \overset{\mathbf{N}}{F},
\tag{4.48}
$$

where

$$
\overset{\mathbf{N}}{F} := \left(\overset{\mathbf{N}}{F}_{10},\cdots,\overset{\mathbf{N}}{F}_{1N_1},\overset{\mathbf{N}}{F}_{20},\cdots,\overset{\mathbf{N}}{F}_{2N_2},\overset{\mathbf{N}}{F}_{30},\cdots,\overset{\mathbf{N}}{F}_{3N_3}\right)\,,
\tag{4.49}
$$

$$
\begin{aligned}
\overset{\mathbf{N}}{F}_{\underline{i}m_i} &= -\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} f_{\underline{i}}(x,x_3)P_{m_i}(ax_3-b)dx_3 + g_i^+\sqrt{1+(\nabla_x\overset{(+)}{h})^2}\\
&+(-1)^{m_i}g_{\underline{i}}^-\sqrt{1+(\nabla_x\overset{(-)}{h})^2}\quad \text{for } m_i=\overline{0,N_i} \ \text{ and } i=1,2,3\,.
\end{aligned}
\tag{4.50}
$$

The system (4.48) represents the vector form for the **N**–th approximation. For $N_1 = N_2 = N_3 = N$ we have I. Vekua's $N$–th approximation.

From (4.2), (7.8), and (7.24), where $w$ is replaced by $u$ we conclude that

$$\overset{\mathbf{N}}{U}_{\underline{i}r_i} = \left(r_i + \frac{1}{2}\right) a u_{\underline{i}r_i} \quad \text{for } r_i = \overline{0, N_i} \ \text{ and } i = 1, 2, 3\,,$$

and

$$v_{\underline{i}r_i} = \left(r_{\underline{i}} + \frac{1}{2}\right)^{-1} a^{-1} h^{-r_i - 1} \overset{\mathbf{N}}{U}_{\underline{i}r_i}\,. \tag{4.51}$$

Therefore we can rewrite the system (4.48) in the form

$$\overset{\mathbf{N}}{\mathcal{A}}(\partial)\mathcal{B}\,\overset{\mathbf{N}}{V} = \overset{\mathbf{N}}{F} \quad \text{in } \omega, \tag{4.52}$$

where $\mathcal{B}$ is the diagonal matrix of order $N_1 + N_2 + N_3 + 3$

$$\begin{aligned} \mathcal{B} \ &:= \ \operatorname{diag}\Big[\Big(\frac{2r_1+1}{2}\, a\, h^{r_1+1}\Big)_{r_1=0}^{N_1}, \ \Big(\frac{2r_2+1}{2}\, a\, h^{r_2+1}\Big)_{r_2=0}^{N_2}, \\ &\qquad \Big(\frac{2r_3+1}{2}\, a\, h^{r_3+1}\Big)_{r_3=0}^{N_3}\Big], \\ \overset{\mathbf{N}}{V} \ &:= \ (v_{10}, \cdots, v_{1N_1},\ v_{20}, \cdots, v_{2N_2},\ v_{30}, \cdots, v_{3N_3})^T \end{aligned}$$

with $v_{\underline{i}r_i}$ given by (4.51).

The results obtained in Section 3 now lead to the following main consequence:

**THEOREM 4.1** *The BVP*

$$\overset{\mathbf{N}}{\mathcal{A}}(\partial)\mathcal{B}\overset{\mathbf{N}}{V} = \overset{\mathbf{N}}{F}, \quad \overset{\mathbf{N}}{V} \in H^1_N(\overset{(+)}{h}, \overset{(-)}{h}, \omega, \gamma) \tag{4.53}$$

*has a unique solution, where the system of differential equations is understood in the weak sense.*

We can reformulate the above (BVP) (4.53) as follows:

**PROBLEM 4.2** *Find a solution (in the weak sense)*

$$\overset{\mathbf{N}}{V} \in H^1_N(\overset{(+)}{h}, \overset{(-)}{h}, \omega)$$

*of the system* (4.52) *satisfying the Dirichlet condition* $\overset{\mathbf{N}}{V}|_\gamma = 0$ *on* $\gamma$ *in the trace sense.*

## Appendix 1A

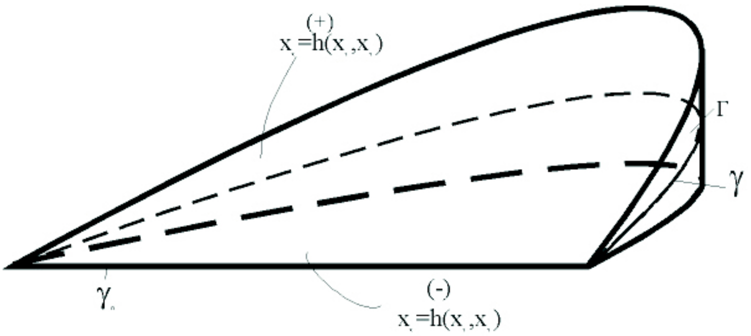


Fig. 1

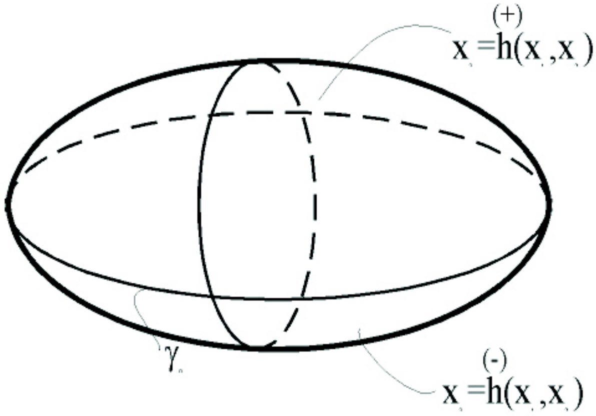


Fig. 2

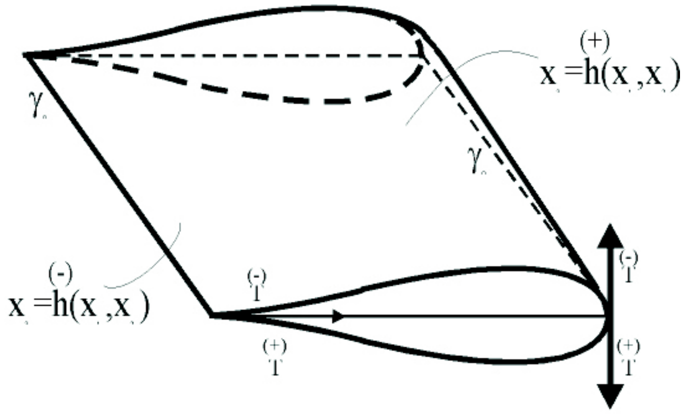


Fig. 3

## Appendix 1B

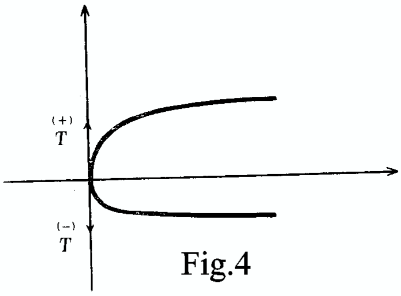


Fig.4

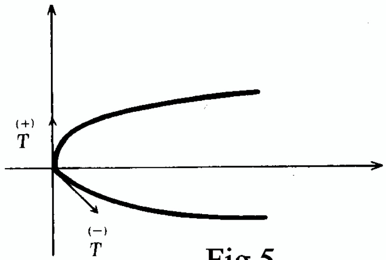


Fig.5

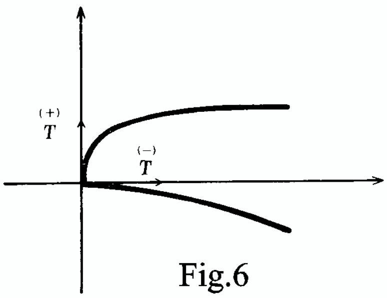


Fig.6

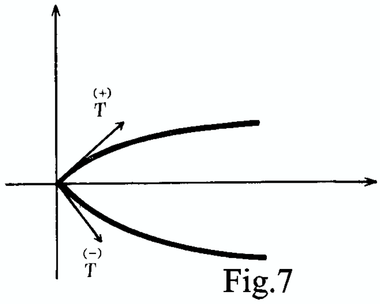


Fig.7

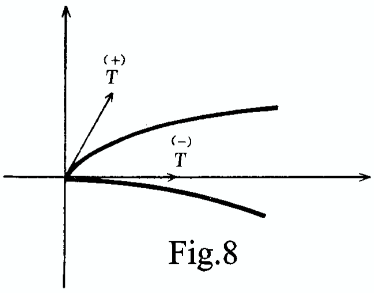


Fig.8

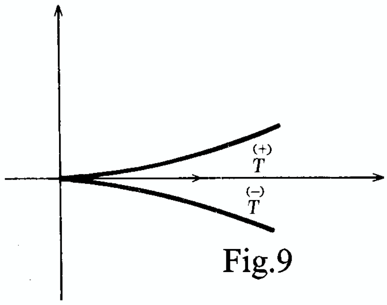


Fig.9

$\overset{(+)}{T}$ and $\overset{(-)}{T}$ denote tangents to the upper and lower profile curves at points of the cusped edges.

Figure 4 corresponds to a blant cusped edge, Figures 5-8 correspond to angularly cusped edges, and Figure 9 corresponds to a sharply cusped edge (a real mathematical cusp).

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## PART 2

## HIERARCHICAL MODELS FOR CUSPED BEAMS

Variational hierarchical one–dimensional models are constructed for cusped elastic beams. With the help of the variational methods the existence and uniqueness theorems for the corresponding one–dimensional boundary value problems are proved in appropriate weighted function spaces. By means of the solutions of these one–dimensional boundary value problems the sequence of approximate solutions in the corresponding three-dimensional region is constructed. It is established that this sequence converges (in the sense of the Sobolev space $H^1$) to the solution of the original three-dimensional boundary value problem.

## List of Notation

$\mathbb{N} := \{1, 2, \cdots\}, \qquad \mathbb{N}_0 := \{0, 1, 2, \cdots\}$

$\mathbb{R}^m \qquad m-$dimensional Euclidean space ($m \in \mathbb{N}$)

$$\Omega := \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ \ x_1 \in ]0, L[ \subset \mathbb{R}^1, \ \ \overset{(-)}{h}_j(x_1) < x_j < \overset{(+)}{h}_j(x_1), \ \ j = 2, 3\Big\}$$

$\overline{\Omega} = \Omega \cup \partial\Omega \qquad$ beam of variable cross–section

$[0, L] \subset R^1$ axis of a beam $\overline{\Omega}$

$2h_j(x_1) := \overset{(+)}{h}_j(x_1) - \overset{(-)}{h}_j(x_1) \begin{cases} > 0, & x_1 \in ]0, L[ \\ \geq 0, & x_1 \in \{0, L\} \end{cases}$ thickness ( for $j = 3$)
and width (for $j = 2$) of a beam at the point $x_1 \in [0, L]$

$$S_j^{\pm} := \Big\{ \Big( x_1, \delta_{j3} x_2 + \delta_{j2} \overset{(\pm)}{h}_2(x_1), \delta_{j2} x_3 + \delta_{j3} \overset{(\pm)}{h}_3(x_1) \Big) \in \mathbb{R}^3 : \\ x_1 \in ]0, L[, \ \ \overset{(-)}{h}_{5-j}(x_1) < x_{5-j} < \overset{(+)}{h}_{5-j}(x_1), \quad j = 2, 3 \Big\},$$

$\delta_{ij} := \begin{cases} 1 & \text{for} \ \ i = j \\ 0 & \text{for} \ \ i \neq j \end{cases}$ Kronecker's delta

$2\widetilde{h}_j(x_1) := \overset{(+)}{h}_j(x_1) + \overset{(-)}{h}_j(x_1), \quad j = 2, 3$

$a_j(x_1) := \dfrac{1}{h_j(x_1)}, \quad j = 2, 3$

$b_j(x_1) := \dfrac{\widetilde{h}_j(x_1)}{h_j(x_1)}, \quad j = 2, 3$

$\Delta_m := \dfrac{\partial^2}{\partial x_1^2} + ... + \dfrac{\partial^2}{\partial x_m^2}$

$P_n \qquad$ Legendre polynomial of order $n$

$C^\infty(\Omega, \Gamma) := \{\varphi \in C^\infty(\Omega) : \varphi|_\Gamma = 0, \ \Gamma \subset \partial\Omega\}$

$H^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n) = W^s(\mathbb{R}^n) \qquad$ Bessel potential
and Sobolev-Slobodetski space on $\mathbb{R}^n$ ($s \in \mathbb{R}$)

$H^s(\Omega) = W^s(\Omega)$ space of restrictions to $\Omega \subset \mathbb{R}^n$ of distributions
from $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$)

$\overset{\sim}{H}{}^s(\Omega) := \{\varphi \in H^s(\mathbb{R}^n) : \text{supp} \ \ \varphi \subset \bar{\Omega}\}$ ($s \in \mathbb{R}$)

$H^s(\partial\Omega)$ Sobolev-Slobodetski space on $\partial\Omega$ $(s \in \mathbb{R})$

$H^s(S_j^{\pm}),\ j = 2, 3,$ space of restrictions to $S_j^{\pm}$ of distributions from $H^s(\partial\Omega)$ $(s \in \mathbb{R})$

$H^s(\Omega, \Gamma) := \{\varphi \in H^s(\Omega) : \ \ \varphi = 0 \ \text{ on } \ \Gamma\}$ $(s \in \mathbb{R})$

$u := (u_1, u_2, u_3)^\top$ displacement vector

$e_{ij}(u) := \dfrac{1}{2}(u_{j,i} + u_{i,j}), \quad i, j = 1, 2, 3,$ strain tensor

$\sigma_{ij}(u) := \lambda\delta_{ij}e_{kk}(u) + 2\mu e_{ij}(u), \quad i, j = 1, 2, 3,$ stress tensor

$\lambda, \mu$ Lamé constants

$T(\partial, n)u$ stress vector

$[T(\partial, n)u]_j := \sigma_{ji}(u)n_i$ the $j-$th component of the stress vector $T(\partial, n)u$ $(j = 1, 2, 3)$

$(\cdot)^\top$ transposition operation

$X_1 \times X_2 \times \cdots \times X_m$ direct product of spaces $X_j,\ j = 1, ..., m$

$X^m := \underbrace{X \times \cdots \times X}_{m\ times}$

$\partial := (\partial_1, \partial_2, \cdots, \partial_n)$

$\partial_j := \dfrac{\partial}{\partial x_j},\ j = 1, \cdots, n$

$u_{i,j}(u) := \dfrac{\partial u_i}{\partial x_j}, \qquad u_{i,jk}(u) := \dfrac{\partial^2 u_i}{\partial x_j \partial x_k}, \quad i, j, k = 1, 2, 3,$

$C^m(\Omega)$, $(C^m(\bar{\Omega}))$ $m$ times continuously differentiable functions in $\Omega$ $(\overline{\Omega})$

$C(\Omega) := C^0(\Omega)$, $C(\bar{\Omega}) := C^0(\bar{\Omega})$

$C^{m,\kappa}(\Omega)$ $(C^{m,\kappa}(\bar{\Omega}))$ $m$ times continuously differentiable functions whose $m$-th order derivatives are Hölder continuous in $\Omega$ $(\bar{\Omega})$ with the exponent $\kappa \in (0, 1]$

$C^{0,1}(\Omega)$ $(C^{0,1}(\bar{\Omega}))$ space of Lipschitz continuous functions in $\Omega$ $(\bar{\Omega})$

# 5 Introduction

In the fifties of the last century, I.Vekua [33] suggested a new mathematical model for elastic prismatic shells (i.e., of plates of variable thickness) which was based on the expansion of the three-dimensional displacement vector fields and the strain and stress tensors of the linear elasticity into orthogonal Fourier-Legendre series with respect to the variable plate thickness. By taking only the first $N+1$ terms of the expansions, he obtained the so-called *$N$-th approximation*. Each of the approximations for $N = 0, 1, ...$ can be considered as an independent mathematical model of plates. In particular, the approximation for $N = 1$ actually coincides with the classical Kirchhoff–Love plate model. In the sixties, I.Vekua [34] offered the analogous mathematical model for thin shallow shells. All his results concerning plates and shells are collected in his monograph [34]. Works of I.Babuška, D.Gordeziani, V.Guliaev, I.Khoma, A.Khvoles, T.Meunargia, C.Schwab, T.Vashakmadze, V.Zhgenti, and others (see [4], [10], [11], [17], [18], [21], [29], [32], [35] and the references therein) are devoted to further analysis of I.Vekua's models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (to non-shallow shells, to the anisotropic case, etc.). At the same time I.Vekua recommended to investigate also cusped plates, i.e., plates whose thickness vanishes on some part or on the whole boundary of the plate projection (for investigations in this direction see the survey [12], [JKNW], Part 1 of this Lecture Notes, and also I.Vekua's comments in [34], p.86).

If we consider the cylindrical bending of plate, in particular, a cusped one, with the rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq \ell$, then we actually get the corresponding results also for cusped beams with constant widths (see [13], [14] and also [30], [31], [22], [23], [24], [25], [26], [5], [6]).

In [Jai4], by expanding fields of displacements, strains, and stresses of the three-dimensional theory of linear elasticity into double Fourier-Legendre series with respect to the variables of bar thickness and width, hierarchical models of beams with variable rectangular cross-sections are constructed. It is allowed that the thickness and width become zero at some points of the beam axis.

This part deals with the existence, uniqueness, and regularity properties of solutions to the boundary value problems of the hierarchical models of cusped beams.

In practice such plates and beams are encountered in calculation of spatial structures with partly fixed edges (e.g., stadium ceiling, aircraft wings etc), in machine-tool constructions (e.g., cutting-machine, planning-machine etc), in the astronautics, and other spheres of practical engineering.

This part is organized as follows.

In Section 2 we collect well-known auxilary material from the three-dimensional elasticity theory and the theory of double Fourier-Legendre series.

Section 3 deals with the construction of hierarchical models which reduce the original three-dimensional boundary value problem for cusped beam type elastic bodies (with rectangular cross-section) to one-dimensional problems.

We recall that in the *regular* case (i.e., when the area of the beam cross-section does not vanish anywhere), the double Fourier-Legendre coefficients of the displacement vector $u$, which solves the original three-dimensional problem in the space $H^1(\Omega)$, automatically belong to the space $H^1(]0,L[)$ (see [Av]). Moreover, all the double moments $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{i00},\cdots,\ \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{iN_i^{(2)}N_i^{(3)}}$ for $i=1,2,3$, and $\mathbf{N}^{(j)} := \left(N_1^{(j)}, N_2^{(j)}, N_3^{(j)}\right)$, $j=2,3$, determined by the corresponding one-dimensional hierarchical models belong to the space $H^1(]0,L[)$, while the approximation of the displacement vector $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}$ represented by means of these moments belong to the space $H^1(\Omega)$. In the case of cusped beams, the double Fourier-Legendre coefficients of the displacement vector $u \in H^1(\Omega)$ do not belong to the space $H^1(]0,L[)$ any more, in general. Moreover, in general, the space of approximate vectors $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}$ represented by the double moments $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{i00},\cdots,\ \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{iN_i^{(2)}N_i^{(3)}}$, $i=1,2,3$, of the class $H^1(]0,L[)$, do not belong to the space $H^1(\Omega)$ either. Therefore it is necessary to choose a function space for double moment functions $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{i00},\cdots,\ \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{iN_i{}^{(2)}N_i{}^{(3)}}$ defined on $]0,L[$ such that the corresponding linear combinations of these moments with the Legendre polynomials as coefficients, belong to the space $H^1(\Omega)$. This is done in Subsection 3.2.

In Subsection 3.3 we establish the uniqueness and existence results in the corresponding function spaces for the obtained one-dimensional variational hierarchical models. We remark here that the well-known approach of previous authors [2],[3] needs some modifications which are connected with the above mentioned peculiarities of the appropriate function spaces where we look for the unknown moments.

Subsection 3.4 describes the convergence in the space $H^1(\Omega)$ of the approximate solution $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}$ to the exact solution $u$ of the three-dimensional original problem. There are given the abstract error estimates with respect to the number pair of approximation order $\mathbf{N}^{(2)},\mathbf{N}^{(3)}$ and the maximum of the thickness and width $2h_j$ $(j=2,3)$ of a beam.

# 6 Preliminary material

### 2.1. Cusped beams with rectangular cross–sections

Let an *elastic beam* type three-dimensional body (later on called "beam") occupy a bounded region $\bar{\Omega}$ with the boundary $\partial\Omega$

$$\Omega := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ 0 < x_1 < L, \ \overset{(-)}{h_j}(x_1) < x_j < \overset{(+)}{h_j}(x_1), j = 2, 3\}, \quad (6.1)$$

where $]0, L[$ is the so-called *axis* of the beam $\bar{\Omega}$, $\bar{\Omega} = \Omega \cup \partial\Omega$, and $[0, L] = ]0, L[ \cup \{0\} \cup \{L\}$.

In what follows we assume that

$$\begin{aligned} &\overset{(\pm)}{h_j}(x_1) \in C^2(]0, L[) \cap C([0, L]), \\ &2h_j(x_1) := \overset{(+)}{h_j}(x_1) - \overset{(-)}{h_j}(x_1) \begin{cases} > 0 \text{ for } x_1 \in ]0, L[, \\ \geq 0 \text{ for } x_1 \in \{0, L\}, \end{cases} \end{aligned} \qquad j = 2, 3,$$

$2h_3(x_1)$ is the *thickness* and $2h_2(x_1)$ is the *width* of the beam $\bar{\Omega}$ at the point $x_1 \in [0, L]$.

Further, let

$$\Gamma_0(\Gamma_L) := \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 = 0(L), \ \overset{(-)}{h_j}(x_1) \leq x_j \leq \overset{(+)}{h_j}(x_1), \ j = 2, 3\Big\},$$

$$\begin{aligned} S_j^{\pm} := \bigg\{ \Big( & x_1, \delta_{j3} x_2 + \delta_{j2} \overset{(\pm)}{h_2}(x_1), \delta_{j2} x_3 + \delta_{j3} \overset{(\pm)}{h_3}(x_1) \Big) \in \mathbb{R}^3 : \\ & x_1 \in ]0, L[, \ \overset{(-)}{h}_{5-j}(x_1) < x_{5-j} < \overset{(+)}{h}_{5-j}(x_1), \quad j = 2, 3 \bigg\}, \end{aligned}$$

and the plane measures

$$m(\Gamma_0) := \Big[\overset{(+)}{h_2}(0) - \overset{(-)}{h_2}(0)\Big]\Big[\overset{(+)}{h_3}(0) - \overset{(-)}{h_3}(0)\Big] \geq 0,$$

$$m(\Gamma_L) := \Big[\overset{(+)}{h_2}(L) - \overset{(-)}{h_2}(L)\Big]\Big[\overset{(+)}{h_3}(L) - \overset{(-)}{h_3}(L)\Big] \geq 0.$$

If only one of the differences $\overset{(+)}{h_j}(x_1) - \overset{(-)}{h_j}(x_1)$ $(j = 2, 3)$ vanishes at $x_1 = 0$ and (or) at $x_1 = L$, then the corresponding ends $\bar{\Gamma}_0$ and $\bar{\Gamma}_L$ of the beam turn into a segment; if the both ones become zero, then the ends turn into a point.

Let

$$\Gamma := \begin{cases} \Gamma_0 \ \text{ when } m(\Gamma_0) > 0, \ m(\Gamma_L) = 0, \\ \Gamma_L \ \text{ when } m(\Gamma_0) = 0, \ \ m(\Gamma_L) > 0, \\ \Gamma_0 \cup \Gamma_L \ \text{ when } m(\Gamma_0) > 0, \ \ m(\Gamma_L) > 0. \end{cases}$$

Obviously,

$$\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_L \cup \bar{S}_2^+ \cup \bar{S}_2^- \cup \bar{S}_3^+ \cup \bar{S}_3^-,$$

where $S_3^+$ and $S_3^-$ are the *upper* and *lower face surfaces* and, $S_2^+$ and $S_2^-$ are the lateral surfaces of the beam. Note that, in general, $\partial\Omega$ is not a Lipschitz surface.

If at least one of $\bar{S}_j^+ \cap \bar{S}_j^- \neq \emptyset$ $(j = 2, 3)$ is valid then a beam is called *a cupsed beam* and appropriately $\Gamma_0$ and $\Gamma_L$ will be referred to as a *cusped end* of a cusped beam.

In Figures 1-5 are given some examples of cusped beams (see Appendix C).

In Figures 6-11 are presented all possible profiles and projections of cusped beams (see Appendix D).

### 2.2. Variational formulation of the basic three-dimensional problem for beam type bodies

The system of statics of the three-dimensional linear theory of isotropic elasticity in terms of the displacement vector reads as follows

$$A(\partial)u := \mu\Delta_3 u + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\; u = f, \qquad (6.2)$$

where $A(\partial)$ is a strongly elliptic differential operator

$$A(\partial) = [A_{kj}(\partial)]_{3\times3} := \Big[\mu\delta_{kj}\Delta_3 + (\lambda + \mu)\frac{\partial^2}{\partial x_k \partial x_j}\Big]_{3\times3},$$

$u := (u_1, u_2, u_3)^T$ is the displacement vector, $\lambda$ and $\mu$ are Lamé constants, $\delta_{kj}$ is Kronecker's symbol; the vector $-f$ corresponds to a volume force.

By

$$e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = \frac{1}{2}(u_{j,i} + u_{i,j})$$

and $\sigma_{ij}(u)$ we denote the strain and stress tensors, respectively. They are related by Hooke's law

$$\sigma_{ij}(u) = \lambda\delta_{ij}e_{kk}(u) + 2\mu e_{ij}(u) = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}),\ i, j = 1, 2, 3.$$

Here and in what follows, for brevity, we often employ the abridged notation:

i) repeated indices imply summation if they are not underlined (Greek letters run from 1 to 2, and Latin letters run from 1 to 3, unless stated otherwise);

ii) subscripts preceded by a comma will mean partial derivatives with respect to the corresponding coordinates (see the list of notation).

By $T(\partial, n)u$ we denote the stress vector calculated on the surface element with the unit normal vector $n = (n_1, n_2, n_3)$:

$$[T(\partial, n)u]_k := \sigma_{kj}(u)n_j,\ k = 1, 2, 3.$$

Recall that (6.2) can be written in the form

$$[A(\partial)u]_k = \sigma_{kj,j}(u) = f_k,\ k = 1, 2, 3.$$

Let us consider the boundary value problem (BVP):

$$A(\partial)u = f \text{ in } \Omega, \tag{6.3}$$

$$Tu = \overset{(+)}{g}{}^{(j)} \text{ on } S_j^+, \ \ j = 2, 3, \tag{6.4}$$

$$Tu = \overset{(-)}{g}{}^{(j)} \text{ on } S_j^-, \ \ j = 2, 3, \tag{6.5}$$

$$u = 0 \text{ on } \Gamma. \tag{6.6}$$

We look for a solution of the BVP (2.3)-(2.6) in the Sobolev space $[W^1(\Omega)]^3 = [H^1(\Omega)]^3$. Assuming $\Omega$ to be a Lipschitz domain, we have the natural constraints on the data (cf., e.g., [McL], Ch.4)

$$f \in \left[\tilde{H}^{-1}(\Omega)\right]^3, \quad \overset{(\pm)}{g}{}^{(j)} \in \left[H^{-\frac{1}{2}}(S_j^\pm)\right]^3, \ \ j = 2, 3, \tag{6.7}$$

which, in the case when $\bar{S}_j^+ \cap \bar{S}_j^- \neq \emptyset$ means that there exists a functional $g^{(j)}$ on

$$S := \partial\Omega \backslash (\bar{\Gamma}_0 \cup \bar{\Gamma}_L)$$

such that $g^{(j)} \in \left[H^{-\frac{1}{2}}(S)\right]^3$ and $g^{(j)}|_{S_j^\pm} = \overset{(\pm)}{g}{}^{(j)}$ on $S_j^\pm$, $j = 2, 3$ (for the tilda spaces see the list of notation).

Equation (6.3) is understood in the distributional sense. The Dirichlet type condition (6.6) is understood in the trace sense ([19], [20]). The conditions (6.4) and (6.5) are understood in the sense of the functional space $[H^{-\frac{1}{2}}(S^\pm)]^3$. Note that for $u \in [H^1(\Omega)]^3$ with $Au \in [\tilde{H}^{-1}(\Omega)]^3$, the functional $Tu \in [H^{-\frac{1}{2}}(\Gamma)]^3$ is correctly defined by the equality (Green's identity)

$$\langle Tu, u^* \rangle_{\partial\Omega} := \int_\Omega \sigma_{ij}(u) e_{ij}(u^*) dx + \langle f, u^* \rangle_\Omega \ \ \forall u^* \in [H^1(\Omega)]^3,$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes a duality between $[H^{-\frac{1}{2}}(\partial\Omega)]^3$ and $[H^{\frac{1}{2}}(\partial\Omega)]^3$, while $\langle \cdot, \cdot \rangle_\Omega$ denotes a duality between $[\tilde{H}^{-1}(\Omega)]^3$ and $[H^1(\Omega)]^3$ (cf. [20], Ch.4; [7]).

Denote

$$H^1(\Omega, \Gamma) := \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma\}.$$

The BVP (6.3)-(6.6) is equivalent to the following variational formulation.

*Problem* (I): Find $u \in [H^1(\Omega; \Gamma)]^3$ such that

$$B(u, u^*) = \mathcal{F}(u^*) \ \ \forall u^* \in [H^1(\Omega, \Gamma)]^3, \tag{6.8}$$

where

$$B(u, u^*) := \int_\Omega \sigma_{ij}(u) e_{ij}(u^*) dx, \tag{6.9}$$

$$\mathcal{F}(u^*) := -\langle f, u^* \rangle_\Omega + \sum_{j=2}^{3} \left[ \langle \overset{(+)}{g}{}^{(j)}, u^* \rangle_{S_j^+} + \langle \overset{(-)}{g}{}^{(j)}, u^* \rangle_{S_j^-} \right]; \tag{6.10}$$

here $\langle \cdot, \cdot \rangle_M$ is a duality pairing between the spaces $H^r(M)$ and $\tilde{H}^{-r}(M)$, where $r = 1$ for $M = \Omega$ and $r = 1/2$ for $M = S_j^+, S_j^-, S$.

The both above formulations are equivalent to the *minimization problem:* Find $u \in [H^1(\Omega, \Gamma)]^3$ such that

$$E(u^*) \geq E(u) \quad \forall u^* \in [H^1(\Omega, \Gamma)]^3, \tag{6.11}$$

where

$$E(u^*) := \frac{1}{2} B(u^*, u^*) - \mathcal{F}(u^*). \tag{6.12}$$

The following existence and uniqueness results are well-known (see, e.g., [8], [27], [20]).

**THEOREM 6.1** *Let $\Omega$ be a Lipschitz domain, $\Gamma \neq \emptyset$, and the conditions* (6.7) *are fulfilled. Then the BVP* (6.3) $-$ (6.6) (*i.e., the equation* (6.8) *and the minimization problem* (6.11), (6.12)) *has a unique solution $u \in [H^1(\Omega, \Gamma)]^3$ and*

$$||u||_{[H^1(\Omega)]^3} \leq C \Big\{ ||f||_{[\tilde{H}^{-1}(\Omega)]^3} + \sum_{j=2}^{3} [|| \overset{(+)}{g}{}^{(j)} ||_{[H^{-\frac{1}{2}}(S_j^+)]^3} + || \overset{(-)}{g}{}^{(j)} ||_{[H^{-\frac{1}{2}}(S_j^-)]^3} ] \Big\},$$

*where $C$ is a positive constant independent of $u, f, \overset{(\pm)}{g}{}^{(j)}$.*

The proof of the theorem is based on the Lax-Milgram lemma since :
i) $\mathcal{F}$ is a bounded linear functional;
ii) the bilinear form $B(\cdot, \cdot)$ is bounded

$$B(u, u^*) \leq \delta_1 ||u||_{[H^1(\Omega)]^3} \; ||u^*||_{[H^1(\Omega)]^3}, \quad \delta_1 = const > 0; \tag{6.13}$$

iii) $B(\cdot, \cdot)$ is coercive (due to the Korn's inequality)

$$B(u, u) \geq \delta_2 ||u||^2_{[H^1(\Omega)]^3} \quad \forall u \in [H^1(\Omega, \Gamma)]^3, \quad \delta_2 = const > 0, \tag{6.14}$$

(see e.g., [20], Theorems, 10.1 and 10.2, [28], Theorem 2.5).

Due to the regularity properties of solutions to the BVP (6.3)-(6.6), on some subsets of $\bar{\Omega}$ we get a higher smoothness for the solution by improving the smoothness of the right-hand side vector (volume force) $f$ and the prescribed stress vectors $\overset{(\pm)}{g}{}^{(j)}$.

More precisely, let $\overset{(\pm)}{g}{}^{(j)} \in [H^{r+\frac{1}{2}}(S_j^\pm)]^3$, $f \in [H^r(\Omega)]^3$, $S_j^\pm \in C^{r+1,1}$ $(j = 2, 3)$, where $r \geq 0$ is an integer. Then

$$u \in [H^{r+2}(\Omega^*)]^3,$$

where $\Omega^*$ is an arbitrary subdomain of $\bar{\Omega}$ such that $\bar{\Omega}^* \cap (\bar{\Gamma}_0 \cup \bar{\Gamma}_L \cup \partial S_2^+ \cup \partial S_2^-) = \emptyset$. Moreover, there exists a constant $C = C(\Omega^*) > 0$ such that

$$||u||_{[H^{r+2}(\Omega^*)]^3} \le C\Big\{ ||f||_{[H^r(\Omega)]^3} + \sum_{j=2}^{3} [|| \overset{(+)}{g}{}^{(j)} ||_{[H^{r+\frac{1}{2}}(S_j^+)]^3} + || \overset{(-)}{g}{}^{(j)} ||_{[H^{r+\frac{1}{2}}(S_j^-)]^3}] \Big\}.$$

In addition, if $f \in C^{0,\kappa}(\Omega)$, $\overset{(\pm)}{g}{}^{(j)} \in C^{1,\kappa}(S^{\pm})$, $S_j^{\pm} \in C^{2,\kappa}$, then $u \in C^{2,\kappa}(\bar{\Omega}^*)$ with $0 < \kappa < 1$ (cf., e.g., [20], [1], [9]).

### 2.3. Double Fourier-Legendre series

Let

$$\varphi(x_1, \cdot, \cdot) \in L_2\left( \left[ \overset{(-)}{h}_2(x_1), \overset{(+)}{h}_2(x_1) \right] \times \left[ \overset{(-)}{h}_3(x_1), \overset{(+)}{h}_3(x_1) \right] \right),$$
$$2h_j(x_1) := \overset{(+)}{h}_j(x_1) - \overset{(-)}{h}_j(x_1) > 0, \; x_1 \in ]0, L[, \quad j = 2, 3.$$

The function $\varphi(x_1, \cdot, \cdot)$ can be then represented in the form of double Fourier-Legendre series

$$\varphi(x_1, x_2, x_3) = \sum_{k_2, k_3 = 0}^{\infty} \left(k_2 + \tfrac{1}{2}\right) \left(k_3 + \tfrac{1}{2}\right) \; a_2(x_1) a_3(x_1) \; \varphi_{k_2 k_3}(x_1) \times P_{k_2}(a_2 x_2 - b_2) P_{k_3}(a_3 x_3 - b_3),$$

which converges in the $L_2$-sense and where

$$a_j := a_j(x_1) = \frac{1}{h_j(x_1)}, \; b_j := b_j(x_1) = \frac{\tilde{h}_j(x_1)}{h_j(x_1)},$$
$$2\tilde{h}_j(x_1) := \overset{(+)}{h}_j(x_1) + \overset{(-)}{h}_j(x_1), \; j = 2, 3,$$

$$\varphi_{k_2 k_3}(x_1) = \int\limits_{\overset{(-)}{h}_2(x_1)}^{\overset{(+)}{h}_2(x_1)} \int\limits_{\overset{(-)}{h}_3(x_1)}^{\overset{(+)}{h}_3(x_1)} \varphi(x_1, x_2, x_3) P_{k_2}(a_2 x_2 - b_2) P_{k_3}(a_3 x_3 - b_3) dx_2 dx_3,$$

$$k_2, k_3 = \overline{0, \infty}.$$

Note that

$$\int\limits_{\overset{(-)}{h}_j}^{\overset{(+)}{h}_j} P_k(a_j x_j - b_j) P_l(a x_j - b_j) a_j(x_1) dx_j = \begin{cases} 0 & \text{for } k \ne l, \\ \dfrac{2}{2k+1} & \text{for } k = l, \end{cases} \qquad (6.15)$$

$$t_j := a_j x_j - b_j = \begin{cases} 1 \quad \text{for} \;\; x_j = \overset{(+)}{h}_j, \\ -1 \;\; \text{for} \;\; x_j = \overset{(-)}{h}_j, \end{cases} \qquad j = 2, 3. \tag{6.16}$$

# 7 Hierarchical method for cusped beams: reduction to one-dimensional models

### 3.1. Double Legendre moments

Let $\Omega$ be a Lipschitz domain as described in Subsection 2.1, $f \in [C^{0,\alpha}(\bar{\Omega})]^3$, and $u \in [H^1(\Omega)]^3 \cap [C^{2,\alpha}(\Omega)]^3 \cap [C^2(\Omega \cup S_2^+ \cup S_2^- \cup S_3^+ \cup S_3^-)]^3$ be a unique solution to the BVP (6.3)-(6.6). Then $u_i$ can be expanded into the Fourier-Legendre series in $\Omega$

$$u_i(x_1, x_2, x_3) = \sum_{k_2, k_3 = 0}^{\infty} \left(k_2 + \frac{1}{2}\right) \left(k_3 + \frac{1}{2}\right) a_2 a_3 u_{ik_2k_3}(x_1)$$

$$\times P_{k_2}(a_2 x_2 - b_2) P_{k_3}(a_3 x_3 - b_3), \quad i = 1, 2, 3,$$

$$x_1 \in ]0, L[, \quad \overset{(-)}{h}_j(x) < x_j < \overset{(+)}{h}_j(x), \; j = 2, 3,$$

where

$$\begin{aligned} &\overset{(+)}{h}_j(x_1) - \overset{(-)}{h}_j(x_1) > 0 \quad \text{for} \quad x_1 \in ]0, L[, \\ &\overset{(+)}{h}_j(x_1) - \overset{(-)}{h}_j(x_1) \geq 0 \quad \text{for} \quad x = 0, L, \end{aligned} \qquad j = 2, 3,$$

$$u_{ik_2k_3}(x_1) = \int_{\overset{(-)}{h}_2(x_1)}^{\overset{(+)}{h}_2(x_1)} \int_{\overset{(-)}{h}_3(x_1)}^{\overset{(+)}{h}_3(x_1)} u_i(x_1, x_2, x_3) P_{k_2}(a_2 x_2 - b_2)$$

$$\times P_{k_3}(a_3 x_3 - b_3) dx_2 dx_3, \tag{7.1}$$

$$k_2, k_3 = \overline{0, \infty}, \; i = 1, 2, 3.$$

It is evident, that (6.6) implies

$$u_{ik_2k_3}(0) = 0 \;\; \text{and} \;\; u_{ik_2k_3}(L) = 0, \;\; \text{when} \;\; m(\Gamma_0) > 0 \;\; \text{and} \;\; m(\Gamma_L) > 0,$$

respectively, due to the trace theorem.

Note that with the help of (6.16) the Fourier coefficient (7.1) can be rewritten as follows

$$u_{ik_2k_3}(x_1) = \frac{1}{a_2\, a_3} \int\limits_{-1}^{+1}\int\limits_{-1}^{+1} u_i\left(x_1, \frac{t_2+b_2}{a_2}, \frac{t_3+b_3}{a_3}\right)\ P_{k_2}(t_2)P_{k_3}(t_3)dt_2dt_3$$

$$= h_2(x_1)h_3(x_1) \int\limits_{-1}^{+1}\int\limits_{-1}^{+1} u_i(x_1, h_2(x_1)t_2 + \tilde{h}(x_1), h_3(x_1)t_3 + \tilde{h}(x_1))$$

$$\times P_{k_2}(t_2)P_{k_3}(t_3)dt_2dt_3,$$

which shows that if $u_i$ is bounded in the spatial vicinity of $x_1 = 0$ and $x_1 = L$, where $m(\Gamma_0) = 0$ and $m(\Gamma_L) = 0$, respectively, then

$$u_{ik_2k_3}(0) = 0, \ \text{ and } \ u_{ik_2k_3}(L) = 0, \qquad k_2, k_3 = \overline{0,\infty},\ i = 1,2,3, \tag{7.2}$$

since

$$\frac{1}{a_j(x_1)} = h_j(x_1) = \frac{1}{2}\left[\overset{(+)}{h}_j(x_1) - \overset{(-)}{h}_j(x_1)\right],\ j = 2,3,$$

vanish at $x_1 = 0$ and $x_1 = L$.

Clearly, in general, $u_i \in H^1(\Omega)$ is not bounded and the condition (7.2) does not hold.

### 3.2. Approximating function space

Let us fix

$$\mathbf{N}^{(j)} := \left(N_1^{(j)}, N_2^{(j)}, N_3^{(j)}\right) \in [\mathbb{N}_0]^3, \quad j = 2,3,$$

and consider the combinations

$$w_i(x_1,x_2,x_3) \equiv \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w_i}(x_1,x_2,x_3) := \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} a_2a_3\left(r_i^{(2)}+\frac{1}{2}\right)\left(r_i^{(3)}+\frac{1}{2}\right)$$

$$\times P_{r_i^{(2)}}(a_2x_2 - b_2)P_{r_i^{(3)}}(a_3x_3 - b_3) \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1),\ \ i = 1,2,3, \tag{7.3}$$

where $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1) \equiv w_{\underline{i}r_i^{(2)}r_i^{(3)}} \in H^1_{loc}(]0,l[)$. The functions $w_{\underline{i}r_i^{(2)}r_i^{(3)}}$ are called double moments of the function $w_i$.

Let $w \equiv \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w} := (w_1, w_2, w_3)$.

We recall that an underlined index means that the corresponding repeated indices do not imply summation.

Denote by

$$\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega) := \widetilde{V}_{N_1^{(2)},N_1^{(3)}} \times \widetilde{V}_{N_2^{(2)},N_2^{(3)}} \times \widetilde{V}_{N_3^{(2)},N_3^{(3)}}$$

the set of vector-functions with components of type (7.3) which belong to $H^1(\Omega)$.

Let $h_2(0)\cdot h_3(0)=0, \quad h_2(x_1)\cdot h_3(x_1)\,\big|_{x_1\in]0,L]}>0$ ,
and

$$w_i \in \widetilde{V}_{N_i^{(2)},N_i^{(3)}}(\Omega,\Gamma_L) \subset H^1(\Omega), \text{i.e., } w \in \widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L) \subset [H^1(\Omega)]^3, \quad (7.4)$$

where

$$\widetilde{V}_{N_{\underline{i}}^{(2)},N_i^{(3)}}(\Omega,\Gamma_L) := \left\{ w_i \in \widetilde{V}_{N_{\underline{i}}^{(2)},N_i^{(3)}}(\Omega):\ w_i = 0 \ \text{ on } \ \Gamma_L \right\},$$

$$\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L) := \left\{ w \in \widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega):\ w = 0 \ \text{ on } \ \Gamma_L \right\}.$$

If $h_2(x_1)\cdot h_3(x_1)|_{x_1\in[0,L]}>0$, then we consider $\widetilde{V}_{N_{\underline{i}}^{(2)},N_i^{(3)}}(\Omega,\Gamma_0\cup\Gamma_L)$.

Our aim is to choose the corresponding function spaces for the double moments $w_{\underline{i}r_i^{(2)}r_i^{(3)}}$.

Taking into account (7.4), (7.3), and using the standard limiting procedure, we easily get

$$\int\limits_{\Omega} |w_i(x_1,x_2,x_3)|^2 d\Omega = \int\limits_0^L \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} \left(r_i^{(2)}+\frac{1}{2}\right)\left(r_i^{(3)}+\frac{1}{2}\right)$$

$$\times a_2 a_3 |w_{ir_i^{(2)}r_i^{(3)}}|^2 dx_1 < +\infty, \quad (7.5)$$

whence

$$a_2^{\frac{1}{2}}\, a_3^{\frac{1}{2}}\, w_{\underline{i}r_i^{(2)}r_i^{(3)}} \in L_2(]0,L[), \quad r_i^{(j)} = \overline{0,N_i^{(j)}}, \quad i=1,2,3; \quad j=2,3.$$

Denote

$$v_{\underline{i}r_i^{(2)}r_i^{(3)}} := \frac{w_{\underline{i}r_i^{(2)}r_i^{(3)}}}{h_2^{r_i^{(2)}+1} h_3^{r_i^{(3)}+1}}. \quad (7.6)$$

Clearly,

$$h_2^{r_i^{(2)}+\frac{1}{2}} h_3^{r_i^{(3)}+\frac{1}{2}} v_{\underline{i}r_i^{(2)}r_i^{(3)}} \in L_2(]0,L[).$$

The functions $v_{\underline{i}r_i^{(2)}r_i^{(3)}}$ are called *weighted double moments* of the function $w_i$.

Analogously, applying the formula [3]

$$P'_{r^{(\underline{j})}}(a_j x_j - b_j) = \sum_{s^{(\underline{j})}=0}^{r^{(j)}-1} \left(2s^{(j)} + \frac{1}{2}\right) \left[1 - (-1)^{r^{(j)}+s^{(j)}}\right] P_{s^{(\underline{j})}}(a_j x_j - b_j)$$

for $j = 2, 3$, we obtain

$$\int_\Omega |w_{i,3}(x_1, x_2, x_3)|^2 d\Omega$$

$$= \int_0^L \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left\{ \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} a_2 a_3^2 \left(r_i^{(2)} + \frac{1}{2}\right) \left(r_i^{(3)} + \frac{1}{2}\right) \right.$$

$$\times P_{r_i^{(2)}}(a_2 x_2 - b_2) \sum_{s_{\underline{i}}^{(3)}=0}^{r_i^{(3)}-1} \left(s_i^{(3)} + \frac{1}{2}\right) \left[1 - (-1)^{r_i^{(3)}+s_i^{(3)}}\right]$$

$$\left. \times P_{s_i^{(3)}}(a_3 x_3 - b_3) w_{i r_i^{(2)} r_i^{(3)}}(x_1) \right\}^2 dx_3 dx_2 dx_1$$

$$= \int_0^L \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left\{ \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{s_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} \left( \sum_{r_{\underline{i}}^{(3)}=s_i^{(3)}}^{N_i^{(3)}} \left(r_i^{(3)} + \frac{1}{2}\right) \right. \right.$$

$$\left. \times \left[1 - (-1)^{r_i^{(3)}+s_i^{(3)}}\right] w_{\underline{i} r_i^{(2)} r_i^{(3)}}(x_1) \right)$$

$$\times a_2 a_3^2 \left(r_i^{(2)} + \frac{1}{2}\right) \left(s_i^{(3)} + \frac{1}{2}\right) P_{r_i^{(2)}}(a_2 x_2 - b_2)$$

$$\left. \times P_{s_i^{(3)}}(a_3 x_3 - b_3) \right\}^2 dx_3 dx_2 dx_1$$

$$= \int_0^L \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{s_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} a_2 a_3^3 \left(r_i^{(2)} + \frac{1}{2}\right) \left(s_i^{(3)} + \frac{1}{2}\right)$$

$$\times \left\{ \sum_{r_i^{(3)}=s_{\underline{i}}^{(3)}}^{N_i^{(3)}} \left(r_i^{(3)} + \frac{1}{2}\right) \left[1 - (-1)^{r_i^{(3)}+s_i^{(3)}}\right] w_{\underline{i} r_i^{(2)} r_i^{(3)}}(x_1) \right\}^2 dx_1 < +\infty, \tag{7.7}$$

whence

$$a_2^{1/2} a_3^{3/2} \sum_{r_{\underline{i}}^{(3)}=s_i^{(3)}}^{N_i^{(3)}} \left(r_i^{(3)} + \frac{1}{2}\right) \left[1 - (-1)^{r_i^{(3)}+s_i^{(3)}}\right] w_{\underline{i} r_i^{(2)} r_i^{(3)}} \in L_2(]0, L[),$$

[3] Here and in what follows we assume that $\sum\limits_{s=k}^{m} (\cdot) = 0$ for $m < k$.

$$r_i^{(j)}=\overline{0,N_i^{(j)}},\quad i=1,2,3;\quad j=2,3.$$

In turn these inclusions (with $s_i^{(3)}=N_i^{(3)}-1,\ s_i^{(3)}=N_i^{(3)}-2,\cdots$) yield

$$a_2^{1/2}a_3^{3/2}w_{\underline{i}r_i^{(2)}r_i^{(3)}}\in L_2(]0,L[),\ \ r_i^{(j)}=\overline{0,N_i^{(j)}},\ \ i=1,2,3;\ j=2,3,$$

i.e., by virtue of (7.6),

$$h_2^{r_i^{(2)}+\frac{1}{2}}h_3^{r_i^{(3)}-\frac{1}{2}}v_{\underline{i}r_i^{(2)}r_i^{(3)}}\in L_2(]0,L[),\quad r_i^{(j)}=\overline{0,N_i^{(j)}},\ \ i=1,2,3,\ \ j=2,3.$$

Similarly,

$$\int\limits_{\Omega}|w_{i,2}(x_1,x_2,x_3)|^2d\Omega$$

$$=\int\limits_0^L\sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}}\sum_{s_{\underline{i}}^{(2)}=0}^{N_i^{(2)}}\left\{\sum_{r_{\underline{i}}^{(2)}=s_i^{(2)}}^{N_i^{(2)}}\left(r_i^{(2)}+\frac{1}{2}\right)\left[1-(-1)^{r_i^{(2)}+s_i^{(2)}}\right]w_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1)\right\}^2$$

$$\times\ a_2^3a_3\left(r_i^{(3)}+\frac{1}{2}\right)\left(s_i^{(2)}+\frac{1}{2}\right)dx_1<+\infty,\tag{7.8}$$

and

$$a_2^{3/2}a_3^{1/2}w_{\underline{i}r_i^{(2)}r_i^{(3)}}\in L_2(]0,L[),\ \ r_i^{(j)}=\overline{0,N_i^{(j)}};\ \ i=1,2,3;\ \ j=1,2,$$

i.e.,

$$h_2^{r_i^{(2)}-\frac{1}{2}}h_3^{r_i^{(3)}+\frac{1}{2}}v_{\underline{i}r_i^{(2)}r_i^{(3)}}\in L_2(]0,L[),$$
$$r_i^{(j)}=\overline{0,N_i^{(j)}};\quad i=1,2,3;\quad j=2,3.$$

Analogously, applying the formulas

$$P_{r^{(\underline{j})},1}(a_jx_j-b_j)=P'_{r^{(\underline{j})}}(a_jx_j-b_j)(a_{j,1}x_j-b_{j,1})$$
$$=\sum_{q^{(j)}=1}^{r^{(j)}}A^j_{q^{(\underline{j})}}(2r^{(j)}-2q^{(j)}+1)\,P_{r^{(j)}-q^{(j)}}(a_jx_j-b_j)$$
$$+A_0^jr^{(\underline{j})}P_{r^{(j)}}(a_jx_j-b_j),\ \ j=2,3,$$

with

$$A_q^j:=-\frac{\overset{(+)}{h}_{j,1}-(-1)^q\overset{(-)}{h}_{j,1}}{2h_{\underline{j}}}=A^j_{q+2s}\quad j=2,3;\ \ q=0,1,\cdots,$$

and taking into account that

$$s_i^{(j)} := r_i^{(j)} - q_i^{(j)}, \quad i = 1,2,3, \quad j = 2,3, \tag{7.9}$$

we arrive at the relation

$$\int_\Omega |w_{i,1}(x_1,x_2,x_3)|^2 d\Omega$$

$$= \int_0^L \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \Big( \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} (r_i^{(2)} + \frac{1}{2})(r_i^{(3)} + \frac{1}{2})$$

$$\times \{ (a_{2,1}a_3 + a_2 a_{3,1}) P_{r_i^{(2)}}(a_2x_2 - b_2) P_{r_i^{(3)}}(a_3x_3 - b_3) w_{\underline{i}r_i^{(2)}r_i^{(3)}}$$

$$+ a_2a_3[A_0^2 r_{\underline{i}}^{(2)} P_{r_i^{(2)}}(a_2x_2 - b_2) + \sum_{q_{\underline{i}}^{(2)}=1}^{r_i^{(2)}} A^2_{q_i^{(2)}}(2r_i^{(2)} - 2q_i^{(2)} + 1)$$

$$\times P_{r_i^{(2)} - q_i^{(2)}}(a_2x_2 - b_2)] P_{r_i^{(3)}}(a_3x_3 - b_3) w_{\underline{i}r_i^{(2)}r_i^{(3)}}$$

$$+ a_2a_3[A_0^3 r_{\underline{i}}^{(3)} P_{r_i^{(3)}}(a_3x_3 - b_3) + \sum_{q_{\underline{i}}^{(3)}=1}^{r_i^{(3)}} A^3_{q_i^{(3)}}(2r_i^{(3)} - 2q_i^{(3)}$$

$$+1) P_{r_i^{(3)} - q_i^{(3)}}(a_3x_3 - b_3)] P_{r_i^{(2)}}(a_2x_2 - b_2) w_{\underline{i}r_i^{(2)}r_i^{(3)}}$$

$$+ a_2a_3 P_{r_i^{(2)}}(a_2x_2 - b_2) P_{r_i^{(3)}}(a_3x_3 - b_3) w_{\underline{i}r_i^{(2)}r_i^{(3)},1} \} \Big)^2 dx_3 dx_2 dx_1$$

$$= \int_0^L \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \Big( \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} a_2a_3 (r_i^{(2)} + \frac{1}{2})(r_i^{(3)} + \frac{1}{2})$$

$$\times \{ [(a_{2,1}a_2^{-1} + A_0^2 r_i^{(2)} + a_{3,1}a_3^{-1} + A_0^3 r_i^{(3)}) w_{\underline{i}r_i^{(2)}r_i^{(3)}} + w_{\underline{i}r_i^{(2)}r_i^{(3)},1}]$$

$$\times P_{r_{\underline{i}}^{(2)}}(a_2x_2 - b_2) P_{r_i^{(3)}}(a_3x_3 - b_3)$$

$$+2 \sum_{s_{\underline{i}}^{(2)}=0}^{r_i^{(2)}-1} A^2_{r_i^{(2)} - s_i^{(2)}}(s_i^{(2)} + \frac{1}{2}) P_{s_i^{(2)}}(a_2x_2 - b_2) P_{r_i^{(3)}}(a_3x_3 - b_3) w_{\underline{i}r_i^{(2)}r_i^{(3)}}$$

$$+2 \sum_{s_{\underline{i}}^{(3)}=0}^{r_i^{(3)}-1} A^3_{r_i^{(3)} - s_i^{(3)}}(s_i^{(3)} + \frac{1}{2}) P_{s_i^{(3)}}(a_3x_3 - b_3)$$

$$\times P_{r_i^{(2)}}(a_2x_2 - b_2) w_{\underline{i}r_i^{(2)}r_i^{(3)}} \} \Big)^2 dx_3 dx_2 dx_1 < +\infty. \tag{7.10}$$

In view of

$$A_0^j = -h_{\underline{j},1} h_j^{-1} = a_{\underline{j},1} a_j^{-1}, \quad j = 2,3;$$

and (3.6), we have

$$\left(a_{2,1}a_2^{-1} + A_0^2 r_i^{(2)} + a_{3,1}a_3^{-1} + A_0^3 r_i^{(3)}\right) w_{\underline{i}r_i^{(2)}r_i^{(3)}} + w_{\underline{i}r_i^{(2)}r_i^{(3)},1}$$

$$= \left[\left(1 + r_i^{(2)}\right) a_{2,1}a_2^{-1} + \left(1 + r_i^{(3)}\right) a_{3,1}a_3^{-1}\right] w_{\underline{i}r_i^{(2)}r_i^{(3)}} + w_{\underline{i}r_i^{(2)}r_i^{(3)},1}$$

$$= a_2^{-r_i^{(2)}-1} a_3^{-r_i^{(3)}-1} \left(a_2^{r_i^{(2)}+1} a_3^{r_i^{(3)}+1} w_{\underline{i}\ r_i^{(2)}r_i^{(3)}}\right)_{,1}$$

$$= h_2^{r_i^{(2)}+1} h_3^{r_i^{(3)}+1} v_{\underline{i}\ r_i^{(2)}r_i^{(3)},1}.$$

Hence, from (3.10) we obtain

$$\int_\Omega |w_{i,1}(x_1,x_2,x_3)|^2 d\Omega$$

$$= \int_0^L \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left\{ \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} a_2 a_3 \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) \right.$$

$$\times \left[ h_2^{r_i^{(2)}+1} h_3^{r_i^{(3)}+1} v_{\underline{i}\ r_i^{(2)}r_i^{(3)},1} \right.$$

$$\times P_{r_i^{(2)}}(a_2x_2 - b_2) P_{r_i^{(3)}}(a_3x_3 - b_3)$$

$$+2 \sum_{s_i^{(2)}=0}^{r_i^{(2)}-1} A^2_{r_i^{(2)}-s_i^{(2)}} \left(s^{(2)} + \frac{1}{2}\right) P_{s_i^{(2)}}(a_2x_2 - b_2) P_{r_i^{(3)}}(a_3x_3 - b_3) w_{\underline{i}r_i^{(2)}r_i^{(3)}}$$

$$+2 \sum_{s_i^{(3)}=0}^{r_i^{(3)}-1} A^3_{r_i^{(3)}-s_i^{(3)}} \left(s^{(3)} + \frac{1}{2}\right) P_{r_i^{(2)}}(a_2x_2 - b_2)$$

$$\left.\left. \times\ P_{s_i^{(3)}}(a_3x_3 - b_3) w_{\underline{i}r_i^{(2)}r_i^{(3)}} \right]\right\}^2 dx_3 dx_2 dx_1 < +\infty. \tag{7.11}$$

Introduce the notation

$$\left(r_{\underline{i}}^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) \Psi^{\left(0,0,r_i^{(2)},r_i^{(3)}\right)} := h_2^{r_i^{(2)}+1} h_3^{r_i^{(3)}+1} v_{\underline{i}r_i^{(2)}r_i^{(3)},1} \tag{7.12}$$

$$\text{for } r_i^{(2)} = s_i^{(2)}, \quad r_i^{(3)} = s_i^{(3)},$$

$$\left(s_{\underline{i}}^{(3)} + \frac{1}{2}\right) \Psi^{\left(r_i^{(2)}-s_i^{(2)},0,r_i^{(2)},r_i^{(3)}\right)} := 2A^2_{r_i^{(2)}-s_i^{(2)}} w_{\underline{i}r_i^{(2)}r_i^{(3)}} \tag{7.13}$$

$$\text{for } r_i^{(2)} - s_i^{(2)} > 0, \quad r_i^{(3)} = s_i^{(3)},$$

$$\left(s_{\underline{i}}^{(2)} + \frac{1}{2}\right) \Psi^{\left(0,r_i^{(3)}-s_i^{(3)},r_i^{(2)},r_i^{(3)}\right)} := 2A^3_{r_i^{(3)}-s_i^{(2)}} w_{\underline{i}r_i^{(2)}r_i^{(3)}} \tag{7.14}$$

$$\text{for } r_i^{(3)} - s_i^{(3)} > 0, \quad r_i^{(2)} = s_i^{(2)},$$

$$\left(s_{\underline{i}}^{(2)} + \frac{1}{2}\right)\left(s_i^{(3)} + \frac{1}{2}\right)\Psi^{\left(r_i^{(2)}-s_i^{(2)}, r_i^{(3)}-s_i^{(3)}, r_i^{(2)}, r_i^{(3)}\right)} := 0 \tag{7.15}$$

$$\text{for } r_i^{(2)} - s_i^{(2)} > 0, \quad r_i^{(3)} - s_i^{(3)} > 0.$$

Then we get from (3.11)

$$\int_\Omega |w_{i,1}(x_1,x_2,x_3)|^2 d\Omega$$

$$= \int_0^L \int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} \{ \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} a_2 a_3 \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right)$$

$$\times \sum_{s_{\underline{i}}^{(2)}=0}^{r_i^{(2)}} \sum_{s_{\underline{i}}^{(3)}=0}^{r_i^{(3)}} \Psi^{(r_i^{(2)}-s_i^{(2)}, r_i^{(3)}-s_i^{(3)}, r_i^{(2)}, r_i^{(3)})}$$

$$\times \left(s_i^{(2)} + \frac{1}{2}\right)\left(s_i^{(3)} + \frac{1}{2}\right) P_{s_i^{(2)}}(a_2x_2 - b_2) P_{s_i^{(3)}}(a_3x_3 - b_3)\}^2 dx_3 dx_2 dx_1$$

$$= \int_0^L \int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} \{ \sum_{s_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{s_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} [ \sum_{r_{\underline{i}}^{(2)}=s_{\underline{i}}^{(2)}}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=s_i^{(3)}}^{N_i^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)$$

$$\times \left(r_i^{(3)} + \frac{1}{2}\right) \Psi^{\left(r_i^{(2)}-s_i^{(2)}, r_i^{(3)}-s_i^{(3)}, r_i^{(2)}, r_i^{(3)}\right)} ]$$

$$\times a_2 a_3 \left(s_{\underline{i}}^{(2)} + \frac{1}{2}\right)\left(s_{\underline{i}}^{(3)} + \frac{1}{2}\right) P_{s_i^{(2)}}(a_2x_2 - b_2)$$

$$\times P_{s_i^{(3)}}(a_3x_3 - b_3)\}^2 dx_3 dx_2 dx_1$$

$$= \int_0^L \int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} \{ \sum_{s_i^{(2)}=0}^{N_i^{(2)}} \sum_{s_i^{(3)}=0}^{N_i^{(3)}} \Phi^{\left(s_i^{(2)}, s_i^{(3)}, N_i^{(2)}, N_i^{(3)}\right)} a_2 a_3$$

$$\times \left(s_i^{(2)} + \frac{1}{2}\right)\left(s_i^{(3)} + \frac{1}{2}\right) P_{s_i^{(2)}}(a_2x_2 - b_2)$$

$$\times P_{s_i^{(3)}}(a_3x_3 - b_3)\}^2 dx_3 dx_2 dx_1 < +\infty,$$

where

$$\Phi^{\left(s_i^{(2)}, s_i^{(3)}, N_i^{(2)}, N_i^{(3)}\right)} := \sum_{r_{\underline{i}}^{(2)}=s_{\underline{i}}^{(2)}}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=s_{\underline{i}}^{(3)}}^{N_i^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)$$

$$\times\left(r_i^{(3)}+\frac{1}{2}\right)\Psi^{\left(r_i^{(2)}-s_i^{(2)},r_i^{(3)}-s_i^{(3)},r_i^{(2)},r_i^{(3)}\right)}. \tag{7.16}$$

Further, by virtue of (2.15),

$$\int\limits_{\Omega}|w_{i,1}(x_1,x_2,x_3)|^2 d\Omega=\int\limits_0^L\sum_{s_{\underline{i}}^{(2)}=0}^{N_i^{(2)}}\sum_{s_{\underline{i}}^{(3)}=0}^{N_i^{(3)}}\left[\Phi^{\left(s_i^{(2)},s_i^{(3)},N_i^{(2)},N_i^{(3)}\right)}\right]^2$$

$$\times a_2a_3\left(s_i^{(2)}+\frac{1}{2}\right)\left(s_i^{(3)}+\frac{1}{2}\right)dx_1. \tag{7.17}$$

Consequently,

$$a_2^{1/2}a_3^{1/2}\Phi^{\left(s_i^{(2)},s_i^{(3)},N_i^{(2)},N_i^{(3)}\right)}\in L_2(]0,L[),$$

$$s_i^{(j)}=\overline{0,N_i^{(j)}},\quad i=1,2,3;\quad j=2,3.$$

Taking into account (3.6), from (7.5), (7.7), (7.8), and (7.12)-(7.17) it follows that

$$\|\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\|^2_{[H^1(\Omega)]^3}:=\sum_{i=1}^{3}||w_i||^2_{H^1(\Omega)}$$

$$=\sum_{i=1}^{3}\{\sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}}\sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}}\left(r_i^{(2)}+\frac{1}{2}\right)\left(r_i^{(3)}+\frac{1}{2}\right)$$

$$\times\left|\left|h_2^{r_i^{(2)}+\frac{1}{2}}h_3^{r_i^{(3)}+\frac{1}{2}}v_{\underline{i}r_i^{(2)}r_i^{(3)}}\right|\right|^2_{L_2(]0,L[)}$$

$$+\sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}}\sum_{s_{\underline{i}}^{(3)}=0}^{N_i^{(3)}}\left(r_i^{(2)}+\frac{1}{2}\right)\left(s_i^{(3)}+\frac{1}{2}\right)$$

$$\times\|\sum_{r_{\underline{i}}^{(3)}=s_i^{(3)}}^{N_i^{(3)}}\left(r_i^{(3)}+\frac{1}{2}\right)\left[1-(-1)^{r_i^{(3)}+s_i^{(3)}}\right]$$

$$\times h_2^{r_i^{(2)}+\frac{1}{2}}h_3^{r_i^{(3)}-\frac{1}{2}}v_{\underline{i}r_i^{(2)}r_i^{(3)}}\|^2_{L_2(]0,L[)}$$

$$+\sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}}\sum_{s_{\underline{i}}^{(2)}=0}^{N_i^{(2)}}\left(r_i^{(3)}+\frac{1}{2}\right)\left(s_i^{(2)}+\frac{1}{2}\right)$$

$$\times\|\sum_{r_{\underline{i}}^{(2)}=s_i^{(2)}}^{N_i^{(2)}}\left(r_i^{(2)}+\frac{1}{2}\right)\left[1-(-1)^{r_i^{(2)}+s_i^{(2)}}\right]$$

$$\times h_2^{r_i^{(2)}-\frac{1}{2}} h_3^{r_i^{(3)}+\frac{1}{2}} v_{\underline{i}r_i^{(2)}r_i^{(3)}} \|^2_{L_2(]0,L[)}$$

$$+ \sum_{s_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{s_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} \left(s_i^{(2)} + \frac{1}{2}\right) \left(s_i^{(3)} + \frac{1}{2}\right)$$

$$\times \| h_2^{s_i^{(2)}+\frac{1}{2}} h_3^{s_i^{(3)}+\frac{1}{2}} v_{\underline{i}s_i^{(2)}s_i^{(3)},1} + 2 \sum_{r_{\underline{i}}^{(2)}=s_i^{(2)}+1}^{N_i^{(2)}} \left(r_i^{(2)} + \frac{1}{2}\right) A^2_{r_i^{(2)}-s_i^{(2)}}$$

$$\times h_2^{r_i^{(2)}+\frac{1}{2}} h_3^{s_i^{(3)}+\frac{1}{2}} v_{\underline{i}r_i^{(2)}s_i^{(3)}} + 2 \sum_{r_{\underline{i}}^{(3)}=s_i^{(3)}+1}^{N_i^{(3)}} \left(r_i^{(3)} + \frac{1}{2}\right) A^3_{r_i^{(3)}-s_i^{(3)}}$$

$$\times h_2^{s_i^{(2)}+\frac{1}{2}} h_3^{r_i^{(3)}+\frac{1}{2}} v_{is_i^{(2)}r_i^{(3)}} \|^2_{L_2(]0,L[)} \}$$

$$=: ||v||^2_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\, \overset{(+)}{h}_3,\, \overset{(-)}{h}_2,\, \overset{(-)}{h}_3, ]0,L[\right)}. \tag{7.18}$$

**DEFINITION 7.1** *By* $H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\, \overset{(+)}{h}_3,\, \overset{(-)}{h}_2,\, \overset{(-)}{h}_3, ]0,L[\right)$ *we denote the following set of vector-functions*

$$v := \left(v_{100}, \cdots, v_{1N_1^{(2)}N_1^{(3)}}, v_{200}, \cdots, v_{2N_2^{(2)}N_2^{(3)}}, v_{300}, \cdots, v_{3N_3^{(2)}N_3^{(3)}}\right)^T$$

*from* $\mathbf{H}^1_{loc}(]0,L[) := [H^1_{loc}(]0,L[)]^{(N_1^{(2)}+1)(N_1^{(3)}+1)+(N_2^{(2)}+1)(N_2^{(3)}+1)+(N_3^{(2)}+1)(N_3^{(3)}+1)}$, *where the components* $v_{\underline{i}r_i^{(2)}r_i^{(3)}}$ *are given by* (7.6) *and for which the norm in the right-hand side of* (7.18) *is finite.*

**LEMMA 7.2** *The space* $H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\, \overset{(+)}{h}_3,\, \overset{(-)}{h}_2,\, \overset{(-)}{h}_3, ]0,L[\right)$ *is complete.*

*Proof.* The norm in the space

$$H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\, \overset{(+)}{h}_3,\, \overset{(-)}{h}_2,\, \overset{(-)}{h}_3, ]\varepsilon,L[\right), \quad \varepsilon > 0, \tag{7.19}$$

is equivalent to the norm

$$|| \cdot ||_{\mathbf{H}^1(]\varepsilon,L[)}, \tag{7.20}$$

since $h_j(x_1) > 0, j = 2,3$, for $x_1 \in ]\varepsilon, L[$.

Let $\{v_n\} \subset H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\, \overset{(+)}{h}_3,\, \overset{(-)}{h}_2,\, \overset{(-)}{h}_3, ]0,L[\right)$ be a fundamental sequence and show that it converges to some vector-function

$$v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\, \overset{(+)}{h}_3,\, \overset{(-)}{h}_2,\, \overset{(-)}{h}_3, ]0,L[\right).$$

Due to the above equivalence, the sequence $\{v_n\}$ is also fundamental in the space $\mathbf{H}^1(]\varepsilon, L[)$ for arbitrary $\varepsilon > 0$. Since $\mathbf{H}^1(]\varepsilon, L[)$ is complete, there exists a vector-function $v^{(\varepsilon)} \in \mathbf{H}^1(]\varepsilon, L[)$ such that

$$||v_n - v^{(\varepsilon)}||_{\mathbf{H}^1(]\varepsilon,L[)} \to 0 \quad \text{as } n \to +\infty.$$

Note that for $\varepsilon_1 < \varepsilon_2$ (i.e., $]\varepsilon_1, L[ \supset ]\varepsilon_2, L[$) there holds

$$v^{(\varepsilon_1)}|_{]\varepsilon_2,L[} = v^{(\varepsilon_2)},$$

where the symbol $|_{]\varepsilon_2,L[}$ denotes the restriction onto $]\varepsilon_2, L[$. Therefore, there exists a vector-function

$$v_0 \in \mathbf{H}^1_{loc}(]0, L[)$$

such that for $\forall \varepsilon > 0$

$$v_0|_{]\varepsilon,L[} = v^{(\varepsilon)}.$$

It is evident that

$$||v_n||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)} \le M \quad \text{for} \quad \forall n \in N,$$

where $M$ is a positive constant.

Obviously,

$$||v_n||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]\varepsilon,L[\right)}$$

$$\le ||v_n||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)} \le M \quad \text{for} \quad \forall \varepsilon > 0,$$

and, in view of the equivalence of norms (7.19) and (7.20),

$$||v_0||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]\varepsilon,L[\right)} = ||v^{(\varepsilon)}||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]\varepsilon,L[\right)}$$

$$= \lim_{n\to+\infty} ||v_n||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]\varepsilon,L[\right)} \le M.$$

Hence,

$$||v_0||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)}$$

$$= \lim_{\varepsilon\to 0} ||v_0||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]\varepsilon,L[\right)} \le M,$$

which implies that

$$v_0 \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[\right).$$

It is easy to show that $v_n \to v_0$ in the space (7.18). Indeed , for arbitrary $\delta > 0$ there exists a number $N_0(\delta)$, such that

$$\begin{aligned} &||v_n - v_m||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \\ &\leq ||v_n - v_m||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[\right)} < \delta \end{aligned}$$

for $\forall n, m > N_0(\delta)$. Passing to the limit as $m \to +\infty$ for a fixed $n$ we get

$$\begin{aligned} &||v_n - v_m||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \\ &\leq ||v_n - v^{(\varepsilon)}||_{H^1_{\mathbf{N}(2),\mathbf{N}(3)}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \\ &+||v_m - v^{(\varepsilon)}||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \\ &\to ||v_n - v^{(\varepsilon)}||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \end{aligned}$$

and

$$\begin{aligned} &||v_n - v^{(\varepsilon)}||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \\ &= ||v_n - v_0||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]\varepsilon, L[\right)} \leq \delta. \end{aligned}$$

Note that the left–hand side expression is bounded, increasing function of $\varepsilon$. Therefore sending $\varepsilon$ to zero we conclude that

$$||v_n - v_0||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[\right)} \leq \delta,$$

which completes the proof. □

**COROLLARY 7.3** $H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[\right)$ *is a Hilbert space. It is embeded in* $\mathbf{H}^1_{loc}(]0, L[)$*, i.e., for arbitrary* $\varepsilon > 0$ *and*

$$v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[\right)$$

*there holds* $v \in \mathbf{H}^1(]\varepsilon, L[)$.

**COROLLARY 7.4** $\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega)$ *is a closed subspace of* $[H^1(\Omega)]^3$.

We introduce the space

$$H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[, L\right)$$
$$:= \left\{ v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[\right) \; : \; v(L) = 0 \right\}.$$

**LEMMA 7.5** *If* $w \in \widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega, \Gamma_L)$ *then the corresponding vector of weighted moments* $v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[, L\right)$.

*Proof.* Due to the trace theorem we have (see also (7.3), (7.6))

$$0 = \tau_{\Gamma_L}(w_i) = \sum_{r_i^{(2)}=0}^{N_i^{(2)}} \sum_{r_i^{(3)}=0}^{N_i^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right)$$
$$\times [P_{r_i^{(2)}}(a_2 x_2 - b_2)\, P_{r_i^{(3)}}(a_3 x_3 - b_3)]_{\Gamma_L} \left[ h_2^{r_i^{(2)}} h_3^{r_i^{(3)}} v_{\underline{i} r_i^{(2)} r_i^{(3)}} \right]_{x_1 = L}, \quad (7.21)$$

where $\tau_{\Gamma_L}$ is the trace operator on $\Gamma_L$.

Whence the equalities

$$v_{\underline{i} r_i^{(2)} r_i^{(3)}}(L) = 0$$

follow, since

$$h_2(L) h_3(L) > 0.$$

This proves the lemma. □

**REMARK 7.6** *It is easy to see that*

$$\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega, \Gamma_L) \;\text{ and }\; H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[, L)$$

*are closed subspaces of*

$$[H^1(\Omega, \Gamma_L)]^3 \;\text{ and }\; H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[),$$

*respectively, and they represent Hilbert spaces with respect to the natural scalar products induced from*

$$[H^1(\Omega)]^3 \;\text{ and }\; H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0, L[).$$

### 3.3. Existence results

For any pair of elements

$$v, v^* \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0,L], L\right)$$

we construct $w,\ w^* \in \overset{\sim}{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L)$ according to the formulas (7.3) and (7.6):

$$w_i(x_1,x_2,x_3) = \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) h_2^{r_i^{(2)}}(x_1)\, h_3^{r_i^{(3)}}(x_1)$$
$$\times v_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1) P_{r_i^{(2)}}\left(a_2(x_1)x_2 - b_2(x_1)\right)\ P_{r_i^{(3)}}\left(a_3(x_1)x_3 - b_3(x_1)\right), \tag{7.22}$$

$$w_i^*(x_1,x_2,x_3) = \sum_{r_{\underline{i}}^{(2)}=0}^{N_i^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N_i^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) h_2^{r_i^{(2)}}(x_1)\, h_3^{r_i^{(3)}}(x_1)$$
$$\times v^*_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1) P_{r_2^{(2)}}(a_2(x_1)x_2 - b_2(x_1)) P_{r_i^{(3)}}(a_3(x_1)x_3 - b_3(x_1)). \tag{7.23}$$

Consider the following variational
*Problem* $(I^{\Omega}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}})$. Find $w \in \overset{\sim}{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L)$ such that

$$B(w,w^*) = \mathcal{F}(w^*) \quad \forall w^* \in \overset{\sim}{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L), \tag{7.24}$$

where the bilinear form $B(\cdot\ ,\cdot)$ and the linear functional $\mathcal{F}(\cdot)$ are given by (6.9) and (6.10), respectively, with $f$ and $g$ satisfying the following conditions $f_k \in \overset{\sim}{H}^{-1}(\Omega)$, $g_k \in H^{-\frac{1}{2}}(S)$, $k = 1,2,3$.

Due to the coercivity property (6.14) and the Lax-Milgram lemma along with Corollary 3.4 it follows

**LEMMA 7.7** *The Variational Problem* $(I^{\Omega}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}})$ *has a unique solution.*

Let, for simplicity

$$v_{\underline{i}r_i^{(2)}r_i^{(3)}} = 0$$

if at least one of the following conditions $N^{(j)} \geq r_i^{(j)} > N_i^{(j)}$, $i = 1,2,3$; $j = 2,3$, with $N^{(2)} := \max\left\{N_1^{(2)}, N_2^{(2)}, N_3^{(2)}\right\}$, $N^{(3)} := \max\left\{N_1^{(3)}, N_2^{(3)}, N_3^{(3)}\right\}$ be fulfilled.

Further, we reduce the three-dimensional variational Problem $(I^{\Omega}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}})$ (see (7.24)) to the one-dimensional variational problem for the vector-function of weighted moments. To this end we have to substitute (7.22) and (7.23) into (7.24), apply (6.9), (2.10), and formulas (see [15])

$$
\begin{aligned}
e_{ij}(w)(x_1,x_2,x_3) &= \sum_{r^{(2)}=0}^{N^{(2)}} \sum_{r^{(3)}=0}^{N^{(3)}} \left(r^{(2)}+\frac{1}{2}\right)\left(r^{(3)}+\frac{1}{2}\right) a_2(x_1)a_3(x_1) \\
&\times\ e_{ijr^{(2)}r^{(3)}}(v)(x_1)P_{r^{(2)}}(a_2(x_1)x_2-b_2(x_1)) \\
&\times\ P_{r^{(3)}}(a_3(x_1)x_3-b_3(x_1)),
\end{aligned}
\tag{7.25}
$$

$$
\begin{aligned}
\sigma_{ij}(w)(x_1,x_2,x_3) &= \sum_{r^{(2)}=0}^{N^{(2)}} \sum_{r^{(3)}=0}^{N^{(3)}} \left(r^{(2)}+\frac{1}{2}\right)\left(r^{(3)}+\frac{1}{2}\right) a_2(x_1)a_3(x_1) \\
&\times\ \sigma_{ijr^{(2)}r^{(3)}}(v)(x_1)P_{r^{(2)}}(a_2(x_1)x_2-b_2(x_1)) \\
&\times\ P_{r^{(3)}}(a_3(x_1)x_3-b_3(x_1)),
\end{aligned}
\tag{7.26}
$$

$$
\begin{aligned}
\sigma_{ijr^{(2)}r^{(3)}} = \lambda\delta_{ij}e_{kkr^{(2)}r^{(3)}} &+ 2\mu e_{ijr^{(2)}r^{(3)}}, \quad i,j=1,2,3, \\
& r^{(k)}=\overline{0,N^{(k)}}, \quad k=2,3,
\end{aligned}
\tag{7.27}
$$

along with the orthogonality property of the Legendre polynomials (6.15). By equalities (2.9), (2.10), (7.25)-(7.27) we get

$$
\begin{aligned}
B(w,w^*) &= \int_0^L dx_1 \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} dx_2 \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \sigma_{ij}(w)e_{ij}(w^*)\,dx_3 \\
&= \int_0^L dx_1 \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} dx_2 \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \sum_{r^{(2)}=0}^{N^{(2)}} \sum_{r^{(3)}=0}^{N^{(3)}} \sum_{s^{(2)}=0}^{N^{(2)}} \sum_{s^{(3)}=0}^{N^{(3)}} \left(r^{(2)}+\frac{1}{2}\right) \\
&\quad \times\left(r^{(3)}+\frac{1}{2}\right)\left(s^{(2)}+\frac{1}{2}\right)\left(s^{(3)}+\frac{1}{2}\right) \\
&\quad \times a_2^2a_3^2\sigma_{ijr^{(2)}r^{(3)}}(v)e_{ijs^{(2)}s^{(3)}}(v^*) \\
&\quad \times P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3) \\
&\quad \times P_{s^{(2)}}(a_2x_2-b_2)P_{s^{(3)}}(a_3x_3-b_3) \\
&= \int_0^L \sum_{r^{(2)}=0}^{N^{(2)}} \sum_{r^{(3)}=0}^{N^{(3)}} \left(r^{(2)}+\frac{1}{2}\right)\left(r^{(3)}+\frac{1}{2}\right) \\
&\quad \times a_2a_3\sigma_{ijr^{(2)}r^{(3)}}(v)e_{ijr^{(2)}r^{(3)}}(v^*)dx_1 \\
&= \int_0^L \sum_{r^{(2)}=0}^{N^{(2)}} \sum_{r^{(3)}=0}^{N^{(3)}} \left(r^{(2)}+\frac{1}{2}\right)\left(r^{(3)}+\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times a_2 a_3 \left[\lambda \delta_{ij} e_{kk\underline{r}^{(2)}\underline{r}^{(3)}}(v) e_{ijr^{(2)}r^{(3)}}(v^*)\right. \\
&\quad \left. +2\ \mu e_{ij\underline{r}^{(2)}\underline{r}^{(3)}}(v) e_{ijr^{(2)}r^{(3)}}(v^*)\right] dx_3 \\
= &\ \sum_{r^{(2)}=0}^{N^{(2)}} \sum_{r^{(3)}=0}^{N^{(3)}} \left(r^{(2)} + \frac{1}{2}\right)\left(r^{(3)} + \frac{1}{2}\right) \\
&\quad \times \int_0^L a_2 a_3 \left[\lambda e_{kk\underline{r}^{(2)}\underline{r}^{(3)}}(v) e_{iir^{(2)}r^{(3)}}(v^*)\right. \\
&\quad \left. +2\ \mu e_{ij\underline{r}^{(2)}\underline{r}^{(3)}}(v) e_{ijr^{(2)}r^{(3)}}(v^*)\right] dx_3 \\
=: &\ B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v, v^*),
\end{aligned}
\tag{7.28}
$$

$$
\begin{aligned}
\mathcal{F}(w^*) = &\ \sum_{i=1}^{3} \sum_{r_i^{(2)}=0}^{N^{(2)}} \sum_{r_i^{(3)}=0}^{N^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) \int_0^L a_2 a_3 \left\{ -f_{\underline{i}r_i^{(2)}r_i^{(3)}} \right. \\
&+ \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left[ \overset{(+)}{g}{}_i^{(2)} \sqrt{1 + \left(\overset{(+)}{h}_{2,1}\right)^2} + (-1)^{r_i^{(2)}} \overset{(-)}{g}{}_i^{(2)} \sqrt{1 + \left(\overset{(-)}{h}_{2,1}\right)^2} \right] \\
&\times P_{r^{(3)}_{\underline{i}}}(a_3 x_3 - b_3) dx_3 \\
&+ \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \left[ \overset{(+)}{g}{}_i^{(3)} \sqrt{1 + \left(\overset{(+)}{h}_{3,1}\right)^2} + (-1)^{r_i^{(2)}} \overset{(-)}{g}{}_i^{(3)} \sqrt{1 + \left(\overset{(-)}{h}_{3,1}\right)^2} \right] dx_2 \\
&\left. \times P_{r^{(2)}_{\underline{i}}}(a_2 x_2 - b_2) dx_2 \right\} h_2^{r_i^{(2)}+1} h_3^{r_i^{(3)}+1} v^*_{\underline{i}r_i^{(2)}r_i^{(3)}} dx_1 \\
= &\ \sum_{i=1}^{3} \sum_{r_i^{(2)}=0}^{N^{(2)}} \sum_{r_i^{(3)}=0}^{N^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) \int_0^L a_2 a_3 \left\{ -f_{\underline{i}r_i^{(2)}r_i^{(3)}} \right. \\
&+ \overset{(+)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} \sqrt{1 + \left(\overset{(+)}{h}_{2,1}\right)^2} + (-1)^{r_i^{(2)}} \overset{(-)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} \sqrt{1 + \left(\overset{(-)}{h}_{2,1}\right)^2} \\
&\left. + \overset{(+)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}} \sqrt{1 + \left(\overset{(+)}{h}_{3,1}\right)^2} + (-1)^{r_i^{(3)}} \overset{(-)}{g}{}^{(3)}_{\underline{i}r_{i^{(2)}}} \sqrt{1 + \left(\overset{(-)}{h}_{3,1}\right)^2} \right\} \\
&\times h_2^{r_i^{(2)}+1} h_3^{r_i^{(3)}+1} v^*_{\underline{i}r_i^{(2)}r_i^{(3)}} dx_1 \\
=: &\ \mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v^*),
\end{aligned}
\tag{7.29}
$$

where

$$f_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1) = \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} f_{\underline{i}}(x_1, x_2, x_3) P_{r_i^{(2)}}(a_2x_2 - b_2)\, P_{r_i^{(3)}}(a_3x_3 - b_3) dx_2 dx_3,$$

$$i = 1, 2, 3, \qquad r_i^{(j)} = \overline{0, N_i^{(j)}}, \quad j = 2, 3,$$

are double moments of $f_i$ and

$$\overset{(\pm)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} := \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} g_{\underline{i}}^{(2)} \left( x_1, \overset{(\pm)}{h}_2(x_1), x_3 \right) P_{r_i^{(3)}}(a_3x_3 - b_3) dx_3,$$

$$\overset{(\pm)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}} := \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} g_{\underline{i}}^{(3)} \left( x_1, x_2, \overset{(\pm)}{h}_3(x_1) \right) P_{r_i^{(2)}}(a_2x_2 - b_2) dx_2,$$

are single moments with respect to $P_{r_i^{(5-j)}}(a_{5-j}x_{5-j} - b_{5-j}), \quad j = 2, 3$, of the restrictions of $g_i^{(j)}$ on $x_j = \overset{(\pm)}{h}_j(x_1), \quad j = 2, 3$. Here we assume

$$\begin{gathered} f_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1) = 0, \quad \overset{(\pm)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}}(x_1) = 0, \quad \text{and} \quad \overset{(\pm)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}}(x_1) = 0 \\ \text{for} \;\; N^{(j)} \geq r_i^{(j)} > N_i^{(j)}, \quad i = 1, 2, 3; \;\; j = 2, 3. \end{gathered} \tag{7.30}$$

Thus, (7.24) is equivalent to the following one-dimensional variational

*Problem* $(I^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}})$ . Find $v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2, \overset{(+)}{h}_3, \overset{(-)}{h}_2, \overset{(-)}{h}_3, ]0, L[, L\right)$ such that

$$\begin{gathered} B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v, v^*) = \mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v^*) \\ \forall v^* \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2, \overset{(+)}{h}_3, \overset{(-)}{h}_2, \overset{(-)}{h}_3, ]0, L[, L\right). \end{gathered} \tag{7.31}$$

There holds

**THEOREM 7.8** *If*

$$\begin{aligned} \psi_{\underline{i}r_i^{(2)}r_i^{(3)}} \;\; := \;\; & h_2^{-\frac{1}{2}} h_3^{-\frac{1}{2}} \{ - f_{\underline{i}r_i^{(2)}r_i^{(3)}} \\ + \;\; & \overset{(+)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} \sqrt{1 + \left(\overset{(+)}{h}_{2,1}\right)^2} + (-1)^{r_i^{(2)}} \overset{(-)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} \sqrt{1 + \left(\overset{(-)}{h}_{2,1}\right)^2} \end{aligned}$$

$$+ \quad \overset{(+)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}} \sqrt{1+\left(\overset{(+)}{h}_{3,1}\right)^2}$$

$$+ \quad (-1)^{r_i^{(3)}} \overset{(-)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}} \sqrt{1+\left(\overset{(-)}{h}_{3,1}\right)^2} \} \in L_2(]0,L[), \qquad (7.32)$$

*then Problem* $(I^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}})$ *has a unique solution* $v$ *and there holds the estimate*

$$||v||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right).} \leq \frac{||\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}||}{\delta_2},$$

*where* $||\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}||$ *is the norm of the functional* $\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}$, *and* $\delta_2$ *is the constant involved in* (6.14).

*Proof.* Due to the equality (7.28), the coercivity and boundedness of $B(\cdot,\cdot)$ (see inequalities (6.13) and (6.14)) and isometry (7.18) (see Definition 3.1) it follows that the bilinear form $B^{]0,L[}_{N^{(2)},N^{(3)}}(\cdot,\cdot)$ is coercive and bounded:

$$\begin{aligned}
B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v,v) &= B(w,w) \geq \delta_2 ||w||^2_{[H^1(\Omega)]^3} \\
&= \delta_2 ||v||^2_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)}, \\
B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v,v^*) &= B(w,w^*) \leq \delta_1 ||w||_{[H^1(\Omega)]^3} ||w^*||_{[H^1(\Omega)]^3} \\
&= \delta_1 ||v||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)} ||v^*||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)}.
\end{aligned}$$

The equalities (3.29) and (3.32) imply that $\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\cdot)$ is a bounded linear functional in the space $H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2,\overset{(+)}{h}_3,\overset{(-)}{h}_2,\overset{(-)}{h}_3,]0,L[\right)$. Indeed,

$$\begin{aligned}
\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v^*) \;&=\; \sum_{i=1}^{3} \sum_{r_i^{(2)}=0}^{N^{(2)}} \sum_{r_i^{(3)}=0}^{N^{(3)}} \left(r_i^{(2)}+\frac{1}{2}\right)\left(r_i^{(3)}+\frac{1}{2}\right) \int_0^L \psi_{\underline{i}r_i^{(2)}r_i^{(3)}} \\
&\times\; h_2^{r_i^{(2)}+\frac{1}{2}} h_3^{r_i^{(3)}+\frac{1}{2}} v^*_{\underline{i}r_i^{(2)}r_i^{(3)}} dx_1,
\end{aligned}$$

and therefore

$$|\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v^*)| \leq \sum_{i=1}^{3} \sum_{r_i^{(2)}=0}^{N^{(2)}} \sum_{r_i^{(3)}=0}^{N^{(3)}} \left(r_i^{(2)}+\frac{1}{2}\right)^{1/2}\left(r_i^{(3)}+\frac{1}{2}\right)^{1/2}$$

$$\times ||\psi_{\underline{i}r_i^{(2)}r_i^{(3)}}||_{L_2(]0,L[)}$$

$$\times \left|\left| \left(r_i^{(2)}+\frac{1}{2}\right)^{1/2} \left(r_i^{(3)}+\frac{1}{2}\right)^{1/2} h_2^{r_i^{(2)}+\frac{1}{2}} h_3^{r_i^{(3)}+\frac{1}{2}} v^*_{\underline{i}r_i^{(2)}r_i^{(3)}} \right|\right|_{L_2(]0,L[)}$$

$$\leq \left[\sum_{i=1}^{3}\sum_{r_i^{(2)}=0}^{N^{(2)}}\sum_{r_i^{(3)}=0}^{N^{(3)}} \left(r_i^{(2)}+\frac{1}{2}\right)^{1/2} \left(r_i^{(3)}+\frac{1}{2}\right)^{1/2} ||\psi_{\underline{i}r_i^{(2)}r_i^{(3)}}||_{L_2(]0,L[)}\right]$$

$$\times ||v^*||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2},\overset{(+)}{h_3},\overset{(-)}{h_2},\overset{(-)}{h_3},]0.L[\right)},$$

since, in view of (3.18),

$$\left|\left| \left(r_i^{(2)}+\frac{1}{2}\right)^{1/2} \left(r_i^{(3)}+\frac{1}{2}\right)^{1/2} h_2^{r_i^{(2)}+\frac{1}{2}} h_3^{r_i^{(3)}+\frac{1}{2}} v^*_{\underline{i}r_i^{(2)}r_i^{(3)}} \right|\right|_{L_2(]0,L[)}$$

$$\leq ||v^*||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2},\overset{(+)}{h_3},\overset{(-)}{h_2},\overset{(-)}{h_3},]0,L[\right)},$$

$$r_i^{(j)} = \overline{0, N^{(j)}}, \quad i=1,2,3; \quad j=2,3.$$

Taking into account the inequalities

$$\left(r_i^{(j)}+\frac{1}{2}\right)^{\frac{1}{2}} \leq \left(N^{(j)}+\frac{1}{2}\right)^{\frac{1}{2}}, \quad i=1,2,3; \quad j=2,3,$$

we get

$$|\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v^*)| \leq ||v^*||_{H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2},\overset{(+)}{h_3},\overset{(-)}{h_2},\overset{(-)}{h_3},]0,L[\right)}$$

$$\times \left[\left(N^{(2)}+\frac{1}{2}\right)^{1/2} \left(N^{(3)}+\frac{1}{2}\right)^{1/2} \sum_{i=1}^{3}\sum_{r_i^{(2)}=0}^{N^{(2)}}\sum_{r_i^{(3)}=0}^{N^{(3)}} ||\psi_{\underline{i}r_i^{(2)}r_i^{(3)}}||_{L_2(]0,L[)}\right].$$

Now, Remark 3.6 and the Lax-Milgram lemma completes the proof. □

**REMARK 7.9** *If $f_i \in L_2(\Omega)$, then for almost all $x_1 \in ]0,L[$:*

$$\begin{aligned} f_i(x_1,x_2,x_3) &= \sum_{r_{\underline{i}}^{(2)}=0}^{\infty}\sum_{r_{\underline{i}}^{(3)}=0}^{\infty} \left(r_{\underline{i}}^{(2)}+\frac{1}{2}\right)\left(r_i^{(3)}+\frac{1}{2}\right) a_2a_3 f_{\underline{i}r_i^{(2)}r_i^{(3)}}(x_1) \\ &\times \; P_{r_{\underline{i}}^{(2)}}(a_2x_2-b_2)P_{r_i^{(3)}}(a_3x_3-b_3) \end{aligned}$$

*and*

$$h_2^{-\frac{1}{2}}h_3^{-\frac{1}{2}} f_{\underline{i}r_i^{(2)}r_i^{(3)}} \in L_2(]0,L[), \quad r_i^{(j)} = \overline{0,\infty}; \quad i=1,2,3; \quad j=2,3,$$

*due to (3.30) and the inequalities*

$$\sum_{r_i^{(2)}=0}^{N^{(2)}} \sum_{r_{\underline{i}}^{(3)}=0}^{N^{(3)}} \left(r_i^{(2)} + \frac{1}{2}\right)\left(r_i^{(3)} + \frac{1}{2}\right) \int_0^L a_2 a_3 |f_{\underline{i} r_i^{(2)} r_i^{(3)}}|^2 dx_1 \leq \int_\Omega |f_i|^2 d\Omega, \quad i = 1,2,3.$$

*In this case* (7.32) *is equivalent to the conditions*

$$h_2^{-\frac{1}{2}} h_3^{-\frac{1}{2}} \left\{ \overset{(+)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} \sqrt{1 + \left(\overset{(+)}{h}_{2,1}\right)^2} + (-1)^{r_i^{(2)}} \overset{(-)}{g}{}^{(2)}_{\underline{i}r_i^{(3)}} \sqrt{1 + \left(\overset{(-)}{h}_{2,1}\right)^2} \right.$$

$$\left. + \overset{(+)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}} \sqrt{1 + \left(\overset{(+)}{h}_{3,1}\right)^2} + (-1)^{r_i^{(3)}} \overset{(-)}{g}{}^{(3)}_{\underline{i}r_i^{(2)}} \sqrt{1 + \left(\overset{(-)}{h}_{3,1}\right)^2} \right\} \in L_2(]0, L[)$$

$$\text{for } r_i^{(j)} = \overline{0, N^{(j)}}, \quad i = 1,2,3, \quad j = 2,3.$$

**LEMMA 7.10** *Let* $\Omega$ *be a Lipschitz domain described in (2.1). Then the union* $\bigcup_{N_1^{(2)}, N_1^{(3)}=0}^{\infty} \tilde{V}_{N_1^{(2)}, N_1^{(3)}}(\Omega, \Gamma_L)$ *is dense in the space* $H^1(\Omega, \Gamma_L)$.

The proof of the lemma is given in Appendix A.

### 3.4. Convergence results

First we prove the following convergence

**THEOREM 7.11** *Assume that* $f \in [L_2(\Omega)]^3$ *and*

$$h_2^{-\frac{1}{2}} h_3^{-\frac{1}{2}} \left[1 + \left(\overset{(\pm)}{h}_{\underline{j},1}\right)^2\right]^{\frac{1}{2}} \overset{(\pm)}{g}{}^{(j)}_{\underline{i}r_i^{(5-j)}} \in L_2(]0, L[), \ i = 1,2,3; \ j = 2,3.$$

*For* $\mathbf{N}^{(2)}, \mathbf{N}^{(3)} \in [\mathbb{N}_0]^3$ *let* $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v} \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h}_2, \overset{(+)}{h}_3, \overset{(-)}{h}_2, \overset{(-)}{h}_3, ]0, L[, L\right)$ *be a unique solution to the problem* $(I^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}})$ *(see* (7.31)*) and*

$$\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w} \in \tilde{V}_{\mathbf{N}^{(2)}\mathbf{N}^{(3)}}(]\Omega, \Gamma_L[) \subset [H^1(\Omega, \Gamma_L)]^3$$

*be the corresponding vector constructed by* (7.22)*, which represents a solution to the variational problem* (7.24)*.*

*And finally, let* $u \in [H^1(\Omega, \Gamma_L)]^3$ *be a unique solution of the three-dimensional variational problem* (6.8)*.*

*Then*

$$\|\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w} - u\|_{[H^1(\Omega)]^3} \to 0 \ \text{ as } \ N^{(j)}_{\min} := \min\left\{N_1^{(j)}, N_2^{(j)}, N_3^{(j)}\right\} \to +\infty, \quad j = 2,3.$$

*Proof.* By standard arguments it can be easily shown that $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}$ minimizes the functional

$$J_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v) := \frac{1}{2}B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v,v) - \mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v) \qquad \forall v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0,L[, L\right), \tag{7.33}$$

i.e.,

$$J_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}) \le J_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v) \qquad \forall v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0,L[, L\right). \tag{7.34}$$

Note that (see (7.31), (7.29), and (7.24))

$$B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}, v\right) = \mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v) = \mathcal{F}(w) = B\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}, w\right) \tag{7.35}$$

$$\forall v \in H^1_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{(+)}{h_2}, \overset{(+)}{h_3}, \overset{(-)}{h_2}, \overset{(-)}{h_3}, ]0,L[, L\right);$$

here and in what follows $w$ corresponds to $v$ via formula (7.22) and is an arbitrary element of the space $\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega, \Gamma_L)$.

Further, with the help of (6.8),(7.28), and (7.33)–(7.35), we easily derive

$$\begin{aligned}
& B\left(u - \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}, u - \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) = B(u,u) - 2B\left(u, \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) \\
& + B\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}, \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) = B(u,u) - 2\mathcal{F}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) \\
& + B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}, \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}\right) - 2\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}\right) \\
& + 2\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}\right) \le B(u,u) - 2\mathcal{F}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) \\
& + B^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v,v) - 2\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(v) + 2\mathcal{F}^{]0,L[}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{v}\right) \\
& = B(u,u) + B(w,w) - 2\mathcal{F}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) - 2\mathcal{F}(w) \\
& + 2\mathcal{F}\left(\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right) = B(u,u) + B(w,w) - 2\mathcal{F}(w) \\
& = B(u,u) + B(w,w) - 2B(u,w) \\
& = B(u-w, u-w) \qquad \forall w \in \widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega, \Gamma_L),
\end{aligned}$$

that is,

$$B\left(u-\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w},u-\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right)\le B(u-w,u-w)\quad\forall w\in\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L).\tag{7.36}$$

We proceed as follows. From (7.36) and the coercivity property of the bilinear form $B$ (see (6.14)) it follows that

$$\delta_2\left\|u-\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}\right\|^2_{[H^1(\Omega,\Gamma_L)]^3}\le B(u-\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w},u-\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w})\le\varepsilon_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\tag{7.37}$$

with

$$\varepsilon_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}:=\inf_{w\in\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L)}B(u-w,u-w)\ge 0.$$

Since

$$\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L)\subset\widetilde{V}_{\overset{*}{\mathbf{N}}^{(2)},\overset{*}{\mathbf{N}}^{(3)}}(\Omega,\Gamma_L)$$

for

$$N_i^{(j)}\le\overset{*}{N}{}_i^{(j)},\quad i=1,2,3;\;\; j=2,3,$$

we conclude that $\varepsilon_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\ge\varepsilon_{\overset{*}{\mathbf{N}}^{(2)},\overset{*}{\mathbf{N}}^{(3)}}$ and therefore there exists the limit

$$\lim_{N^{(2)}_{\min},N^{(3)}_{\min}\to+\infty}\varepsilon_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}=\varepsilon\ge 0.$$

Due to (7.37) it remains to show that $\varepsilon=0$. We prove it by contrudiction. Assume that $\varepsilon>0$.

Note that, under our assumption,

$$\varepsilon_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\ge\varepsilon>0\quad\forall N^{(j)}\in[\mathbb{N}_0]^3,\quad j=2,3.\tag{7.38}$$

By Lemma 3.10 the union

$$\bigcup_{(\mathbf{N}^{(2)},\mathbf{N}^{(3)})}\widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}\ \text{ is dense in }\ [H^1(\Omega,\Gamma_L)]^3.\tag{7.39}$$

Because of (7.39), there exists a vector–function $\overset{*}{w}$ and vectors $\overset{*}{\mathbf{N}}{}^{(j)}$, $j=2,3$, such that

$$\overset{*}{w}\in\widetilde{V}_{\overset{*}{\mathbf{N}}^{(2)},\overset{*}{\mathbf{N}}^{(3)}}(\Omega,\Gamma_L)\ \text{ and }\ \left\|\overset{*}{w}-u\right\|^2_{[H^1(\Omega)]^3}<\frac{\varepsilon}{2\delta_1}$$

with the same $\delta_1$ as in (6.13).

Therefore,

$$\varepsilon_{\overset{*}{\mathbf{N}}^{(2)},\overset{*}{\mathbf{N}}^{(3)}}\le B(u-\overset{*}{w},\ u-\overset{*}{w})\le\delta_1\left\|u-\overset{*}{w}\right\|^2_{[H^1(\Omega)]^3}\le\frac{\varepsilon}{2},$$

which contradicts to (7.38). Thus, $\varepsilon=0$ and the result follows. □

**THEOREM 7.12** *Let $u$ and $\overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w}$ be as in Theorem 3.11 and, in addition, let the conditions*

$$\begin{aligned} &h_j^{-1}u \in [L_2(\Omega)]^3, \quad h_{\underline{j}}^{-1}\overset{(\pm)}{h}_{j,1}u \in [L_2(\Omega)]^3, \\ &u_{,1} \in [L_2(\Omega)]^3, \quad \frac{(h_2h_3)_{,1}}{h_2h_3}u \in [L_2(\Omega)]^3, \\ &\overset{(\pm)}{h}_{j,1}u_{,j} \in [L_2(\Omega)]^3, \quad \overset{*}{h}_j{}^{1/2}u \in [L_2(\Omega)]^3, \quad j=2,3, \end{aligned} \tag{7.40}$$

*be fulfilled with*

$$\begin{aligned} \overset{*}{h}_j := \tfrac{1}{3}a_{j,1}^2 &\left(\overset{(+)}{h}{}^2_j + \overset{(+)}{h_j}\,\overset{(-)}{h_j} + \overset{(-)}{h}{}^2_j\right) \\ &-2a_{j,1}b_{j,1}\tilde{h}_j + b_{j,1}^2, \quad j=2,3. \end{aligned} \tag{7.41}$$

*Moreover, let $s \geq 2$ and*

$$\begin{aligned} &\partial_i^k u \in [L_2(\Omega)]^3, \quad k=\overline{1,s}; \quad i=2,3, \\ &\partial_i^k \partial_j u \in [L_2(\Omega)]^3, \quad k=\overline{1,s}; \quad i=2,3; \quad j=1,2,3, \\ &h_{j,1}\partial_i^k \partial_j u \in [L_2(\Omega)]^3, \quad k=\overline{1,s}; \quad i=2,3; \quad j=2,3, \\ &\tilde{h}_{j,1}\partial_i^k \partial_j u \in [L_2(\Omega)]^3, \quad k=\overline{1,s}; \quad i=2,3; \quad j=2,3. \end{aligned} \tag{7.42}$$

*Then for $N^{(j)} - N^{(j)}_{min} \leq C_0$ with $C_0$ independent of $\mathbf{N}^{(j)}$ $(j=2,3)$, we have*

$$\left\| u - \overset{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}{w} \right\|^2_{[H^1(\Omega)]^3} \leq \frac{(\overset{0}{h}_2)^{2s}}{(N^{(2)}_{\min})^{2s-3}}\, q^{(2)}(\mathbf{N}^{(2)}) + \frac{(\overset{0}{h}_3)^{2s}}{(N^{(3)}_{\min})^{2s-3}}\, q^{(3)}(\mathbf{N}^{(3)}), \tag{7.43}$$

$$\overset{0}{h}_j := \max_{[0,L]} h_j(x_1), \quad q^j(\mathbf{N}^{(\underline{j})}) \to 0 \quad \text{as} \quad \mathbf{N}^{(j)} \to \infty, \quad j=2,3.$$

*Proof.* It can be shown (see Appendix B) that the conditions (7.40) imply that the partial double Fourier-Legendre sums $S_{N_{\underline{i}}^{(2)},N_i^{(3)}}$ of the vector–function $u$

$$\begin{aligned} S_{N_{\underline{i}}^{(2)},N_i^{(3)}}(x_1,x_2,x_3) := \sum_{r^{(2)}=0}^{N_i^{(2)}} \sum_{r^{(3)}=0}^{N_i^{(3)}} a_2a_3\left(r^{(2)}+\frac{1}{2}\right)\left(r^{(3)}+\frac{1}{2}\right)u_{\underline{i}r^{(2)}r^{(3)}}(x_1) \\ \times P_{r^{(2)}}(a_2x_2-b_2)\,P_{r^{(3)}}(a_3x_3-b_3), \quad N_i^{(j)} = \overline{0,\infty}; \quad j=2,3, \quad i=1,2,3, \end{aligned}$$

where

$$u_{ir^{(2)}r^{(3)}}(x_1) := \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i(x_1,x_2,x_3)P_{r^{(2)}}(a_2x_2-b_2)$$

$$\times P_{r^{(3)}}(a_3x_3 - b_3)dx_2dx_3, \quad r^{(j)} = \overline{0,\infty}; \quad j = 2,3, \ (7.44)$$

belong to the space $\overset{\sim}{V}_{N_i^{(2)},N_i^{(3)}}(\Omega,\Gamma_L) \subset H^1(\Omega,\Gamma_L)$.

Introduce

$$\varepsilon_{N_{\underline{i}}^{(2)},N_i^{(3)}}(x_1,x_2,x_3) := u_i(x_1,x_2,x_3) - S_{N_{\underline{i}}^{(2)},N_i^{(3)}}(x_1,x_2,x_3)$$
$$= \sum_{r^{(2)}=N_{\underline{i}}^{(2)}+1}^{\infty} \sum_{r^{(3)}=N_i^{(3)}+1}^{\infty} a_2a_3\left(r^{(2)}+\frac{1}{2}\right)\left(r^{(3)}+\frac{1}{2}\right)$$
$$\times u_{ir^{(2)}r^{(3)}}(x_1)P_{r^{(2)}}\left(a_2x_2-b_2\right)P_{r^{(3)}}\left(a_3x_3-b_3\right).$$

Our goal is to estimate the norm of $\varepsilon_{N_i{}^{(2)},N_i^{(3)}}$ in $H^1(\Omega)$.

Applying the recurrence relation

$$P_r(t) = \frac{1}{2r+1}\left[P'_{r+1}(t) - P'_{r-1}(t)\right], \qquad r \geq 1,$$

from (7.44) we get

$$u_{ir^{(2)}r^{(3)}}(x_1) = \frac{1}{a_2(2r^{(2)}+1)}\Big[(\partial_2u_i)_{r^{(2)}-1\ r^{(3)}} - (\partial_2u_i)_{r^{(2)}+1\ r^{(3)}}\Big]$$
$$= \frac{1}{a_3(2r^{(3)}+1)}\Big[(\partial_3u_i)_{r^{(2)}\ r^{(3)}-1} - (\partial_3u_i)_{r^{(2)}\ r^{(3)}+1}\Big]. \ (7.45)$$

From (7.45) it follows

$$\left|\left|(h_2h_3)^{-\frac{1}{2}}u_{ir^{(2)}r^{(3)}}\right|\right|^2_{L_2(]0,L[)}$$
$$\leq \frac{\tilde{c}_1\left(\overset{0}{h}_2\right)^{2s}}{(r^{(2)})^{2s}} \sum_{l^{(2)}=r^{(2)}-s}^{r^{(2)}+s} \left|\left|(h_2h_3)^{-\frac{1}{2}}(\partial_2^s u_i)_{l^{(2)}r^{(3)}}\right|\right|^2_{L_2(]0,L[)}, \quad (7.46)$$

$$\left|\left|(h_2h_3)^{-\frac{1}{2}}u_{ir^{(2)}r^{(3)}}\right|\right|^2_{L_2(]0,L[)}$$
$$\leq \frac{\tilde{c}_2\left(\overset{0}{h}_3\right)^{2s}}{(r^{(3)})^{2s}} \sum_{l^{(3)}=r^{(3)}-s}^{r^{(3)}+s} \left|\left|(h_2h_3)^{-\frac{1}{2}}(\partial_3^s u_i)_{r^{(2)}l^{(3)}}\right|\right|^2_{L_2(]0,L[)}, \quad (7.47)$$

with constants $\tilde{c}_1$ and $\tilde{c}_2$, independent of $u_i, r^{(j)}, \overset{0}{h}_j \quad (i=1,2,3,\ j=2,3)$.

By virtue of (7.46) and (7.47) we have

$$\left\| \left(r^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} (h_2h_3)^{-\frac{1}{2}} u_{ir^{(2)}r^{(3)}} \right\|^2_{L_2(]0,L[)}$$

$$\leq \frac{c_1 \left(\overset{0}{h}_2\right)^{2s}}{(r^{(2)})^{2s}} \sum_{l^{(2)}=r^{(2)}-s}^{r^{(2)}+s} \left\| \left(l^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} \right.$$

$$\left. \times (h_2h_3)^{-\frac{1}{2}} \left(\partial_2^s u_i\right)_{l^{(2)}r^{(3)}} \right\|^2_{L_2(]0,L[)}$$

$$+\frac{c_2 \left(\overset{0}{h}_3\right)^{2s}}{(r^{(3)})^{2s}} \sum_{l^{(3)}=r^{(3)}-s}^{r^{(3)}+s} \left\| \left(r^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(l^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} \right.$$

$$\left. \times (h_2h_3)^{-\frac{1}{2}} \left(\partial_3^s u_i\right)_{r^{(2)}l^{(3)}} \right\|^2_{L_2(]0,L[)} \tag{7.48}$$

with constants $c_1$ and $c_2$, independent of $u_i, r^{(j)}, \overset{0}{h}_j, \ (i=1,2,3, \ j=2,3)$.

Analogously, due to (7.48) we derive

$$\left\| \left(r^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} (h_2h_3)^{-\frac{1}{2}} \left(\partial_k u_i\right)_{r^{(2)}r^{(3)}} \right\|^2_{L_2(]0,L[)}$$

$$\leq \frac{c_1 \left(\overset{0}{h}_2\right)^{2s}}{(r^{(2)})^{2s}} \sum_{l^{(2)}=r^{(2)}-s}^{r^{(2)}+s} \left\| \left(l^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} \right.$$

$$\left. \times (h_2h_3)^{-\frac{1}{2}} \left(\partial_2^s \partial_k u_i\right)_{l^{(2)}r^{(3)}} \right\|^2_{L_2(]0,L[)}$$

$$+\frac{c_2 \left(\overset{0}{h}_3\right)^{2s}}{(r^{(3)})^{2s}} \sum_{l^{(3)}=r^{(3)}-s}^{r^{(3)}+s} \left\| \left(r^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(l^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} \right.$$

$$\left. \times (h_2h_3)^{-\frac{1}{2}} \left(\partial_3^s \partial_k u_i\right)_{r^{(2)}l^{(3)}} \right\|^2_{L_2(]0,L[)}, \tag{7.49}$$

$$i,k=1,2,3,$$

$$\left\| \left(r^{(2)}+\frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)}+\frac{1}{2}\right)^{\frac{1}{2}} (h_2h_3)^{-\frac{1}{2}} \left(h_{\underline{j},1} \partial_j u_i\right)_{r^{(2)}r^{(3)}} \right\|_{L_2(]0,L[)}$$

$$\leq \frac{c_1 \left(\overset{0}{h}_2\right)^{2s}}{(r^{(2)})^{2s}} \sum_{l^{(2)}=r^{(2)}-s}^{r^{(2)}+s} \Bigg\| \left(l^{(2)} + \frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$\times (h_2 h_3)^{-\frac{1}{2}} \left(h_{\underline{j},1} \partial_2^s \partial_j u_i\right)_{l^{(2)} r^{(3)}} \Bigg\|^2_{L_2(]0,L[)}$$

$$+ \frac{c_2 \left(\overset{0}{h}_3\right)^{2s}}{(r^{(3)})^{2s}} \sum_{l^{(3)}=r^{(3)}-s}^{r^{(3)}+s} \Bigg\| \left(r^{(2)} + \frac{1}{2}\right)^{\frac{1}{2}} \left(l^{(3)} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$\times (h_2 h_3)^{-\frac{1}{2}} \left(h_{\underline{j},1} \partial_3^s \partial_j u_i\right)_{r^{(2)} l^{(3)}} \Bigg\|^2_{L_2(]0,L[)}, \tag{7.50}$$

$$i = 1, 2, 3, \quad j = 2, 3,$$

$$\Bigg\| \left(r^{(2)} + \frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)} + \frac{1}{2}\right)^{\frac{1}{2}} (h_2 h_3)^{-\frac{1}{2}} \left(\tilde{h}_{\underline{j},1} \partial_j u_i\right)_{r^{(2)} r^{(3)}} \Bigg\|_{L_2(]0,L[)}$$

$$\leq \frac{c_1 \left(\overset{0}{h}_2\right)^{2s}}{\left(r^{(2)}\right)^{2s}} \sum_{l^{(2)}=r^{(2)}-s}^{r^{(2)}+s} \Bigg\| \left(l^{(2)} + \frac{1}{2}\right)^{\frac{1}{2}} \left(r^{(3)} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$\times (h_2 h_3)^{-\frac{1}{2}} \left(\tilde{h}_{\underline{j},1} \partial_2^s \partial_j u_i\right)_{l^{(2)} r^{(3)}} \Bigg\|^2_{L_2(]0,L[)}$$

$$+ \frac{c_2 \left(\overset{0}{h}_3\right)^{2s}}{(r^{(3)})^{2s}} \sum_{l^{(3)}=r^{(3)}-s}^{r^{(3)}+s} \Bigg\| \left(r^{(2)} + \frac{1}{2}\right)^{\frac{1}{2}} \left(l^{(3)} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$\times (h_2 h_3)^{-\frac{1}{2}} \left(\tilde{h}_{\underline{j},1} \partial_3^s \partial_j u_i\right)_{r^{(2)} l^{(3)}} \Bigg\|^2_{L_2(]0,L[)}, \tag{7.51}$$

$$j = 2, 3.$$

By the Parseval equality along with evident calculations we arrive at the relations (without loss of generality we assume that $N_i^{(j)}$, $j = 2, 3$, are odd)

$$\left\| \varepsilon_{N_{\underline{i}}^{(2)}, N_i^{(3)}} \right\|^2_{L_2(\Omega)} = \sum_{r^{(2)}=N_{\underline{i}}^{(2)}+1}^{\infty} \; \sum_{r^{(3)}=N_i^{(3)}+1}^{\infty} \int_0^L h_2^{-1} h_3^{-1} \left(r^{(2)} + \frac{1}{2}\right)$$

$$\times \left(r^{(3)} + \frac{1}{2}\right) (u_{ir^{(2)} r^{(3)}})^2 dx_1, \tag{7.52}$$

$$\left\|\partial_2 \varepsilon_{N_{\underline{i}}^{(2)},N_i^{(3)}}\right\|^2_{L_2(\Omega)} = \left( \sum_{r^{(2)}=N_{\underline{i}}^{(2)}+1}^{\infty} \sum_{r^{(3)}=N_i^{(3)}+1}^{\infty} \int_0^L h_2^{-1} h_3^{-1} \left( r^{(2)} + \frac{1}{2} \right) \right.$$

$$\times \left( r^{(3)} + \frac{1}{2} \right) \left[ \left( \partial_2 u_i \right)_{r^{(2)} r^{(3)}} \right]^2 dx_1$$

$$+ \sum_{r^{(3)}=0}^{N_{\underline{i}}^{(3)}} \left( r^{(3)} + \frac{1}{2} \right) \left\{ \frac{N_i^{(2)}(N_i^{(2)}-1)}{4} \int_0^L h_2^{-1} h_3^{-1} \left[ \left( \partial_2 u_i \right)_{N_i^{(2)} r^{(3)}} \right]^2 dx_1 \right.$$

$$\left.\left. + \frac{N_{\underline{i}}^{(2)}(N_i^{(2)}+1)}{4} \int_0^L h_2^{-1} h_3^{-1} \left[ \left( \partial_2 u_i \right)_{N_i^{(2)}+1 \ r^{(3)}} \right]^2 dx_1 \right\} \right), \tag{7.53}$$

$$\left\|\partial_3 \varepsilon_{N_{\underline{i}}^{(2)},N_i^{(3)}}\right\|^2_{L_2(\Omega)} = \left( \sum_{r^{(2)}=N_{\underline{i}}^{(2)}+1}^{\infty} \sum_{r^{(3)}=N_i^{(3)}+1}^{\infty} \int_0^L h_2^{-1} h_3^{-1} \left( r^{(2)} + \frac{1}{2} \right) \right.$$

$$\times \left( r^{(3)} + \frac{1}{2} \right) \left[ \left( \partial_3 u_i \right)_{r^{(2)} r^{(3)}} \right]^2 dx_1$$

$$+ \sum_{r^{(2)}=0}^{N_i^{(2)}} \left( r^{(2)} + \frac{1}{2} \right) \left\{ \frac{N_i^{(3)}(N_i^{(3)}-1)}{4} \int_0^L h_2^{-1} h_3^{-1} \left[ \left( \partial_3 u_{\underline{i}} \right)_{r^{(2)} N_i^{(3)}} \right]^2 dx_1 \right.$$

$$\left.\left. + \frac{N_i^{(3)}(N_i^{(3)}+1)}{4} \int_0^L h_2^{-1} h_3^{-1} \left[ \left( \partial_3 u_{\underline{i}} \right)_{r^{(2)} N_i^{(3)}+1} \right]^2 dx_1 \right\} \right), \tag{7.54}$$

$$hskip6cm i = 1,2,3, \quad j = 2,3,$$

$$\left\|\partial_1 \varepsilon_{N_{\underline{i}}^{(2)},N_i^{(3)}}\right\|^2_{L_2(\Omega)} \le 5 \left( \sum_{r^{(2)}=N_i^{(2)}+1}^{\infty} \left( r^{(2)} + \frac{1}{2} \right) \left( r^{(3)} + \frac{1}{2} \right) \right.$$

$$\times \int_0^L h_2^{-1} h_3^{-1} \left[ \left( \partial_1 u_{\underline{i}} \right)_{r^{(2)} r^{(3)}} \right]^2 dx_1$$

$$+\sum_{r^{(3)}=0}^{N_i^{(3)}} \int_0^L h_2^{-1} h_3^{-1} \left(r^{(3)}+\frac{1}{2}\right) \frac{N_i^{(2)}+1}{4} \left\{ \left[ \left(\partial_2 u_{\underline{i}}\right)_{N_i^{(2)} r^{(3)}} \right]^2 \right.$$

$$\times \left[ N_i^{(2)} \left(h_{2,1}\right)^2 + \left(N_i^{(2)}+2\right) \left(\tilde{h}_{2,1}\right)^2 \right] + \left[ \left(\partial_2 u_{\underline{i}}\right)_{N_i^{(2)}+1 r^{(3)}} \right]^2$$

$$\left. \times \left[ \left(N_i^{(2)}+2\right) \left(h_{2,1}\right)^2 + N_i^{(2)} \left(\tilde{h}_{2,1}\right)^2 \right] \right\} dx_1$$

$$+\sum_{r^{(2)}=0}^{N_i^{(2)}} \int_0^L h_2^{-1} h_3^{-1} \left(r^{(2)}+\frac{1}{2}\right) \frac{N_i^{(3)}+1}{4} \left\{ \left[ \left(\partial_3 u_{\underline{i}}\right)_{r^{(2)} N_i^{(3)}} \right]^2 \right.$$

$$\times \left[ N_i^{(3)} \left(h_{3,1}\right)^2 + \left(N_i^{(3)}+2\right) \left(\tilde{h}_{3,1}\right)^2 \right] + \left[ \left(\partial_3 u_{\underline{i}}\right)_{r^{(2)} N_i^{(3)}+1} \right]^2$$

$$\left. \left. \times \left[ \left(N_i^{(3)}+2\right) \left(h_{3,1}\right)^2 + N_i^{(3)} \left(\tilde{h}_{3,1}\right)^2 \right] \right\} dx_1 \right), \ i=1,2,3. \tag{7.55}$$

From (7.42), (7.52), and (7.48) we have

$$\left|\left|\varepsilon_{N_{\underline{i}}^{(2)}, N_i^{(3)}}\right|\right|^2_{L_2(\Omega)} \le \frac{\left(\overset{0}{h}_2\right)^{2s}}{\left(N_i^{(2)}\right)^{(2s)}} \ q_{i0}^{(2)} \left(N_{\underline{i}}^{(2)}\right) + \frac{\left(\overset{0}{h}_3\right)^{2s}}{\left(N_i^{(3)}\right)^{(2s)}} \ q_{i0}^{(3)} \left(N_{\underline{i}}^{(3)}\right); \tag{7.56}$$

here and in what follows $q_{ik}^{(j)}\left(N_{\underline{i}}^{(j)}\right) \to 0$ as $N_i^{(j)} \to \infty$, $j=2,3$; $i = 1,2,3$, $k=\overline{0,3}$.

Quite similarly, taking into account the relation

$$h_2^{\alpha} h_3^{\beta} u_{i,23} = \left(h_2^{\alpha} h_3^{\beta} u_i\right)_{,23},$$

from (7.42), (7.53)–(7.55), and (7.49)–(7.51) we arrive at inequalities

$$\left|\left|\partial_k \varepsilon_{N_{\underline{i}}^{(2)}, N_i^{(3)}}\right|\right|^2_{L_2(\Omega)} \le \frac{\left(\overset{0}{h}_2\right)^{2s}}{\left(N_i^{(2)}\right)^{2s-3}} \ q_{\underline{i}k}^{(2)} \left(N_i^{(2)}\right) + \frac{\left(\overset{0}{h}_3\right)^{2s}}{\left(N_i^{(3)}\right)^{2s-3}} \ q_{\underline{i}k}^{(3)} \left(N_i^{(3)}\right), \quad k=1,2,3. \tag{7.57}$$

Combining (7.56) and (7.57) we have

$$\left|\left|\varepsilon_{N_{\underline{i}}^{(2)},N_i^{(3)}}\right|\right|^2_{H^1(\Omega)} \leq \frac{\left(\overset{0}{h}_2\right)^{2s}}{\left(N^{(2)}_{\min}\right)^{2s-3}} \sum_{k=0}^{3} q^{(2)}_{\underline{i}k}\left(N_i^{(2)}\right) + \frac{\left(\overset{0}{h}_3\right)^{2s}}{\left(N^{(3)}_{\min}\right)^{2s-3}} \sum_{k=0}^{3} q^{(3)}_{\underline{i}k}\left(N_i^{(3)}\right). \qquad (7.58)$$

Invoking inequality (7.36) with

$$w = \left(S_{\mathbf{N}_1^{(2)},\mathbf{N}_1^{(3)}}, S_{\mathbf{N}_2^{(2)},\mathbf{N}_2^{(3)}}, S_{\mathbf{N}_3^{(2)},\mathbf{N}_3^{(3)}}\right) \in \widetilde{V}_{\mathbf{N}^{(2)},\mathbf{N}^{(3)}}(\Omega,\Gamma_L)$$

and taking into consideration the coercivity and boundedness properties of the bilinear form $B(\cdot\ ,\cdot)$ we obtain inequality (7.43). □

# Appendix A. Proof of Lemma 3.10.

It is evident that there exist functions $\overset{(-)}{g_j}(x_1)$ and $\overset{(+)}{g_j}(x_1)$, $j = 2, 3$, of the space $C^1([0, L])$ such that

$$\overset{(-)}{g_j}(x_1) \le \overset{(-)}{h_j}(x_1) \text{ and } \overset{(+)}{h_j}(x_1) \le \overset{(+)}{g_j}(x_1) \text{ for } 0 < x_1 < L, \; j = 2, 3; \quad \text{(A.1)}$$

moreover,

$$\left( \overset{(+)}{g_j}(x_1) - \overset{(-)}{g_j}(x_1) \right) \Bigg|_{x_1 \in [0,L]} > 0, \quad j = 2, 3, \quad \text{(A.2)}$$

and $\overset{(\pm)}{g_j}(L) = \overset{(\pm)}{h_j}(L), \quad j = 2, 3.$

Let

$$\Omega^* := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \; : \; 0 < x_1 < L, \; \overset{(-)}{g_j}(x_1) < x_j < \overset{(+)}{g_j}(x_1), \; j = 2, 3\}$$

and

$$\Gamma_L^* := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \; : \; x_1 = L, \; \overset{(-)}{g_j}(L) < x_j < \overset{(+)}{g_j}(L), \; j = 2, 3\}.$$

Due to the inequalities (A.1) then we have

$$\Omega \subset \Omega^*, \qquad \Gamma_L = \Gamma_L^*. \quad \text{(A.3)}$$

Further, let $w_1^0$ be an arbitrary element from the space $H^1(\Omega, \Gamma_L)$. On the one hand, since $\Omega$ is a Lipschitz domain and (A.3) holds, there exists an extension $w_1^*$ of the function $w_1^0$ onto the domain $\Omega^*$ such that $w_1^* \in H^1(\Omega^*, \Gamma_L^*)$ (see, e.g., [20]). On the other hand, since the space $C^\infty(\Omega^*, \Gamma_L^*)$ is dence in $H^1(\Omega^*, \Gamma_L^*)$, for arbitrary $\varepsilon > 0$ there exists a function $w_1 \in H^2(\Omega^*, \Gamma_L^*) := \{w \in H^2(\Omega^*) \; : \; w|_{\Gamma_L^*} = 0\}$ such that

$$||w_1^* - w_1||_{H^1(\Omega^*)} < \frac{\varepsilon}{2}. \quad \text{(A.4)}$$

Let $S_{N_1^{(2)}, N_1^{(3)}}(w_1)$ be the partial double Fourier–Legendre sums of $w_1$, i.e.,

$$S_{N_1^{(2)}, N_1^{(3)}}(w_1) := \sum_{r_1^{(2)}=0}^{N_1^{(2)}} \sum_{r_1^{(3)}=0}^{N_1^{(3)}} a_2 \, a_3 \left( r_1^{(2)} + \frac{1}{2} \right) \left( r_1^{(3)} + \frac{1}{2} \right) P_{r_1^{(2)}}(a_2 x_2 - b_2)$$

$$\times P_{r_1^{(3)}}(a_3 x_3 - b_3) \, w_{1 \, r_1^{(2)} r_1^{(3)}}, \quad \text{(A.5)}$$

where

$$w_{1\, r_1^{(2)}\, r_1^{(3)}} := \int\limits_{\overset{(-)}{g}_3}^{\overset{(+)}{g}_3} \int\limits_{\overset{(-)}{g}_2}^{\overset{(+)}{g}_2} w_1(x_1, x_2, x_3)\ P_{r_1^{(2)}}(a_2 x_2 - b_2)$$

$$\times P_{r_1^{(3)}}(a_3 x_3 - b_3)\ dx_2 dx_3, \tag{A.6}$$

$$0 \le r_1^{(i)} \le N_1^{(i)}, \ \ i = 2, 3,$$

are double moments of the function $w_1$. As for as $w_1 \in H^2(\Omega^*, \Gamma_L^*)$, by (A.2), (A.5), and (A.6) we see that

$$S_{N_1^{(2)}, N_1^{(3)}}(w_1) \in \widetilde{V}_{N_1^{(2)}, N_1^{(3)}}(\Omega^*, \Gamma_L^*), \tag{A.7}$$

since it has the form similar to (7.3). Taking into consideration inequality (A.2) and applying estimate (7.58) for the regular (noncusped) case (when $s = 2$ and $i = 1$), we derive the relation

$$||w_1 - S_{N_1^{(2)}, N_1^{(3)}}(w_1)||_{H^1(\Omega^*)} \le \frac{q_2(N_1^{(2)})}{N_1^{(2)}} + \frac{q_3(N_1^{(3)})}{N_1^{(3)}}, \qquad \lim_{N_1^{(i)} \to \infty} q_i(N_1^{(i)}) = 0, \ i = 2, 3.$$

Therefore there are sufficiently large integers $N_1^{(2)}$ and $N_1^{(2)}$ such that

$$||w_1 - S_{N_1^{(2)}, N_1^{(3)}}(w_1)||_{H^1(\Omega^*)} < \frac{\varepsilon}{2}. \tag{A.8}$$

The inclusions (A.3) and (A.7) imply that

$$S_{N_1^{(2)}, N_1^{(3)}}(w_1)|_\Omega \in \widetilde{V}_{N_1^{(2)}, N_1^{(3)}}(\Omega, \Gamma_L). \tag{A.9}$$

Now from (A.4), (A.8), and (A.9) it follows that

$$||\ w_1^0 - S_{N_1^{(2)}, N_1^{(3)}}(w_1)|_\Omega\ ||_{H^1(\Omega)} \le ||w_1^* - w_1||_{H^1(\Omega^*)} + ||w_1 - S_{N_1^{(2)}, N_1^{(3)}}(w_1)||_{H^1(\Omega^*)} < \varepsilon,$$

which completes the proof. □

# Appendix B. On conditions (3.40)

Here we will find out which kind of conditions have to satisfy a function $u$ ensuring the embedding

$$\begin{aligned} v_i(x_1,x_2,x_3) &:= a_2 a_3 u_{ir^{(2)}r^{(3)}}(x_1) P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3) \\ &\in H^1(\Omega,\Gamma_L), \quad i=1,2,3, \end{aligned} \tag{B.1}$$

where

$$u_{ir^{(2)}r^{(3)}}(x_1) := \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i(x_1,x_2,x_3)P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3)dx_2dx_3. \tag{B.2}$$

From (B.1) it follows

$$\begin{aligned} v_{i,1} = {} & (a_2a_3)_{,1} u_{ir^{(2)}r^{(3)}} P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3) \\ & + a_2a_3 u_{ir^{(2)}r^{(3)},1} P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3) \\ & + a_2a_3 u_{ir^{(2)}r^{(3)}}(a_{2,1}x_2-b_{2,1})P'_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3) \\ & + a_2a_3 u_{ir^{(2)}r^{(3)}}(a_{3,1}x_3-b_{3,1})P_{r^{(2)}}(a_2x_2-b_2)P'_{r^{(3)}}(a_3x_3-b_3), \end{aligned} \tag{B.3}$$

$$v_{i,2} = a_2a_3 u_{ir^{(2)}r^{(3)}} a_2 P'_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3), \tag{B.4}$$

$$v_{i,3} = a_2a_3 u_{ir^{(2)}r^{(3)}} a_3 P'_{r^{(3)}}(a_3x_3-b_3)P_{r^{(2)}}(a_2x_2-b_2). \tag{B.5}$$

It is evident that

$$\begin{aligned} & u_{ir^{(2)}r^{(3)},1}(x_1) \\ & = \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i(x_1,\overset{(+)}{h}_2(x_1),x_3)P_{r^{(2)}}(a_2\overset{(+)}{h}_2(x_1)-b_2)P_{r^{(3)}}(a_3x_3-b_3)dx_3 \times \overset{(+)}{h}_{2,1} \\ & - \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i(x_1,\overset{(-)}{h}_2(x_1),x_3)P_{r^{(2)}}(a_2\overset{(-)}{h}_2(x_1)-b_2)P_{r^{(3)}}(a_3x_3-b_3)dx_3 \times \overset{(-)}{h}_{2,1} \\ & + \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} u_i(x_1,x_2,\overset{(+)}{h}_3(x_1))P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3\overset{(+)}{h}_3(x_1)-b_3)dx_2 \times \overset{(+)}{h}_{3,1} \end{aligned}$$

$$-\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} u_i(x_1,x_2,\overset{(-)}{h}_3(x_1))P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3\overset{(-)}{h}_3(x_1)-b_3)dx_2\times \overset{(-)}{h}_{3,1}$$

$$+\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_{i,1}(x_1,x_2,x_3)P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3)dx_2dx_3$$

$$+\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i(x_1,x_2,x_3)(a_{2,1}x_2-b_{2,1})P'_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3)dx_2dx_3$$

$$+\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i(x_1,x_2,x_3)(a_{3,1}x_3-b_{3,1})P_{r^{(2)}}(a_2x_2-b_2)P'_{r^{(3)}}(a_3x_3-b_3)dx_2dx_3$$

$$=:\sum_{i=1}^{7} I_i. \tag{B.6}$$

Taking into account the inequalities

$$\max_{[-1,1]}|P_{r^{(j)}}(x_1)|\ ,\ \ \max_{[-1,1]}|P'_{r^{(j)}}(x_1)|\le M=const<+\infty,\quad j=2,3, \tag{B.7}$$

and applying Schwarz inequality we get from (B.2)

$$|u_{ir^{(2)}r^{(3)}}(x_1)|^2\le M^4\left(\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} 1\cdot|u_i(x_1,x_2,x_3)|dx_2dx_3\right)^2$$

$$\le M^4\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} 1^2dx_2dx_3\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i^2(x_1,x_2,x_3)dx_2dx_3$$

$$=M^4\left(\overset{(+)}{h}_2-\overset{(-)}{h}_2\right)\left(\overset{(+)}{h}_3-\overset{(-)}{h}_3\right)\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i^2(x_1,x_2,x_3)dx_2dx_3$$

$$=4M^4h_2\,h_3\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i^2(x_1,x_2,x_3)dx_2dx_3. \tag{B.8}$$

With the help of (B.1) and (B.8) we derive

$$||v_i||^2_{L_2(\Omega)} = \int\limits_0^L a_2^2 a_3^2 u^2_{r(2)r(3)} \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} P^2_{r(2)}(a_2x_2 - b_2) P^2_{r(3)}(a_3x_3 - b_3) dx_3 dx_2 dx_1$$

$$\leq 4M^4 \int\limits_0^L a_2^2 a_3^2 u^2_{r(2)r(3)} h_2 h_3 dx_1 = 4M^4 \int\limits_0^L (h_2h_3)^{-1} u^2_{ir(2)r(3)} dx_1$$

$$\leq 4M^4 \int\limits_0^L (h_2h_3)^{-1} \left\{ 4M^4 h_2 h_3 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2(x_1, x_2, x_3) dx_2 dx_3 \right\} dx_1$$

$$= 16M^8 \int\limits_0^L \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2(x_1, x_2, x_3) dx_2 dx_3 dx_1 = 16M^8 ||u_i||^2_{L_2(\Omega)}. \qquad \text{(B.9)}$$

Quite similarly, from (B.4), (B.5), (B.7), and (B.8) we have

$$||v_{i,2}||^2_{L_2(\Omega)} = \int\limits_0^L a_2^4 a_3^2 u^2_{ir(2)r(3)} \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} {P'_{r(2)}}^2(a_2x_2 - b_2) P^2_{r(3)}(a_3x_3 - b_3) dx_3 dx_2 dx_1$$

$$\leq 4M^4 \int\limits_0^L a_2^4 a_3^2 u^2_{ir(2)r(3)} h_2 h_3 dx_1$$

$$\leq 4M^4 \int\limits_0^L h_2^{-3} h_3^{-1} \left\{ 4M^4 h_2 h_3 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2(x_1, x_2, x_3) dx_2 dx_3 \right\} dx_1$$

$$= 16M^8 \int\limits_0^L \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} h_2^{-2} u_i^2(x_1, x_2, x_3) dx_2 dx_3 dx_1$$

$$= 16M^8 ||h_2^{-1} u_i||_{L_2(\Omega)}, \qquad \text{(B.10)}$$

$$||v_{i,3}|| \leq 16M^8 ||h_3^{-1} u_i||_{L_2(\Omega)}. \qquad \text{(B.11)}$$

These inequalities yield

$$\int_{\Omega} \left[(a_2a_3)_{,1}u_{ir^{(2)}r^{(3)}}P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3)\right]^2 dx_3dx_2dx_1$$

$$\le 4M^4\int_0^L [(a_2a_3)_{,1}]^2 u^2_{ir^{(2)}r^{(3)}}h_2h_3dx_1 = 4M^4\int_0^L \frac{[(h_2h_3)_{,1}]^2}{(h_2h_3)^3}u^2_{ir^{(2)}r^{(3)}}dx_1$$

$$\le 4M^4\int_0^L \frac{[(h_2h_3)_{,1}]^2}{(h_2h_3)^3}\left\{4M^4h_2h_3\int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2dx_2dx_3\right\}dx_1$$

$$= 16M^8\int_0^L \frac{[(h_2h_3)_{,1}]^2}{(h_2h_3)^2}\int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2dx_2dx_3dx_1$$

$$= 16M^8\left|\left|\frac{(h_2h_3)_{,1}u_i}{(h_2h_3)}\right|\right|^2_{L_2(\Omega)}, \tag{B.12}$$

$$\int_{\Omega}\left[a_2a_3u_{ir^{(2)}r^{(3)}}(a_{2,1}x_2-b_{2,1})P'_{r^{(2)}}(a_2x_2-b_2)\,P_{r^{(3)}}(a_3x_3-b_3)\right]^2dx_3dx_2dx_1$$

$$\le \int_0^L a_2^2a_3^2u^2_{ir^{(2)}r^{(3)}}\int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} M^4(a_{2,1}x_2-b_{2,1})^2dx_3dx_2dx_1$$

$$= 2\,M^4\int_0^L a_2^2a_3^2u^2_{ir^{(2)}r^{(3)}}\,h_3\int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}(a_{2,1}x_2-b_{2,1})^2dx_2dx_1$$

$$= 2\,M^4\int_0^L a_2^2a_3^2u^2_{ir^{(2)}r^{(3)}}\,h_3\Big[\frac{2}{3}a_{2,1}^2h_2\left(\overset{(+)}{h_2}{}^2+\overset{(+)}{h_2}\overset{(-)}{h_2}+\overset{(-)}{h_2}{}^2\right)$$
$$-4\,a_{2,1}b_{2,1}h_2\tilde{h_2}+2\,b_{2,1}^2\,h_2\Big]dx_1$$

$$\le 16M^8\int_0^L\int_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\int_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}}\Big[\frac{1}{3}a_{2,1}^2\left(\overset{(+)}{h_2}{}^2+\overset{(+)}{h_2}\overset{(-)}{h_2}+\overset{(-)}{h_2}{}^2\right)$$
$$-2a_{2,1}b_{2,1}\tilde{h_2}+b_{2,1}^2\Big]u_i^2dx_3dx_2dx_1$$

$$= 16M^8 \left\| \overset{*}{h}_2^{\frac{1}{2}} u_i \right\|^2_{L_{2(\Omega)}}, \tag{B.13}$$

where

$$\overset{*}{h}_j = \frac{1}{3} a_{j,1}^2 \left( \overset{(+)}{h}_j^{\,2} + \overset{(+)}{h}_j \overset{(-)}{h}_j + \overset{(-)}{h}_j^{\,2} \right) - 2a_{j,1} b_{j,1} \tilde{h}_j + b_{j,1}^2, \quad j = 2, 3. \tag{B.14}$$

By the same type arguments we derive

$$\int_\Omega \left[ a_2 a_3 u_{ir^{(2)}r^{(3)}} (a_{3,1} x_3 - b_{3,1}) P_{r^{(2)}}(a_2 x_2 - b_2) P'_{r^{(3)}}(a_3 x_3 - b_3) \right]^2 dx_3 dx_2 dx_1$$

$$\leq 16M^8 \left\| \overset{*}{h}_3^{\frac{1}{2}} u_i \right\|^2_{L_2(\Omega)}, \tag{B.15}$$

where $\overset{*}{h}_3$ is given by (B.14).

For $x_2 \in \left[ \overset{(-)}{h}_2(x_1), \overset{(+)}{h}_2(x_1) \right]$ we have

$$u_i(x_1, \overset{(+)}{h}_2(x_1), x_3) = u_i(x_1, x_2, x_3) + \int_{x_2}^{\overset{(+)}{h}_2} u_{i,2}(x_1, x_2, x_3) dx_2. \tag{B.16}$$

Whence

$$u_i^2(x_1, \overset{(+)}{h}_2(x_1), x_3) \leq 2u_i^2(x_1, x_2, x_3) + 2 \left( \int_{x_2}^{\overset{(+)}{h}_2} 1 \cdot u_{i,2}(x_1, x_2, x_3) dx_2 \right)^2$$

$$\leq 2u_i^2(x_1, x_2, x_3) + 2 \int_{x_2}^{\overset{(+)}{h}_2} 1^2 dt \int_{x_2}^{\overset{(+)}{h}_2} u_{i,2}^2(x_1, x_2, x_3) dx_2$$

$$\leq 2u_i^2(x_1, x_2, x_3) + 4h_2 \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} u_{i,2}^2(x_1, x_2, x_3) dx_2. \tag{B.17}$$

Integrate both parts of inequality (B.17) with respect to $x_2$ from $\overset{(-)}{h}_2$ to $\overset{(+)}{h}_2$

$$2h_2 u_i^2(x_1, \overset{(+)}{h}_2(x_1), x_3) \leq 2 \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} u_i^2(x_1, x_2, x_3) dx_2$$

$$+8h_2^2 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} u_{i,2}^2(x_1, x_2, x_3)dx_2, \tag{B.18}$$

i.e.,

$$u_i^2(x_1, \overset{(+)}{h_2}(x_1), x_3) \leq h_2^{-1} \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} u_i^2(x_1, x_2, x_3)dx_2$$

$$+4h_2 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} u_{i,2}^2(x_1, x_2, x_3)dx_2. \tag{B.19}$$

Analogously we get the following inequalities

$$u_i^2(x_1, \overset{(-)}{h_2}(x_1), x_3) \leq h_2^{-1} \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} u_i^2(x_1, x_2, x_3)dx_2$$

$$+4h_2 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} u_{i,2}^2(x_1, x_2, x_3)dx_2, \tag{B.20}$$

$$u^2(x_1, x_2, \overset{(\pm)}{h_3}(x_1)) \leq h_3^{-1} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u^2(x_1, x_2, x_3)dx_3$$

$$+4h_3 \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_{,3}^2(x_1, x_2, x_3)dx_3. \tag{B.21}$$

Further we apply relations (B.6), (B.7), and (B.19)–(B.21) to obtain

$$\int\limits_{\Omega} \left[a_2 a_3 I_1(x_1) P_{r^{(2)}}(a_2 x_2 - b_2) P_{r^{(3)}}(a_3 x_3 - b_3)\right]^2 dx_3 dx_2 dx_1$$

$$
\begin{aligned}
&\le M^4 \int\limits_{\Omega} a_2^2 a_3^2 I_1^2(x_1) dx_3 dx_2 dx_1 = M^4 \int\limits_0^L a_2^2 a_3^2 I_1^2(x_1)\, dx_1 \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} 1 dx_2 dx_3 \\
&= 4M^4 \int\limits_0^L a_2^2 a_3^2 I_1^2(x_1) h_2 h_3 dx_1 = 4M^4 \int\limits_0^L (h_2 h_3)^{-1} I_1^2(x) dx_1 \\
&\le 4M^4 \int\limits_0^L (h_2 h_3)^{-1} \left[ \overset{(+)}{h}_{2,1} \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} |u_i(x_1, \overset{(+)}{h}_2(x_1), x_3)| M^2\, dx_3 \right]^2 dx_1 \\
&\le 4M^8 \int\limits_0^L \left\{ (h_2 h_3)^{-1} \left( \overset{(+)}{h}_{2,1} \right)^2 \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} 1^2 dx_3 \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i^2(x_1, \overset{(+)}{h}_2(x_1), x_3) dx_3 \right\} dx_1 \\
&\le 8M^8 \int\limits_0^L \left\{ (h_2 h_3)^{-1} h_3 \left( \overset{(+)}{h}_{2,1} \right)^2 \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left[ h_2^{-1} \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} u_i^2(x_1, x_2, x_3) dx_2 \right.\right. \\
&\qquad\qquad \left.\left. + 4h_2 \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} u_{i,2}^2(x_1, x_2, x_3) dx_2 \right] dx_3 \right\} dx_1 \\
&= 8M^8 \int\limits_0^L \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left( \overset{(+)}{h}_{2,1} \right)^2 h_2^{-2} u_i^2(x_1, x_2, x_3) dx_3 dx_2 dx_1 \\
&\qquad + 32M^8 \int\limits_0^L \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left( \overset{(+)}{h}_{2,1} \right)^2 u_{i,2}^2(x_1, x_2, x_3) dx_3 dx_2 dx_1 \\
&= 8M^8 \left\| h_2^{-1} \overset{(+)}{h}_{2,1} u_i \right\|_{L_2(\Omega)}^2 + 32M^2 \left\| \overset{(+)}{h}_{2,1} u_{i,2} \right\|_{L_2(\Omega)}^2 . \qquad \text{(B.22)}
\end{aligned}
$$

Similarly we have

$$
\int\limits_{\Omega} \left[ a_2 a_3 I_2(x_1) P_{r^{(2)}}(a_2 x_2 - b_2) P_{r^{(3)}}(a_3 x_3 - b_3) \right]^2 dx_3 dx_2 dx_1
$$

$$\leq 8M^8 \left|\left| h_2^{-1} \overset{(-)}{h}_{2,1} u_i \right|\right|^2_{L_2(\Omega)} + 32M^2 \left|\left| \overset{(-)}{h}_{2,1} u_{i,2} \right|\right|^2_{L_2(\Omega)}, \tag{B.23}$$

$$\int_\Omega \left[a_2 a_3\, I_j(x_1)\, P_{r^{(2)}}(a_2 x_2 - b_2) P_{r^{(3)}}(a_3 x_3 - b_3)\right]^2 dx_3 dx_2 dx_1$$

$$\leq 8M^8 \left|\left| h_3^{-1} \overset{(\pm)}{h}_{3,1} u_i \right|\right|^2_{L_2(\Omega)} + 32M^2 \left|\left| \overset{(\pm)}{h}_{3,1} u_{i,3} \right|\right|^2_{L_2(\Omega)}, \quad j = 3, 4, \tag{B.24}$$

$$\int_\Omega \left[a_2 a_3\, I_5(x_1)\, P_{r^{(2)}}(a_2 x_2 - b_2) P_{r^{(3)}}(a_3 x_3 - b_3)\right]^2 dx_3 dx_2 dx_1$$

$$\leq 16M^8 \left|\left| u_{i,1} \right|\right|^2_{L_2(\Omega)}. \tag{B.25}$$

Due to (B.6) and (B.7) we derive

$$\int_\Omega \left[a_2 a_3\, I_6(x_1)\, P_{r^{(2)}}(a_2 x_2 - b_2) P_{r^{(3)}}(a_3 x_3 - b_3)\right]^2 dx_3 dx_2 dx_1$$

$$\leq M^4 \int_\Omega a_2^2 a_3^2 I_6^2(x_1) dx_3 dx_2 dx_1 = M^4 \int_0^L a_2^2 a_3^2 I_6^2(x_1) \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} 1 dx_2 dx_3$$

$$= 4M^4 \int_0^L a_2^2 a_3^2 I_6^2(x_1) h_2 h_3 dx_1 = 4M^4 \int_0^L (h_2 h_3)^{-1} I_6^2(x_1) dx_1. \tag{B.26}$$

Applying Hölder's inequality and arguing as in the proof of inequalities (B.8) and (B.13) we get

$$I_6^2(x_1) \leq M^4 \left( \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} |u_i(x_1, x_2, x_3)|\, |a_{2,1} x_2 - b_{2,1}| dx_2 dx_3 \right)^2$$

$$\leq M^4 \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} u_i^2(x_1, x_2, x_3) dx_2 dx_3 \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} (a_{2,1} x_2 - b_{2,1})^2 dx_2 dx_3, \tag{B.27}$$

$$\int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} (a_{2,1} x_2 - b_{2,1})^2 dx_2 dx_3 = \int_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \int_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} (a_{2,1}^2 x_2^2 - 2a_{2,1} b_{2,1} x_2 + b_{2,1}^2) dx_2 dx_3$$

$$= 2h_3 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} (a_{2,1}^2 x_2^2 - 2a_{2,1}b_{2,1}x_2 + b_{2,1}^2)dx_2$$

$$= 2h_3 (a_{2,1}^2 \frac{x_2^3}{3} - a_{2,1}b_{2,1}x_2^2 + b_{2,1}^2 x_2)\Big|_{x_2=\overset{(-)}{h_2}}^{x_2=\overset{(+)}{h_2}}$$

$$= 4h_3\, h_2 \left[ \frac{1}{3} a_{2,1}^2 \left( \overset{(+)}{h_2}{}^2 + \overset{(+)}{h_2}\,\overset{(-)}{h_2} + \overset{(-)}{h_2}{}^2 \right) - 2a_{2,1}b_{2,1}\tilde{h_2} + b_{2,1}{}^2 \right]. \qquad \text{(B.28)}$$

These inequalities imply

$$I_6^2(x_1) \leq 4M^4 h_2 h_3 \overset{*}{h_2} \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2(x_1, x_2, x_3)dx_2 dx_3, \qquad \text{(B.29)}$$

where $\overset{*}{h_2}$ is given by (B.14).

In accordance with (B.29) we have from (B.26)

$$\int\limits_{\Omega} \left[a_2 a_3\, I_6(x_1)\, P_{r^{(2)}}(a_2x_2 - b_2) P_{r^{(3)}}(a_3x_3 - b_3)\right]^2 dx_3 dx_2 dx_1$$

$$\leq 16M^8 \int\limits_0^L \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} \overset{*}{h_2} u_i^2(x_1, x_2, x_3) dx_3 dx_2 dx_1$$

$$= 16M^8 \,||\, \overset{*}{h_2}{}^{\frac{1}{2}}\, u_i \,||_{L_2(\Omega)}^2 . \qquad \text{(B.30)}$$

Quite analogously we get

$$\int\limits_{\Omega} \left[a_2 a_3 I_7(x_1) P_{r^{(2)}}(a_2x_2 - b_2) P_{r^{(3)}}(a_3x_3 - b_3)\right]^2 dx_3 dx_2 dx_1$$

$$\leq 4M^4 \int\limits_0^L (h_2 h_3)^{-1}\, I_7^2(x_1) dx_1, \qquad \text{(B.31)}$$

$$I_7^2(x_1) \leq M^4 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_i^2(x_1, x_2, x_3) dx_2 dx_3 \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} (a_{3,1}x_3 - b_{3,1})^2 dx_2 dx_3, \qquad \text{(B.32)}$$

$$\int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}}(a_{3,1}x_3-b_{3,1})^2dx_2dx_3=4\,h_2\,h_3\Big[\frac{1}{3}a_{3,1}^2\left(\overset{(+)}{h_3}{}^2+\overset{(+)}{h_3}\,\overset{(-)}{h_3}+\overset{(-)}{h_3}{}^2\right)$$

$$-2a_{3,1}b_{3,1}\tilde{h_3}+b_{3,1}{}^2\Big], \tag{B.33}$$

$$I_7^2(x_1)\leq 4M^4h_2h_3\overset{*}{h}_3\int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}}u_i^2(x_1,x_2,x_3)dx_2dx_3, \tag{B.34}$$

$$\int\limits_{\Omega}\left[a_2a_3I_7(x_1)P_{r^{(2)}}(a_2x_2-b_2)P_{r^{(3)}}(a_3x_3-b_3)\right]^2dx_3dx_2dx_1$$

$$\leq 16M^8\left|\left|\overset{*}{h}_3{}^{\frac{1}{2}}u_i\right|\right|^2_{L_2(\Omega)}, \tag{B.35}$$

where $\overset{*}{h}_3$ is given by (B.14).

Note that the quantity $||v_i||^2_{H^1(\Omega)}$ can be estimated from above by the right-hand side expressions of inequalities (B.9)–(B.13), (B.15), (B.22)–(B.25), (B.30), and (B.35) and, therefore, the embedding (B.1), i.e., the inclusion $v_i\in H^1(\Omega,\Gamma_L)$ will be automatically satisfied if the following conditions are fulfilled

$$\begin{aligned}&h_j^{-1}u\in[L_2(\Omega)]^3\,,\quad h_{\underline{j}}^{-1}\overset{(\pm)}{h}_{j,1}\,u\in[L_2(\Omega)]^3\,,\quad j=2,3,\\&u_{,1}\in[L_2(\Omega)]^3\,,\end{aligned} \tag{B.36}$$

$$\frac{(h_2h_3)_{,1}}{h_2h_3}u\in[L_2(\Omega)]^3\,,\quad \overset{(\pm)}{h}_{j,1}\,u_{,j}\in[L_2(\Omega)]^3\,,\quad j=2,3, \tag{B.37}$$

$$\overset{*}{h}_j^{\frac{1}{2}}\,u\in[L_2(\Omega)]^3\,,\ j=2,3. \tag{B.38}$$

Here $\overset{*}{h}_j$ is given by (B.14).

Remark that in the case of a symmetric beam, i.e., if $h_j=\overset{(+)}{h_j}=-\overset{(-)}{h_j},\quad j=2,3$, we have $b_j=0$ and conditions (B.38) read as follows

$$\frac{h_{j,1}}{h_j}\,u\in[L_2(\Omega)]^3\,,\quad j=2,3, \tag{B.39}$$

i.e.,

$$(\log h_j)_{,j}\,u\in[L_2(\Omega)]^3\,,\quad j=2,3. \tag{B.40}$$

In addition, if $h_j = \overset{(+)}{h}_j = -\overset{(-)}{h}_j = k_j x_1^{\kappa_j}$, $k_j, \kappa_j = const$, $k_j > 0$, $0 \leq \kappa_j \leq 1$, then conditions (B.38) are written as

$$\kappa_j x_1^{-1} u \in [L_2(\Omega)]^3\,, \quad j = 2, 3. \tag{B.41}$$

Moreover, since $0 \leq \kappa_j \leq 1$, conditions (B.36) and (B.37) are equivalent to the inclusions

$$u_{,1} \in [L_2(\Omega)]^3\,, \quad x_1^{\kappa_j - 1} u_{,j} \in [L_2(\Omega)]^3\,, \quad j = 2, 3. \tag{B.42}$$

## Appendix 2C

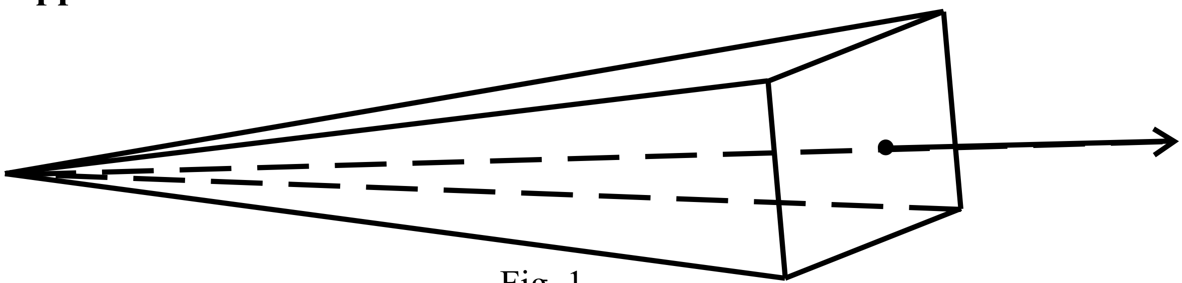

Fig. 1

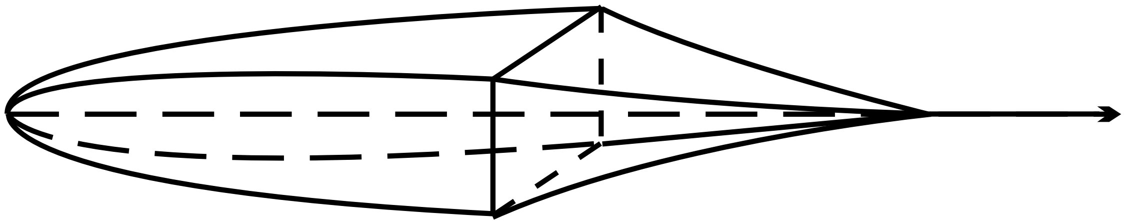

Fig. 2

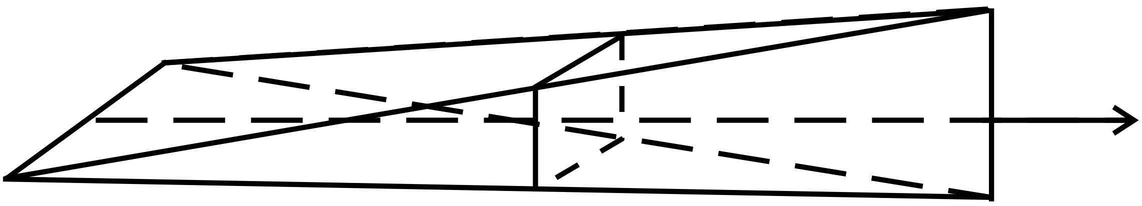

Fig. 3

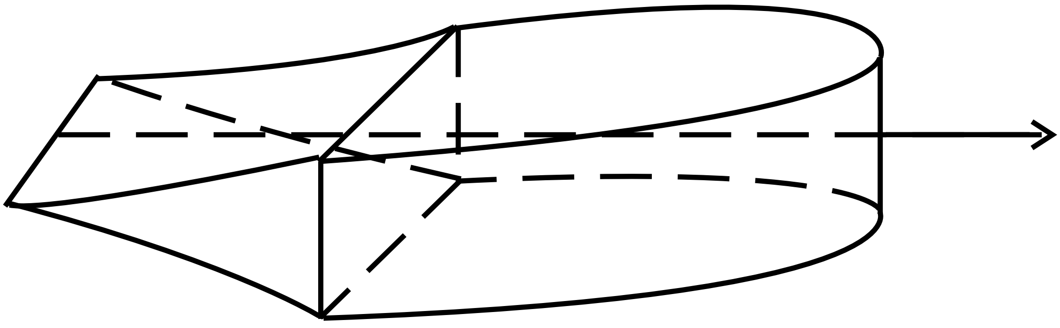

Fig. 4

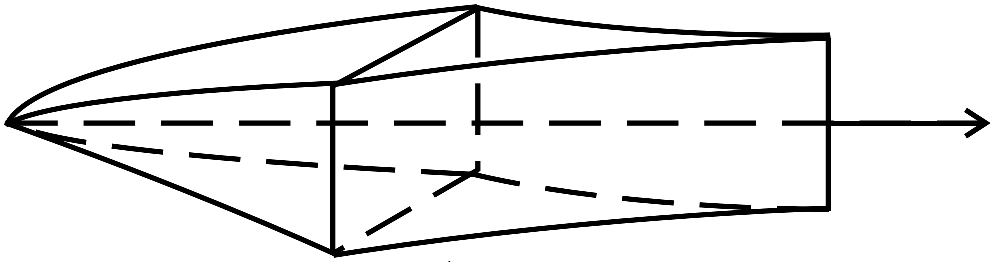

Fig. 5

## Appendix 2D

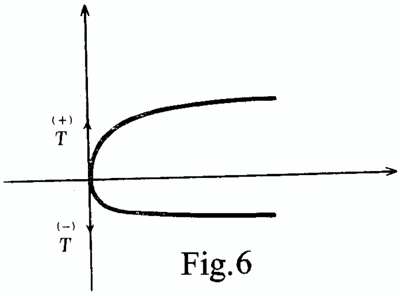

Fig.6

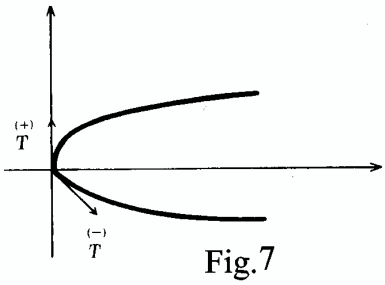

Fig.7

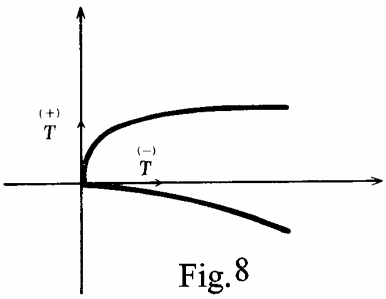

Fig.8

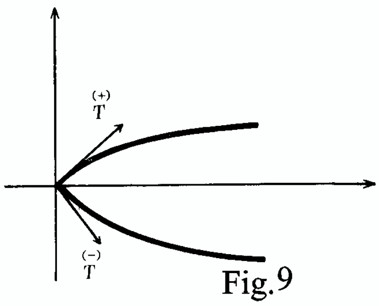

Fig.9

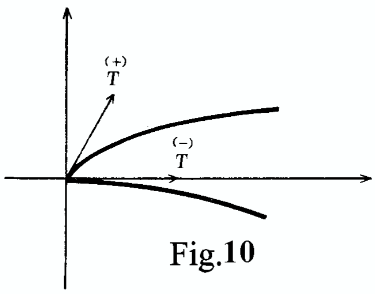

Fig.10

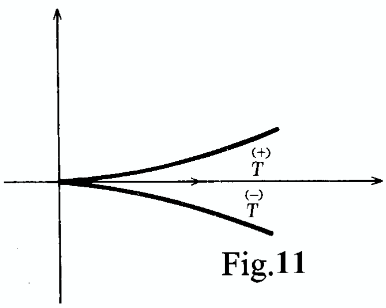

Fig.11

$\overset{(+)}{T}$ and $\overset{(-)}{T}$ denote tangents to the profile and projection curves at points of the cusped ends.

Figure 6 corresponds to a blant cusped end, Figures 7-10 correspond to angularly cusped ends, and Figure 11 corresponds to a sharply cusped end (a real mathematical cusp).

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