ON SOME REFINED THEORIES OF PLATES AND SHELLS

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Abstract. In this paper we consider Reissner-Mindlin's type linear theory and I. Vekua's refined linear theory for plates, as well as, Koiter-Naghdi's and I. Vekua's refined nonlinear theories for non-shallow shells. We also consider Kirsch's well-known problem for plates [1].

Key words and phrases: Non-shallow shells, metric tensor and tensor of curvature, midsurface of the shell.

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1. A complete system of equilibrium equation and the stress-strain relations of the three-dimensional (3D) nonlinear theory of elasticity can be written as:

$$\hat{\nabla}_i \boldsymbol{\sigma}^i + \mathbf{\Phi} = 0, \tag{1}$$
$$\boldsymbol{\sigma}^i = E^{ijpq} e_{pq}(\mathbf{R}_j + \partial_j \mathbf{u}) \quad (i,j,p,q = 1,2,3),$$

where $\hat{\nabla}_i$ are covariant derivatives relative to the space curvilinear coordinates x^i; $\boldsymbol{\sigma}^i$ and $\mathbf{\Phi}$ are, respectively, the contravariant "constituents" of the stress vector and an external force, e_{ij} are covariant components of the strain tensor, $\mathbf{u}$ is the displacement vector,

$$\begin{aligned} 2e_{ij} &= \mathbf{R}_i \partial_j \mathbf{u} + \mathbf{R}_j \partial_i \mathbf{u} + \partial_i \mathbf{u} \partial_j \mathbf{u}, \\ E^{ijpq} &= \lambda g^{ij} g^{pq} + \mu (g^{ip} g^{jq} + g^{iq} g^{jp}) \quad (g^{ij} = \mathbf{R}^i \mathbf{R}^j), \end{aligned} \tag{2}$$

λ and μ are Lame's constants, $\mathbf{R}_i$ and $\mathbf{R}^i$ are covariant and contravariant basis vectors of the surface $\hat{S}(x^3 = const)$ of the 3D domain Ω, which are connected with the basis vectors $\mathbf{r}_i$ and $\mathbf{r}^i$ of the midsurface S ($x^3 = 0$) by the following relations:

$$\begin{aligned} &\mathbf{R}_i = A_{i.}^{\cdot j} \mathbf{r}_j, \quad \mathbf{R}^i = A_{\cdot j}^{i \cdot} \mathbf{r}^j \quad (i,j = 1,2,3), \quad \boldsymbol{R}_3 = \boldsymbol{R}^3 = \boldsymbol{r}_3 = \boldsymbol{r}^3 = \boldsymbol{n}, \\ &A_{\alpha.}^{\cdot \beta} = a_\alpha^\beta - x_3 b_\alpha^\beta, \quad A_{i.}^{\cdot 3} = A_{\cdot 3}^{i \cdot} = \delta_{i3}, \quad A_{\cdot \beta}^{\alpha \cdot} = \vartheta^{-1}[a_\beta^\alpha + x_3(b_\beta^\alpha - 2H a_\beta^\alpha)], \\ &\vartheta = 1 - 2Hx_3 + Kx_3^2, \end{aligned} \tag{3}$$

where $a_\alpha^\beta(a_{\alpha\beta}, a^{\alpha\beta})$ and $b_\alpha^\beta(b_{\alpha\beta}, b^{\alpha\beta})$ are mixed (covariant, contravariant) components of the metric tensor and tensor of curvature of the midsurface S ($x_3 = 0$), $x_3 = x^3$ is the the thickness coordinate and h is the semi-thickness of the shell Ω, H and K are middle and Gaussian curvatures of the midsurface S ($x_3 = 0$), g and a are discriminants of metric tensor of the the surfaces $\hat{S}$ and S.

Shallow and Non-Shallow Shells. The main quadratic forms of the surface S $(x_3 = 0)$ and the surface $\hat{S}$ $(x_3 = const)$ have the form:

$$\begin{aligned} &\mathrm{I} = ds^2 = a_{\alpha\beta} dx^\alpha dx^\beta, \quad \mathrm{II} = k_s ds^2 = b_{\alpha\beta} dx^\alpha dx^\beta, \quad S\ (x_3 = 0), \\ &\mathrm{I} = d\hat{s}^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad \mathrm{II} = \hat{k}_{\hat{s}} d\hat{s}^2 = \hat{b}_{\alpha\beta} dx^\alpha dx^\beta, \quad \hat{S}\ (x_3 = const), \end{aligned} \tag{4}$$

where k_s and $\hat{k}_{\hat{s}}$ are the normal curvatures of the surfaces S and $\hat{S}$:

$$a_{\alpha\beta} = \boldsymbol{r}_\alpha \boldsymbol{r}_\beta, \;\; b_{\alpha\beta} = -\boldsymbol{r}_\alpha \boldsymbol{n}_\beta, \;\; k_s = b_{\alpha\beta} s^\alpha s^\beta, \;\; s^\alpha = \frac{dx^\alpha}{ds}, \;\; S\ (x_3 = 0), \tag{5}$$

$$\begin{aligned} g_{\alpha\beta} &= \boldsymbol{R}_\alpha \boldsymbol{R}_\beta = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 b_{\alpha\gamma} b_\beta^\gamma, \\ \hat{b}_{\alpha\beta} &= (1 - 2Hx_3) b_{\alpha\beta} + x_3 K a_{\alpha\beta}, \quad \hat{S}\ (x_3 = const). \end{aligned}$$

The unit vectors of the tangent $\hat{\boldsymbol{s}}$ and tangential normal $\hat{\boldsymbol{l}}$ are expressed by the following formulas:

$$\hat{\boldsymbol{s}} = [(1 - x_3 k_s)\boldsymbol{s} - x_3 \tau_s \boldsymbol{l}] \frac{ds}{d\hat{s}}, \quad \hat{\boldsymbol{l}} = [(1 - x_3 k_s)\boldsymbol{l} - x_3 \tau_s \boldsymbol{s}] \frac{ds}{d\hat{s}}, \tag{6}$$

$$d\hat{s} = \sqrt{1 - 2x_3 k_s + x_3^2 (k_s^2 + \tau_s^2)}\, ds,$$

where $\boldsymbol{s}$ and $\boldsymbol{l}$ are the the tangent and tangential normal on the midsurface S, ds and $d\hat{s}$ are the linear elements of the surfaces S and $\hat{S}$, and τ_s is the geodesic torsion of the surface S.

Under shallow shells we mean 3D shell-type elastic bodies satisfying the conditions

$$a_\alpha^\beta - x_3 b_\alpha^\beta \cong a_\alpha^\beta, \;\; \Rightarrow \;\; \boldsymbol{R}_\alpha \cong \boldsymbol{r}_\alpha, \;\; \boldsymbol{R}^\alpha \cong \boldsymbol{r}^\alpha, \;\; g_{\alpha\beta} \cong a_{\alpha\beta}, \;\; \hat{b}_{\alpha\beta} \cong b_{\alpha\beta}, \tag{7}$$

i.e., in this case the interior geometry of the shell does not vary in thickness and therefore such kind of shells are usually called the shells with non-varying geometry.

For non-shallow shells in the case of Koiter-Mindlin's theory we have

$$\begin{aligned} &\boldsymbol{R}_\alpha = (a_\alpha^\beta - x_3 b_\alpha^\beta)\boldsymbol{r}_\beta, \;\; \boldsymbol{R}^\alpha = (a_\alpha^\beta + x_3 b_\alpha^\beta)\boldsymbol{r}^\beta, \;\; \boldsymbol{R}_3 = \boldsymbol{n}, \;\; \Rightarrow \\ &g_{\alpha\beta} \cong a_{\alpha\beta} - 2x_3 b_{\alpha\beta}, \quad g^{\alpha\beta} \cong a^{\alpha\beta} + 2x_3 b^{\alpha\beta}, \end{aligned} \tag{8}$$

i.e., in this case only the linear part with respect to x_3 is retained.

In the sequel, by non-shallow shells we mean 3D shell-type elastic bodies satisfying the relations (3), (4), (5), (6).

To reduce the 3D problems of the theory of elasticity to 2D ones, it is necessary to rewrite the relations (3)-(6) in forms of the bases of the midsurface S $(x_3 = 0)$.

The relation (1) can be written as [2]:

$$\nabla_\alpha (\vartheta \boldsymbol{\sigma}^\alpha) + \partial_3 (\vartheta \boldsymbol{\sigma}^3) + \vartheta \boldsymbol{\Phi} = 0, \tag{9}$$

$$\boldsymbol{\sigma}^i = \frac{1}{2} M^{i_1 j_1 p_1 q_1} A^i_{i_1} (A^p_{p_1} \boldsymbol{r}_{q_1} \partial_p \boldsymbol{U} + A^q_{q_1} \boldsymbol{r}_{p_1} \partial_q \boldsymbol{U} + A^p_{p_1} A^q_{q_1} \partial_p \boldsymbol{U} \partial_q \boldsymbol{U}) \boldsymbol{r}_{j_1}, \tag{10}$$

where ∇_α are covariant derivatives on the midsurface S,

$$M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu (a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1}) \ \ (a^{ij} = \boldsymbol{r}^i \boldsymbol{r}^j). \tag{11}$$

2. In the present paper we use I. Vekua's reduction method for the nonlinear theory of non-shallow shells (I. Vekua used the method for linear theory of shallow shells) the essence of which consists, without going into detals, in the following: since the system of Legendre polynomials $P_n(\frac{x_3}{h})$ is complete in the interval $[-h, h]$, for equation (9) the equivalent infinite system of 2D equations is obtained as

$$\nabla_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha} - \frac{2m+1}{h} \left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^{3} + \overset{(m-3)}{\boldsymbol{\sigma}}{}^{3} + ... \right) + \overset{(m)}{\boldsymbol{F}} = 0, \tag{12}$$

where

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^{i}, \overset{(m)}{\boldsymbol{\Phi}} \right) = \frac{2m+1}{2h} \int\limits_{-h}^{h} \left(\vartheta \boldsymbol{\sigma}^i, \vartheta \boldsymbol{\Phi} \right) P_m \left(\tfrac{x_3}{h} \right) dx_3,$$

$$\overset{(m)}{\boldsymbol{F}} = \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h} \left(\overset{(+)}{\vartheta} \overset{(+)}{\boldsymbol{\sigma}}{}_3 - (-1)^m \overset{(-)}{\vartheta} \overset{(-)}{\boldsymbol{\sigma}}{}_3 \right) \quad \left(\overset{(\pm)}{\vartheta} = 1 \pm 2hH + Kh^2 \right).$$

Thus we have obtained the infinite system of 2D equations (12), for which the boundary conditions of the face surfaces ($x_3 = \pm h$) are satisfied, i.e. $\overset{(\pm)}{\boldsymbol{\sigma}}{}^3 = \boldsymbol{\sigma}^3(x^1, x^2, \pm h)$ is the preassigned vector field and is contained in the equilibrium equations.

The equations (10) may be written as:

$$\begin{aligned}
\overset{(m)}{\boldsymbol{\sigma}}{}^{i} &= \frac{2m+1}{2h} \int\limits_{-h}^{h} \vartheta \boldsymbol{\sigma}^i P_m \left(\frac{x_3}{h} \right) dx_3 \\
&= \tfrac{1}{2} M^{i_1 j_1 p_1 q_1} \Bigg[\sum_{m_1=0}^{\infty} \left(\underset{(m_1)}{\overset{(m)}{A}}{}^{i\,p}_{i_1 p_1} \boldsymbol{r}_{q_1} \cdot D_p \overset{(m_1)}{\boldsymbol{U}} + \underset{(m_1)}{\overset{(m)}{A}}{}^{i\,q}_{i_1 q_1} \boldsymbol{r}_{p_1} \cdot D_q \overset{(m_1)}{\boldsymbol{U}} \right) \boldsymbol{r}_{j_1} \cdot \\
&+ \sum_{m_1, m_2=0}^{\infty} \Bigg[\left(\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{i\,j\,p}_{i_1 j_1 p_1} \boldsymbol{r}_{q_1} \cdot D_p \overset{(m_1)}{\boldsymbol{U}} + \underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{i\,j\,q}_{i_1 j_1 q_1} \boldsymbol{r}_{p_1} \cdot D_q \overset{(m_1)}{\boldsymbol{U}} \right) D_j \overset{(m_2)}{\boldsymbol{U}} \\
&+ \underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{i\,p\,q}_{i_1 p_1 q_1} \left(D_p \overset{(m_1)}{\boldsymbol{U}} \cdot D_q \overset{(m_1)}{\boldsymbol{U}} \right) \boldsymbol{r}_{j_1} \Bigg] \\
&+ \sum_{m_1, m_2, m_3=0}^{\infty} \underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{i\,j\,p\,q}_{i_1 j_1 p_1 q_1} \left(D_p \overset{(m_1)}{\boldsymbol{U}} \cdot D_q \overset{(m_2)}{\boldsymbol{U}} \right) D_j \overset{(m_3)}{\boldsymbol{U}} \Bigg],
\end{aligned} \tag{13}$$

where

$$D_i \overset{(m)}{\boldsymbol{U}} = \delta^\beta_i \partial_\beta \overset{(m)}{\boldsymbol{U}} + \delta^3_i \overset{(m)}{\boldsymbol{U}}{}'; \quad \overset{(m)}{\boldsymbol{U}}{}' = \frac{2m+1}{h} \left(\overset{(m+1)}{\boldsymbol{U}} + \overset{(m+3)}{\boldsymbol{U}} + ... \right), \tag{14}$$

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{ij}_{i_1j_1} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}P_{m_1}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3,$$

$$\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ijp}_{i_1j_1p_1} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}A^p_{p_1}P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3, \quad (15)$$

$$\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ijpq}_{i_1j_1p_1q_1} = \frac{2m+1}{2h}$$

$$\times\int\limits_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}A^p_{p_1}A^q_{q_1}P_{m_1}\left(\frac{x_3}{h}\right)P_{m_2}\left(\frac{x_3}{h}\right)P_{m_3}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3.$$

The boundary conditions on the lateral contour take the form:

a) for the stresses

$$\overset{(m)}{\boldsymbol{\sigma}}_{(l)} = \overset{(m)}{\sigma}_{(ll)}\boldsymbol{l} + \overset{(m)}{\sigma}_{(ls)}s + \overset{(m)}{\sigma}_{(ln)}\boldsymbol{n} = \frac{2m+1}{2h}\int\limits_{h}^{h}\boldsymbol{\sigma}_{(\boldsymbol{l})}\frac{d\hat{s}}{ds}P_m\left(\frac{x_3}{h}\right)dx_3, \quad (16)$$

b) for the displacements

$$\overset{m}{\boldsymbol{U}} = \overset{(m)}{U}_{(l)}\boldsymbol{l} + \overset{(m)}{U}_{(s)}\boldsymbol{s} + \overset{(m)}{U}_{3}\boldsymbol{n} = \frac{2m+1}{2h}\int\limits_{h}^{h}\boldsymbol{U}P_m\left(\frac{x_3}{h}\right)dx_3. \quad (17)$$

The passage to finite systems can be realized by various methods one of which consists in considering of a finite series, i.e.

$$(\vartheta\boldsymbol{\sigma}^i, \boldsymbol{U}, \vartheta\boldsymbol{\Phi}) = \sum_{m=0}^{N}\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\boldsymbol{U}}, \overset{(m)}{\Phi}\right)P_m\left(\frac{x_3}{h}\right),$$

where N is a fixed nonnegative number. In other words, it is assumed that

$$\overset{(m)}{\boldsymbol{U}} = 0, \quad \overset{(m)}{\boldsymbol{\sigma}}{}^i = 0 \quad \text{if} \quad m > N.$$

This approximation will be called the Nth order approximation.

The integrals of type (15) can be calculated; for example,

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{\alpha\beta}_{\alpha_1\beta_1} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\vartheta^{-1}B^{\alpha}_{\alpha_1}(x_3)B^{\beta}_{\beta_1}(x_3)P_{m_1}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3$$

$$
= \begin{cases} \frac{2m+1}{2\sqrt{E}h}\left[B^{\alpha}_{\alpha_1}(hy)B^{\beta}_{\beta_1}(hy)\begin{pmatrix} P_{m_1}(y)Q_m(y), \ m_1 \leq m \\ Q_{m_1}(y)P_m(y), \ m_1 \leq m \end{pmatrix}\right]^{y_2}_{y_1} + \frac{L^{\alpha}_{\alpha_1}L^{\beta}_{\beta_1}}{K}\sigma^m_{m_1} \\ \qquad \text{for} \ \ E \neq 0, \ K \neq 0, \\ \\ a^{\alpha}_{11}a^{\beta}_{\beta_1}\delta^m_{m_1} \qquad \text{for} \quad E = H^2 - K, \end{cases} \tag{18}
$$

where $Q_m(y)$ is the Legendre function of the second kind, E is the Euler difference, $B^{\alpha}_{\beta}(x) = B^{\alpha}_{\beta} = a^{\alpha}_{\beta} + xL^{\alpha}_{\beta}, \ L^{\alpha}_{\beta} = b^{\alpha}_{\beta} - 2Ha^{\alpha}_{\beta}$. Under the square brackets we mean the following:

$$
[f(y)]^{y_2}_{y_1} = f(y_2) - f(y_1), \quad y_{1,2} = [(H \mp \sqrt{E})h]^{-1}.
$$

Note that for Koiter-Naghdi's non-shallow shells the following expression

$$
\overset{(m)}{\underset{(m_1)}{A}}{}^{\alpha\beta}_{\alpha_1\beta_1} \cong a^{\alpha}_{\alpha_1}a^{\beta}_{\beta_1}\delta^m_{m_1} + h(a^{\alpha}_{\alpha_1}b^{\beta}_{\beta_1} + a^{\beta}_{\beta_1}b^{\alpha}_{\alpha_1})\Big(\frac{m}{2m-1}\delta^m_{m_1-1} + \frac{m+1}{2m+3}\delta^m_{m_1+1}\Big) \tag{19}
$$

is obtained.

For the integrals containing the product of three Legendre polynomials $P_m = P_m(\frac{x_3}{h})$ we have

$$
\overset{(m)}{\underset{(m_1,m_2)}{A}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \frac{2m+1}{2n}\int\limits_{-h}^{h} \vartheta^{-2}B^{\alpha_1}_{\beta_1}B^{\alpha}_{\beta_2}B^{\alpha}_{\beta_3}P_{m_1}P_{m_2}P_m dx_3 = \frac{2m+1}{K^2h^4}
$$

$$
\times \sum_{r=0}^{min(m_1,m_2)} \gamma_{m_1m_2r}\sum_{n=0}^{3}\overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}h^n\frac{\partial^2}{\partial y_1\partial y_2}\left[\frac{y^n}{y_1-y_2}\begin{pmatrix} P_s(y)Q_m(y), s \leq m \\ Q_s(y)P_m(y), s \geq m \end{pmatrix}\right]^{y_2}_{y_1},
$$

where $s = m_1 + m_2 - 2r$,

$$
\gamma_{pqr} = \frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}}\frac{2(p+q)-4r+1}{2(p+q)-2r+1}, \quad A_p = \frac{1\cdot 3 \cdots 2p-1}{p!},
$$

and $\overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}$ is defined from the relation (see [5])

$$
B^{\alpha_1}_{\beta_1}(x)B^{\alpha_2}_{\beta_2}(x)B^{\alpha_3}_{\beta_3}(x) = \sum_{n=0}^{3}\overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}x^n,
$$

For the integrals containing the product of four Legendre polynomials the corresponding representations can be written similarly.

3. Now we consider various refined theories of plates and the Kirsch's problem for the concentration of stresses near the hole.

The system of Reissner-Mindlin's equations for tension-pressure coincides with the classical theory of generalized plane stress.

For bending of plates the system of Reissner-Mindlin's equations can be written in the complex form [3]:

$$\begin{aligned} &\partial_z(M_{11}-M_{22}+2iM_{12})+\partial_{\bar z}(M_{11}+M_{22})-Q_+=M_+,\\ &\partial_z Q_+ +\partial_{\bar z}\bar Q_+ = M_3, \quad (Q_+=Q_1+iQ_2), \quad (2\partial_z=\partial_1-i\partial_2), \end{aligned} \tag{20}$$

where

$$\begin{aligned} &M_{11}-M_{22}+2iM_{12}=\tfrac{8\mu h^3}{3}\partial_{\bar z}V_+ \quad (V_+=V_1+iV_2),\\ &M_{11}+M_{22}=\tfrac{4(\lambda^*+\mu)}{3}h^3\rho \quad (\rho=2Re\partial_z V_+),\\ &Q_+=\tfrac{5\mu h}{3}(2\partial_{\bar z}V_3+V_+) \quad \text{(Reissner)},\\ &Q_+=\tfrac{4\mu h}{3}(2\partial_{\bar z}V_3+V_+) \quad \text{(Mindlin)}, \end{aligned} \tag{21}$$

$$\begin{aligned} &\mu\Delta V_+ +2(\lambda^*+\mu)\partial_{\bar z}\rho-\tfrac{5\mu}{h^2}(2\partial_{\bar z}V_3+V_+)=\tfrac{3}{2h^3}M_+,\\ &\mu(\Delta V_3+\rho)=\tfrac{3}{5h}M_3, \end{aligned} \tag{22}$$

$$\lambda^* := \frac{2\lambda\mu}{\lambda+2\mu}.$$

The boundary conditions for Kirsch's problem on the hole's contour Γ have the form

$$M_{ll}+iM_{ls}=0, \quad Q_{ln}=0 \tag{23}$$

and at infinity we have

$$M_{11}^\infty=M_1, \quad (M_{12}=M_{22}=Q_+)^\infty=0. \tag{24}$$

Now we consider this problem by I. Vekua's methods.

I. Vekua's first method (so-called "simplified scheme")

$$\begin{aligned} &\partial_z\left(\overset{(m)}{\sigma}_{11}-\overset{(m)}{\sigma}_{22}+2i\overset{(m)}{\sigma}_{12}\right)+\partial_{\bar z}\left(\overset{(m)}{\sigma}_{11}+\overset{(m)}{\sigma}_{22}\right)-\overset{(m)}{\underline{\sigma}}_{+}+\overset{(m)}{F}_{+}=0,\\ &\overset{(m)}{\partial_z\sigma_+}+\partial_{\bar z}\overset{(m)}{\bar\sigma}_+-\overset{(m)}{\underline{\sigma}}_{33}+\overset{(m)}{F}_3=0 \quad \left(\overset{(m)}{\sigma}_+=\overset{(m)}{\sigma}_{13}+i\overset{(m)}{\sigma}_{23}\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned} &\overset{(m)}{\sigma}_{11}-\overset{(m)}{\sigma}_{22}+2i\overset{(m)}{\sigma}_{12}=4\mu\partial_{\bar z}\overset{(m)}{u}_+ \quad \left(\overset{(m)}{u}_+=\overset{(m)}{u}_1+i\overset{(m)}{u}_2\right),\\ &\overset{(m)}{\sigma}_{11}+\overset{(m)}{\sigma}_{22}=2(\lambda+\mu)\overset{(m)}{\theta}+2\lambda D_3\overset{(m)}{u}_3 \quad \left(\overset{(m)}{\theta}=2Re\partial_z\overset{(m)}{u}_+\right),\\ &\overset{(m)}{\sigma}_+=\mu\left(2\partial_{\bar z}\overset{(m)}{u}_3+D_3\overset{(m)}{u}_+\right),\\ &\overset{(m)}{\sigma}_{33}=\lambda\overset{(m)}{\theta}+(\lambda+2\mu)D_3\overset{(m)}{u}_3, \end{aligned} \tag{26}$$

$$\overset{(m)}{\underline{\sigma}}_3=\frac{2m+1}{h}\left(\overset{(m-1)}{\sigma}_3+\overset{(m-3)}{\sigma}_3+\cdots\right), \quad D_3\overset{(m)}{u}\frac{2m+1}{h}\sum_{s=m}^{N}\frac{1-(-1)^{s+m}}{2}\overset{(s)}{u},$$

$$\overset{(m)}{\boldsymbol{\sigma}}{}^{i} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\boldsymbol{\sigma}^{i}P_m\left(\tfrac{x_3}{h}\right)dx_3, \quad \overset{(m)}{\boldsymbol{F}} = \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h}\left(\overset{(+)}{\sigma}_3 - (-1)^m\overset{(-)}{\sigma}_3\right).$$

I. Vekua's second method (so-called "normed moments method"). In this case the expressions $\sigma_{i3}(i=1,2,3)$ are compatible with boundary condition on surface $x_3=\pm h$ (see [2]):

$$\begin{aligned}
&\partial_z\left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12}\right) + \partial_{\bar z}\left(\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22}\right) - \underline{\overset{(m)}{\sigma}}_{+} + \overset{(m)}{Y}_{+} = 0,\\
&\partial_z\overset{(m)}{\sigma}_{+} + \partial_{\bar z}\overset{(m)}{\bar\sigma}_{+} - \underline{\overset{(m)}{\sigma}}_{33} + \overset{(m)}{Y}_3 = 0,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
&\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12} = 4\mu\partial_{\bar z}\overset{(m)}{u}_{+},\\
&\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(m)}{\theta} + 2\lambda D_3\overset{(m)}{u}_3 -\\
&2\varepsilon_{N,m}\sum_{s=0}^{N}(1+(-1)^{s+m})\left(\tfrac{\lambda^2}{\lambda+2\mu}\overset{(s)}{\theta} + \lambda D_3\overset{(s)}{u}_3\right),\\
&\overset{(m)}{\sigma}_{+} = \mu\left(2\partial_{\bar z}\overset{(m)}{u}_3 + D_3\overset{(m)}{u}_{+} - \varepsilon_{N,m}\sum_{s=0}^{N}(1+(-1)^{s+m})\left(2\partial_{\bar z}\overset{(s)}{u}_3 + D_3\overset{(s)}{u}_{+}\right)\right),\\
&\overset{(m)}{\sigma}_{33} = \lambda\overset{(m)}{\theta} + (\lambda+2\mu)D_3\overset{(m)}{u}_3 - \varepsilon_{N,m}\\
&\qquad\times\sum_{s=0}^{N}(1+(-1)^{s+m})\left(\lambda\overset{(s)}{\theta} + (\lambda+2\mu)D_3\overset{(s)}{u}_3\right),\\
&\varepsilon_{N,m} = \frac{2m+1}{N(N+2)}\left(1 - \frac{(-1)^{N+m}}{N+1}\right).
\end{aligned} \tag{28}$$

The Kirsch's problem for these cases can be written as:

a) boundary conditions at infinity:

$$\overset{(0)}{\sigma}_{11}{}^{\infty} = P_1, \quad \overset{(0)}{\sigma}_{22}{}^{\infty} = P_2, \quad \left(\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{3i}\right)^{\infty} = 0 \quad (tension - pressure), \tag{29}$$

or

$$\overset{(1)}{\sigma}_{11}{}^{\infty} = M_1, \quad \overset{(1)}{\sigma}_{22}{}^{\infty} = M_2, \quad \left(\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{3i}\right)^{\infty} = 0 \quad (bending), \tag{30}$$

b) boundary conditions on the circular hole ($|z|=R$):

$$\overset{(m)}{\sigma}_{rr} + i\overset{(m)}{\sigma}_{r\vartheta} = 0, \quad \overset{(m)}{\sigma}_{r3} = 0 \quad (m=0,1,\cdots,N), \tag{31}$$

P_1, P_2, M_1, M_2 are constants.

For the approximation of order N the system of equilibrium equations with respect to components displacement vector $\overset{(m)}{u}_i$ can be written in the matrix forms:

$$\Delta V + AV = X, \tag{32}$$

$$\Delta\Omega + B\Omega = Y, \tag{33}$$

where V and Ω are column-matrices of the form

$$V = \big(\overset{(0)}{V_1}, \overset{(1)}{V_1}, ..., \overset{(N)}{V_1}, \overset{(0)}{V_3}, \overset{(1)}{V_3}, ..., \overset{(N)}{V_3}\big)^T, \qquad \Omega = \big(\overset{(0)}{V_2}, \overset{(1)}{V_2}, ..., \overset{(N)}{V_2}\big)^T,$$

$$\left(\overset{(m)}{u}_+ = \overset{(m)}{u}_1 + i\overset{(m)}{u}_2 = \partial_{\bar{z}}\overset{(m)}{V}_+ = \partial_{\bar{z}}\left(\overset{(m)}{V}_1 + i\overset{(m)}{V}_2\right)\right), \quad \overset{(m)}{u}_3 = \overset{(m)}{V}_3\right).$$

Using now Vekua-Bitsadze's formulas for the homogenous matrix equations (32) and (33) we obtain the complex representation of the general solutions

$$V = Re\left[f(z) + \int_{z_0}^{z} R(z, \bar{z}, t, \bar{t}) f(t) dt\right], \tag{34}$$

$$\Omega = Re\left[\varphi(z) + \int_{z_0}^{z} r(z, \bar{z}, t, \bar{t}) \varphi(t) dt\right], \tag{35}$$

where R and r are the Riemann matrix functions of the equations (32) and (33), $f(z)$ and $\varphi(z)$ are holomorphic column-matrices. R and r can be represented by Bessel's functions of the first kind:

$$f(z) = \Big(f_o(z), f_1(z), \cdots, f_{2N+1}(z)\Big)^T, \quad \varphi(z) = \Big(\varphi_0(z), \varphi_1(z), \cdots, \varphi_N(z)\Big)^T.$$

The particular solutions of the matrix equations (32) and (33) have the form

$$\hat{V}(z, \bar{z}) = \int_{z_0}^{z}\int_{\bar{z}_0}^{\bar{z}} \hat{R}(z, \bar{z}, t, \bar{t}) X(t, \bar{t}) dt d\bar{t}, \quad \overset{\wedge}{\Omega}(z, \bar{z}) = \int_{z_0}^{z}\int_{\bar{z}_0}^{\bar{z}} \hat{r}(z, \bar{z}, t, \bar{t}) Y(t, \bar{t}) dt d\bar{t}.$$

where $\hat{R}$ and $\hat{r}$ are matrix functions, which can be also expressed by the Bessel's functions of the second kind.

Conclusion. 1. a) I. Vekua's approximation of order $N = 0$ (first method) gives the system of plane deformation equations. The coefficient of stress concentration K, coincides with well-known meaning

$$K = \frac{max\, \overset{(0)}{\sigma}_{\vartheta\vartheta}}{P} = 3 \quad (P_1 = P,\ P_2 = 0).$$

b) I. Vekua's approximation of order $N = 0$ (second method) and Reissner's method describe the generalized plane stress, i.e. $K = 3$.

2. a) I. Vekua's approximation of order $N = 1$ (first method) for the tension-pressure gives for K the following formula

$$K = 1 + 2\frac{2\varkappa K_0(\varkappa) + [4 + 5(1 - \sigma^2)\varkappa^2]K_1(\varkappa)}{2(1 - \sigma^2)\varkappa K_0(\varkappa) + [4 + 5(1 - \sigma^2)\varkappa^2]K_1(\varkappa)},$$

where $\varkappa^2 = \frac{\sigma}{1-\sigma}\frac{R^2}{h^2}$, i.e. $K = K(h, R, \sigma)$ depends on h, R, σ (Poisson's coefficient), and when $\frac{h}{R} \to 0$ or $\frac{R}{h} \to \infty \Rightarrow K = 3$ (since for each n when $x \to \infty$ we have $K_n(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}$).

b) for the plate's bending I. Vekua's approximation $N = 1$ and Reissner's method give:
(I. Vekua's $N = 1$)

$$K = 1 + \frac{2K_2(\varkappa)}{K_2(\varkappa) + 2(1-\sigma)K_0(\varkappa)} \quad \Rightarrow \quad K = \frac{5-2\sigma}{3-2\sigma}\left(\frac{h}{R} \to 0, \varkappa = \frac{3R^2}{h^2}\right)$$

(E. Reissner)

$$K = 1 + \frac{2K_2(\varkappa)}{3K_2(\varkappa) + 2\sigma K_0(\varkappa)} \quad \Rightarrow \quad K_{cl} = \frac{5+2\sigma}{3+2\sigma}\left(\frac{h}{R} \to 0, \varkappa = \frac{5R^2}{2h^2}\right)$$

i.e. Reissner's coefficient K coincides with the classical result, when $\frac{h}{R} \to 0$.

3. a) I. Vekua's approximation of order $N = 2$ (first and second methods) for the tension-pressure solves the 3D problem, when $P_1 = P_2 = const$;

b) for bending of plate coincides with the Reissner result.

4. a) I. Vekua's approximation of order $N = 3$ solves the problem for the tension-pressure when $P_1 = const$, $P_2 = 0$ (II-method).

b)I. Vekua's approximation of order $N = 3$ (II-method) for bending of plate solves the problem when $M_1 = const$, $M_2 = 0$.

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