EXPLICIT SOLUTIONS OF THE BVPs OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY FOR THE HALF-SPACE

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Abstract. The purpose of this paper is to be explicitly solved the basic first and the second boundary value problems (BVPs) of the theory of consolidation with double porosity for the half-space. The obtained solutions are represented in quadratures.

Key words and phrases: Porous media, double porosity, half-space

AMS subject classification 2010: 74G25, 74G30

Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. When fluid flows and deformation processes occur simultaneously , three coupled partial differential equations can be derived [1],[2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid. Inertia effects are neglected as they are in Biot's theory.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. In part I of a series of paper on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In the part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [3] provided a related finite element formulation to consider the numerical solution of Aifantis' equations of double porosity (see [1],[2],[3] and references cited therein). The basic results and the historical information on the theory of porous media were summarized by de Boer [4].

The main goal of this investigation is to construct explicitly the solutions of the basic first and the second boundary value problems (BVPs) of the theory of consolidation with double porosity for the half-space.

1. Basic Equations, Boundary Value Problems

The basic Aifantis' equations of statics of the theory of consolidation with double porosity are given by the partial differential equations in the form [1], [2]

$$\mu\Delta u + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\, u - \mathrm{grad}\,(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1.1}$$

$$\begin{aligned} &(m_1\Delta - k)p_1 + kp_2 = 0,\\ &kp_1 + (m_2\Delta - k)p_2 = 0, \end{aligned} \tag{1.2}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores. λ is Lame' constant, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. $m_j = \dfrac{k_j}{\mu^*}, j = 1, 2.$ The constants k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, the constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities $\lambda,\ \mu,\ k,\ \beta_j,\ k_j\ (j = 1, 2)$ and μ^* are all positive constants. $\triangle$ is Laplace operator.

Let D denote the upper half-space $x_3 > 0$ and the boundary of D is S ($x_1 o x_2$ plane). Let us choose the unit normal $n(0, 0, 1)$.

Introduce the definition of a regular vector-function.

Definition 1. A vector-function $U(x) = (u_1, u_2, u_3, p_1, p_2)$ defined in the domain D is called regular if it has integrable continuous second derivatives in D, U itself and its first order derivatives are continuously extendable at every point of the boundary of D, i.e., $U \in C^2(D)\bigcap C^1(\overline{D})$, and the vector $U(x)$, with the components $u_i(x)$, $i = 1, ..., 5$, satisfies the following conditions at infinity:

$$U(x) = O(|x|^{-1}),\ \ \frac{\partial U_i}{\partial x_j} = O(|x|^{-2}),\ \ |x|^2 = x_1^2 + x_2^2 + x_3^2, \tag{1.3}$$

$$i = 1, ..., 5\ j = 1, 2, 3.$$

For the equation (1.1)-(1.2) we pose the following boundary value problems:

Find a regular vector U, satisfying in D equations (1.1)-(1.2), and on the boundary S one of the following conditions is given:

Problem I. The displacement vector and the fluid pressures are given in the form

$$u^+(z) = f(z)^+,\quad p_1^+(z) = f_4^+,\ \ p_2^+(z) = f_5^+(z),\quad z \in S,$$

where $f^+ \in C^{1,\alpha}(S)$, $f_i^+ \in C^{1,\alpha}(S)$, $0 < \alpha \leq 1$, $i = 4, 5$, are given functions.

Problem II. The stress vector and the normal derivatives of the pressure functions $\dfrac{\partial p_j}{\partial n}$ are given in the form

$$(Pu)^+ = f(z)^+, \quad \left(\frac{\partial p_1(z)}{\partial n}\right)^+ = f_4^+, \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^+ = f_5^+(z), \quad z \in S,$$

where $f^+ \in C^{1,\alpha}(S)$, $f_i^+ \in C^{1,\alpha}(S)$, $0 < \alpha \leq 1$, $i = 4, 5$, are given functions, Pu is a stress vector, which acts on an elements of the S with the normal $n = (0, 0, 1)$

$$P(\partial x, n)u = T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.4}$$

where [5]

$$\begin{aligned} &T(\partial x, n) = \| \, T_{kj}(\partial x, n) \, \|_{3\times 3}, \\ &T_{ij}(\partial x, n) = \mu\delta_{ij}\frac{\partial}{\partial n} + \lambda n_i \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_i}, \quad i, j, = 1, 2, 3. \end{aligned} \tag{1.5}$$

Further we assume that p_j is known, when $x \in D$. Substitute the $\beta_1 p_1 + \beta_2 p_2$ in (1.1) and search the particular solution of the following equation

$$\mu\Delta u + (\lambda + \mu)\mathrm{grad}\,\mathrm{div}\, u = \mathrm{grad}\,(\beta_1 p_1 + \beta_2 p_2).$$

We put

$$u_0 = -\frac{1}{4\pi} \int \int_D \int \Gamma(x - y)\mathrm{grad}\,(\beta_1 p_1 + \beta_2 p_2) dv, \tag{1.6}$$

where [5]

$$\Gamma(x - y) = \frac{1}{4\mu(\lambda + 2\mu)} \left\| \frac{(\lambda + 3\mu)\delta_{kj}}{r} + \frac{(\lambda + \mu)(x_k - y_k)(x_j - y_j)}{r^3} \right\|_{3\times 3},$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2.$$

Substituting u_0 into (1.1) we obtain

$$\mu\Delta u_0 + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u_0 = \mathrm{grad}(\beta_1 p_1 + \beta_2 p_2). \tag{1.7}$$

Thus we have proved that $u_0(x)$ is a particular solution of the equation (1.1). $\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2)$ is a continuous vector in D along with its first derivatives and the vector $\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2)$ has to satisfy the following condition at infinity

$$\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2) = O(|x|^{-1-\alpha}), \alpha > 0.$$

Thus the general solution of the equation (1.1) is $u = V + u_0$, where

$$A(\partial x)V = \mu\Delta V + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,V = 0. \tag{1.8}$$

This equation is the equation of an isotropic elastic body, i.e.., we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

2. Solution of the First Boundary Value Problem

A solution of the first BVP will be sought in the domain D in terms of the double layer potential

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{2\pi}\int\!\!\int_S \frac{\partial}{\partial x_3} M(x-y)\varphi(y)dy_1dy_2\ , \tag{2.1}$$

where

$$M(y-x) = \begin{pmatrix} m_2\dfrac{e^{-\lambda_0 r}}{r} - \dfrac{k}{\lambda_0^2}\dfrac{e^{-\lambda_0 r}-1}{r} & -\dfrac{k}{\lambda_0^2}\dfrac{e^{-\lambda_0 r}-1}{r} \\ -\dfrac{k}{\lambda_0^2}\dfrac{e^{-\lambda_0 r}-1}{r} & m_1\dfrac{e^{-\lambda_0 r}}{r} - \dfrac{k}{\lambda_0^2}\dfrac{e^{-\lambda_0 r}-1}{r} \end{pmatrix}, \tag{2.2}$$
$$\lambda_0^2 = \frac{k}{m_1} + \frac{k}{m_2},$$

φ is the unknown real vector function. To determine it, we obtain the integral equation

$$\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi(z) = \frac{1}{2\pi}\int\!\!\int_S \frac{\partial}{\partial x_3} M(y-z)\varphi(y)dy_1dy_2 = \begin{pmatrix} f_4^+(z) \\ f_5^+(z) \end{pmatrix}. \tag{2.3}$$

Takihg into account the fact that $\dfrac{\partial}{\partial x_3}M(y-z) = 0, \quad z_3 = 0$, from the latter equation we have

$$\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi(z) = \begin{pmatrix} f_4^+(z) \\ f_5^+(z) \end{pmatrix}, \tag{2.4}$$

and (2.1) takes the form

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{2\pi}\int\!\!\int_S \frac{\partial}{\partial x_3} M(x-y)\begin{pmatrix} m_2^{-1}f_4^+(z) \\ m_1^{-1}f_5^+(z) \end{pmatrix}dy_1dy_2\ . \tag{2.5}$$

The solution of the equation (1.8)

$$\mu\Delta V + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,V = 0$$

when $V^{\pm} = F^{+} = f^{+} - u_0^{+}$ is given in the form [5]

$$V(x) = \int_S [N(\partial y, n)\Gamma(y-x)]^T F^{+}(y)dy_1dy_2, \quad x \in D, \quad y \in S, \tag{2.6}$$

where [5]

$$[N(\partial y, n)\Gamma(x-y)]^T_{kj} = \frac{\partial}{\partial n}\frac{\delta_{kj}}{r} + \sum_{k=1}^{3} M_{kj}(\partial y, n)\left[\frac{(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{(\lambda+3\mu)r^3}\right],$$

$$M_{kj} = n_j\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial n} = \frac{\partial}{\partial x_3}.$$

We regard the formula (2.6) as an analogue of Poisson's formula in the theory of consolidation with double porosity for the solution of the first BVP for the half-space.

3. Solution of the Second Boundary Value Problem

A solution of the second BVP is sought in the domain D in terms of the simple layer potential

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{2\pi}\int\int_S M(x-y)\varphi(y)dy_1dy_2 \ , \tag{3.1}$$

where $M(x-y)$ is given by (2.2), φ is an unknown real function. To determine it, we obtain the integral equation

$$-\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi(z) = \frac{1}{2\pi}\int\int_S \frac{\partial}{\partial x_3}M(y-z)\varphi(y)dy_1dy_2 = \begin{pmatrix} f_4^{+}(z) \\ f_5^{+}(z) \end{pmatrix}. \tag{3.2}$$

Taking into account the fact that $\dfrac{\partial}{\partial x_3}M(y-z) = 0, \quad z_3 = 0$, from the equation (3.2) we have

$$-\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi(z) = \begin{pmatrix} f_4^{+}(z) \\ f_5^{+}(z) \end{pmatrix}, \tag{3.3}$$

and (2.1) takes the form

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = -\frac{1}{2\pi}\int\int_S M(x-y)\begin{pmatrix} m_2^{-1}f_4^{+}(z) \\ m_1^{-1}f_5^{+}(z) \end{pmatrix}dy_1dy_2 \ . \tag{3.4}$$

The solution of the equation (1.8)

$$\mu\Delta V + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,V = 0$$

(when $[T(\partial x,n)V]^+=F$) is sought in the form

$$V(x)=\frac{1}{2\pi}\int\!\!\int_S[\Gamma(y-x)-\frac{1}{2(\lambda+2\mu)}H(x-y)]g(y)dy_1dy_2,\quad x\in D, \tag{3.5}$$

where

$$\Gamma(x-y)=\frac{1}{2\mu(\lambda+2\mu)}\left\|\frac{(\lambda+3\mu)\delta_{kj}}{r}+\frac{(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{r^3}\right\|_{3\times3},$$

$$r^2=(x_1-y_1)^2+(x_2-y_2)^2+x_3^2,$$

$$H(y-x)=\begin{pmatrix}\frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1\partial x_2} & \frac{\partial^2}{\partial x_1\partial x_3}\\ \frac{\partial^2}{\partial x_1\partial x_2} & \frac{\partial^2}{\partial x_2^2} & \frac{\partial^2}{\partial x_2\partial x_3}\\ -\frac{\partial^2}{\partial x_1\partial x_3} & -\frac{\partial^2}{\partial x_2\partial x_3} & -\frac{\partial}{\partial x_3^2}\end{pmatrix}\Phi(x,y), \tag{3.6}$$

$$\Phi(x,y)=x_3\ln(r+x_3)-x_3.$$

From (3.5) for the stress vector we obtain

$$T(\partial x,n)V(x)=\frac{1}{2\pi}\int\!\!\int_S G(y-x)g(y)dy_1dy_2,\quad x\in D,\quad y\in S, \tag{3.7}$$

where

$$G(y-x)=3(\lambda+\mu)$$

$$\times\begin{pmatrix}\frac{(x_1-y_1)^2x_3}{r^5} & \frac{(x_1-y_1)(x_2-y_2)x_3}{r^5} & \frac{(x_1-y_1)x_3^2}{r^5}\\ \frac{(x_1-y_1)(x_2-y_2)x_3}{r^5} & \frac{(x_2-y_2)^2x_3}{r^5} & \frac{(x_2-y_2)x_3^2}{r^5}\\ \frac{(x_1-y_1)x_3^2}{r^5} & \frac{(x_2-y_2)x_3^2}{r^5} & \frac{(x_3-y_3)x_3}{r^5}\end{pmatrix}. \tag{3.8}$$

For determining g we have integral equation

$$(\lambda+\mu)g(z)+\frac{1}{2\pi}\int\!\!\int_S G(z-y)g(y)dy_1dy_2=F(z),\quad z\in S, \tag{3.9}$$

Note that $G(z-y)=0,\ \ z\in S$. Then $(\lambda+\mu)g(z)=F(z)$, and owing to (3.5) we have

$$V(x)=\frac{1}{2\pi(\lambda+\mu)}\int\!\!\int_S[\Gamma(y-x)-\frac{1}{2(\lambda+2\mu)}H(x-y)]F(y)dy_1dy_2,\qquad x\in D,\quad y\in S, \tag{3.10}$$

The stress vector can be calculated from (3.7).

The formula (3.10) is an analogou of the Poisson's formula for the solution of the second BVP for the half-space.

Acknowledgement

The designated project has been completed by financial support of Georgia National Science Foundation (Grant GNSF/ST 08/3-388). Any idea in this publication is possessed by the authors and may not represent the opinion of Georgian National Science Foundation itself.

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Received 11.05.2010; revised 13.09.2010; accepted 21.10.2010