# LIMIT THEOREMS FOR DISCOUNTED SUMS OF RANDOM VECTORS WITH A VARYING DISCOUNT MATRIX

*Irina Gasviani*[1], *Tengiz Shervashidze*[2,3,1]

[1] *Faculty of Informatics and Control Systems, Georgian Technical University*
[2] *A. Razmadze Mathematical Institute*
[3] *I. Vekua Institute of Applied Mathematics, I. Javakhishvili Tbilisi State University*

**Abstract**. Weak convergence to normality is shown for normalized discounted sum of a sequence of i.i.d. random vectors from $R^m, m \geq 1$, for the case where discount matrix is chosen among a finite number of ones from the certain class of $m \times m$ matrices, at random according to a realization of another sequence of random variables. When $m = 1$ and discount factor varies periodically a rate of convergence is considered.

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We deal here with limit theorems which may be applicable in actuarial and financial modelling.

**1.** Consider a sequence $X_0, X_1, \ldots$ of i.i.d. $m$-dimensional random vectors on a probability space $(\Omega, F, P)$ such that $EX_0 = 0$, $E\|X_0\|^2 = \sigma^2 < \infty$, $\mathrm{cov}(X_0) = R$.

When $0 < A < 1$ and $m = 1$ the Abel sum

$$\eta_A := \sum_{j=0}^{\infty} A^j X_j \tag{1}$$

may be interpreted as the *present value* of the consecutive payments $X_0, X_1, \ldots$ with the *discount factor* $A$ and (1) is often referred to as a *discounted sum.*

Gerber proved in [2] that the probability distribution of the normalized sum

$$\zeta_A := (1 - A^2)^{1/2} \eta_A \tag{2}$$

converges weakly to the normal distribution $N(0, \sigma^2)$ when $A \to 1^-$. See the sources referred to in [6] to have a brief view on the history and some recent advances in investigation of discounted sums (1). We refer here only to [7] as a source in which an approach via partial sums of (1) is emphasized.

In the case $m > 1$ and an $m \times m$-matrix valued $A$ which we call a *discount matrix* the random vector $\eta_A$ is defined still by (1), where now $A^0 = I$, $A^{j+1} = AA^j$, $j \geq 1$, and $I$ stands for the identity $m \times m$-matrix, may be interpreted as vector of different payments or have a lot of other interpretations.

In [6] for a fixed symmetric and positive definite $m \times m$ matrix $R$ and $c, 1 \le c < \infty$, the set of nonsingular $m \times m$ matrices

$$\mathbf{A}(R,c) := \{A : \|A\| < 1,\ A = A^\top,\ AR = RA,\ \|I - A\| \le c(1 - \|A\|)\} \quad (3)$$

is introduced and for the normalized sum

$$\zeta_A := (I - A^2)^{1/2}\eta_A \quad (4)$$

the following assertion is proved.

**Theorem A** *If for fixed $R$ and $c$, $1 \le c < \infty$, $A$ takes its values in the set of $m \times m$-matrices $\mathbf{A}(R,c)$ and $A \to I$ in the sense that $\|I - A\| \to 0$, then the probability distribution of the normalized sum $\zeta_A$ introduced by (4) converges weakly to the normal distribution $N(0,R)$.*

This theorem covers the case of positive scalar matrices $A = aI$ (with $0 < a < 1$ and $c = 1$) and diagonal ones with at least two different diagonal elements both tending to 1 from the left.

When several discount matrices are chosen periodically (e.g., when monthly discount factors depend on seasons), we have the following assertion for normalized sums (6) below(emphasizing by (7) the scalar normalization in most transparent case of the above-mentioned scalar matrices; see [3]).

**Theorem B** *If $s \ge 1$ and $B_i \in \mathbf{A}(R,c)$, the set defined by (3), $B_i \to I$ in the sense that $\|I - B_i\| \to 0$, $i = 1, \dots, s$, then for the discounted sum*

$$\eta_B := \eta_{B_1,\dots,B_s} := \sum_{j=0}^{\infty} A_j^j X_j,\ A_j = B_i,\ j \equiv (i-1) \mod s,\ i = 1, \dots, s, \quad (5)$$

*the probability distribution of the normalized sum*

$$\zeta_B := \zeta_{B_1,\dots,B_s} := \Big[\sum_{i=1}^{s} B_i^{2(i-1)}(I - B_i^{2s})^{-1}\Big]^{-1/2}\eta_B \quad (6)$$

*converges weakly to the normal distribution $N(0,R)$.*

*In the special case of scalar matrices $B_i = b_i I$, $b_i \to 1^-$, $i = 1, \dots, s$, and $\eta_b := \eta_{b_1,\dots,b_s}$ given by (5), the assertion holds for*

$$\zeta_b := \zeta_{b_1,\dots,b_s} := \Big[\sum_{i=1}^{s} b_i^{2(i-1)}(1 - b_i^{2s})^{-1}\Big]^{-1/2}\eta_b. \quad (7)$$

If periodicity of the discount matrices is the same as in Theorem B but powers are assigned to all $B_i$s as to $B_1$ in $\eta_B$, we obtain the following assertion, where $[\cdot]$ stands for the integer part of a real number.

**Theorem B***. *If $s \geq 1$ and $B_i \in \mathbf{A}(R,c)$, the set defined by* (3)*, $B_i \to I$ in the sense that $\|I - B_i\| \to 0$, $i = 1, \dots, s$, then for the discounted sum*

$$\eta_B^* := \eta_{B_1,\dots,B_s}^* := \sum_{j=0}^{\infty} A_j^{[j/s]} X_j, \qquad A_j = B_i, \ \ j \equiv (i-1) \mod s, \ i = 1, \dots, s, \tag{5*}$$

*the probability distribution of the normalized sum*

$$\zeta_B^* := \zeta_{B_1,\dots,B_s}^* := \Big[\sum_{i=1}^{s} (I - B_i^{2s})^{-1}\Big]^{-1/2} \eta_B^* \tag{6*}$$

*converges weakly to the normal distribution $N(0,R)$.*

*In the special case of scalar matrices $B_i = b_i I$, $b_i \to 1^-$, $i = 1, \dots, s$,and $\eta_b^* := \eta_{b_1,\dots,b_s}^*$ given by* (5*)*, the assertion holds for*

$$\zeta_b^* = \zeta_{b_1,\dots,b_s}^* = \Big[\sum_{i=1}^{s} (1 - b_i^{2s})^{-1}\Big]^{-1/2} \eta_b^*. \tag{7*}$$

Though an analog of Berry–Esseen's bound for a discounted sum of i.i.d. random variables due to Gerber [2] could be extended to the case of periodically varying discount factor, using a result by Paditz [4], we try to give a short derivation of an estimate of the rate of convergence directly from [2].

Let $G$ be a random variable with standard normal distribution $N(0,1)$ and corresponding distribution function $N(u)$, $u \in \mathbf{R}$. For finite $\rho = E|X_0|^3$ Gerber proved that for Kolmogorov distance between $\sigma^{-1}\zeta_A$ defined by (2) for $m = 1$ and $G$ the following inequality holds

$$d(\sigma^{-1}\zeta_A, G) = \sup_{u \in \mathbf{R}} |P\{\sigma^{-1}\zeta_A < u\} - N(u)| < c\,\frac{\rho}{\sigma^3}(1-A)^{1/2} \tag{8}$$

with an absolute constant $c$.

We have the following representation of discounted sum with periodically variable discount factor (cf. (5)) as a weighted sum of $s$ independent discounted sums with a constant discount factor

$$\eta_b = \eta_{b_1,\dots,b_s} =: \sum_{i=1}^{s} b_i^{i-1} \eta(i)_{b_i^s}, \ \ \eta(i)_A := \sum_{k=0}^{\infty} A^k X_{sk+i-1}, \ \ i = 1, \dots, s. \tag{9}$$

For two discount factors (9) leads to

$$\sigma^{-1}\zeta_{b_1,b_2} = \frac{\eta_{b_1,b_2}}{\sqrt{D(\eta_{b_1,b_2})}}$$
$$= \sigma^{-1}[(1-b_1^4)^{-1} + b_2^2(1-b_2^4)^{-1}]^{-1/2}\{\eta(1)_{b_1^2} + b_2\eta(2)_{b_2^2}\}$$

$$= \sigma^{-1}[\gamma_1(1-b_1^4)^{1/2}\eta(1)_{b_1^2} + \gamma_2(1-b_2^4)^{1/2}\eta(2)_{b_2^2}], \qquad (10)$$

where positive $\gamma_1$ and $\gamma_2$ are such that $\gamma_1^2 + \gamma_2^2 = 1$. There exist independent standard normal random variables $G_1$ and $G_2$ such that $G = \gamma_1 G_1 + \gamma_2 G_2$ (in distribution).

Using the standard properties of Kolmogorov distance, from (8),(9) and (10) we obtain

$$\begin{aligned} d(\sigma^{-1}\zeta_{b_1,b_2}, G) &= \sup_{u\in\mathbf{R}} |P\{\sigma^{-1}\zeta_{b_1,b_2} < u\} - N(u)| \\ &< c\frac{\rho}{\sigma^3}[(1-b_1^2)^{1/2} + (1-b_2^2)^{1/2}]. \end{aligned}$$

Similar argument for $s > 2$ leads to the estimate

$$\begin{aligned} d(\sigma^{-1}\zeta_{b_1,\dots,b_s}, G) &= \sup_{u\in\mathbf{R}} |P\{\sigma^{-1}\zeta_{b_1,\dots,b_s} < u\} - N(u)| \\ &< c\frac{\rho}{\sigma^3}[(1-b_1^s)^{1/2} + \cdots + (1-b_s^s)^{1/2}] \\ &< c\sqrt{s}\frac{\rho}{\sigma^3}[(1-b_1)^{1/2} + \cdots + (1-b_s)^{1/2}]. \end{aligned}$$

It is evident that the same expressions involving square roots can serve as estimates of the convergence rate to normality for $\zeta^*_{b_1,\dots,b_s}$ from (7*) with

$$\eta^*_b := \eta^*_{b_1,\dots,b_s} =: \sum_{i=1}^{s} \eta(i)_{b_i^s} \qquad (9^*)$$

and $\eta(i)_A$ for $m = 1$ from (9):

$$\begin{aligned} d(\sigma^{-1}\zeta^*_{b_1,\dots,b_s}, G) &= \sup_{u\in\mathbf{R}} |P\{\sigma^{-1}\zeta^*_{b_1,\dots,b_s} < u\} - N(u)| \\ &< c\frac{\rho}{\sigma^3}[(1-b_1^s)^{1/2} + \cdots + (1-b_s^s)^{1/2}] \\ &< c\sqrt{s}\frac{\rho}{\sigma^3}[(1-b_1)^{1/2} + \cdots + (1-b_s)^{1/2}]. \end{aligned}$$

**2.** Let us consider a stationary two-component sequence $(\xi_j, X_j)$, $j = 0, 1, \dots$, where $\xi_j$ takes its values in $\{1, \dots, s\}$ and $X_j \in R^m$; denote

$$\xi = (\xi_0, \xi_1, \dots), \quad \xi_{0n} = (\xi_0, \dots, \xi_n),$$
$$X = (X_0, X_1, \dots), \quad X_{0n} = (X_0, \dots, X_n).$$

**Definition**. $X$ is a sequence of conditionally independent random vectors controlled by a sequence $\xi$ if for any natural $n$ the conditional distribution $P_{X_{0n}|\xi_{0n}}$ of $X_{0n}$ given $\xi_{0n}$ is the direct product of conditional distributions of $X_j$ given only the corresponding $\xi_j$, $j = 0, \dots, n$, i.e.,

$$P_{X_{0n}|\xi_{0n}} = \mathcal{P}_{\xi_0} \times \cdots \times \mathcal{P}_{\xi_n},$$

where $\mathcal{P}_i$ is the conditional distribution of $X_0$ given $\{\xi_0 = i\}$, $i = 1, \dots, s$ (see, e.g., [1],[5]).

Let $\pi_i = P\{\xi_0 = i\}$, $i = 1, \dots, s$, be a common distribution of $\xi_j$s.

For $s = 1$ $X$ becomes a sequence of i.i.d. random variables with $\mathcal{P}_1$ as a common distribution.

An equivalent definition of the conditionally independent random vectors $X$ controlled by the sequence $\xi$ is independence of components of $X$ given any realization of $\xi$ and equality $P_{X_j|\xi} = \mathcal{P}_{\xi_j}$ for any $j = 0, 1, \dots$.

Before formulation of the main result of this note let us introduce some extra notation and give an explanation how the discounted sum is generated. We begin with $I_E$ for the indicator of an event $E$ and

$$\nu_{ji} := I_{(\xi_j = i)}, \quad \sum_{i=1}^{s} \nu_{ji} = 1, \quad i = 1, \dots s, \quad j = 0, 1, \dots .$$

Denote by $\nu_j(i) = \sum_{k=0}^{j} \nu_{ki}$ the frequency of the value $i$ within the first $j+1$ variables $\xi_k$, $i = 1, \dots, s$, $j = 0, 1, \dots$. For every $j$ we have

$$\sum_{i=1}^{s} \nu_j(i) = j + 1, \quad j = 0, 1, \dots . \tag{11}$$

While the $(j+1)$st matrix, say $T_j$, which transforms the $(j+1)$st vector $X_j$ is obtained from the previous one by the constant factor $A$ from the set of matrices $\mathbf{A}(R, c)$ when the choice is deterministic setting $T_0 = I$ for the initial vector (thus we have had for that case $T_j = A^j, j = 0, 1, \dots$), by the random choice according to the stationary control sequence $\xi$, when we choose a factor among the matrices $B_1, \dots, B_s$ belonging to the same set $\mathbf{A}(R, c)$ with the probability distribution $(\pi_1, \dots, \pi_s)$ and a resulting factor would be presented as follows

$$B_{\xi_j} = B_1^{\nu_{j1}} \cdots B_s^{\nu_{js}}, \quad j = 1, 2, \dots, \tag{12}$$

and as a sequence of transformations we obtain

$$T_0 = I, \;\; T_j = \Pi_{k=1}^{j} B_{\xi_k}, \;\; j = 1, 2, \dots . \tag{13}$$

Let us now consider the case of discounting organized similarly to the case considered in Theorem B*.

**Theorem 1** *Let a stationary sequence* $(X_j, \xi_j), j = 0, 1, \dots,$ *be such that* $X$ *is conditionally independent sequence of m-dimensional random vectors controlled by an ergodic sequence* $\xi = (\xi_0, \xi_1, \dots)$ *each member of which takes its values from the set* $\{1, \dots, s\}$ *and conditional distribution* $\mathcal{P}_i$ *has zero mean and covariance matrix* $R, i = 1, \dots, s,$. *Each moment* $j$ *when the control sequence takes the value* $i$ *and this proceeds* $\nu_j(i)$*th time the corresponding vector* $X_j$ *is to be transformed by the matrix* $B_i^{\nu_j(i)-1}$ *with* $B_i \in \mathbf{A}(R, c)$, $i = 1, \dots, s$ *(for* *each $j$ we have (11)). As $B_i \to I$ in the sense that $\|I - B_i\| \to 0$, $i = 1, \dots, s$, then given the control sequence $\xi$ the conditional probability distribution of the random vector*

$$\zeta^*_{B_1,\dots,B_s}(\xi) = \left[\sum_{i=1}^{s}(I - B_i^2)^{-1}\right]^{-1/2} \sum_{i=1}^{s}\sum_{j=0}^{\infty} I_{(\xi_j = i)} B_i^{\nu_j(i)-1} X_j \tag{14}$$

*converges almost surely to the normal distribution $N(0, R)$.*

**Sketch of the proof.** For $m = 1$, this assertion follows from [4]. As for general case, let us behave according to [7]. Due to ergodicity of the control sequence the relative frequency of the event $\{\xi_j = i\}$ among the first $n + 1$ values of $j$ being equal to $\nu_n(i)/(n+1)$ tends $P$-almost surely as $n \to \infty$ to $\pi_i > 0, i = 1, \dots, s$. This means, that if we restrict ourselves by the $(n+1)$th partial sum of (14) each subsum with the fixed $i$ has a number of items growing to infinity and thus as corresponding $X_j$s are independent, due to Theorem A we have asymptotic normality $N(0, R)$ for each subsum normalized by $(I - B_i^2)^{1/2}$, $i = 1, \dots, s$. As the subsums are conditionally independent due to [3], under the normalization given in (14) the total limiting law remains the same $N(0, R)$. As the limit distribution does not depend on $\xi$ by taking an expectation (see, e.g., [1]) one can easily pass to a limit relation for absolute distribution.

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**R e f e r e n c e s**

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