MAIN ARTICLES

THE NEUMANN PROBLEM FOR A DEGENERATE DIFFERENTIAL–OPERATOR EQUATION

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Abstract. We consider the Neumann problem for a degenerate differential–operator equation of higher order. We establish some embedding theorems in weighted Sobolev space W_α^m and show existence and uniqueness of the generalized solution of the Neumann problem. We also give a description of the domain of definition and of the spectrum for the corresponding operator.

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1. Introduction

In present paper we consider the Neumann problem for the operator equation

$$Pu \equiv (-1)^m D_t^m (t^\alpha D_t^m) u + t^\alpha A u = f, \tag{1}$$

where $t \in (0, b)$, $\alpha \geq 0$, $D_t \equiv d/dt$, $f \in L_{2,-\alpha}((0, b), H)$ and A is a linear operator in Hilbert space H and has a complete system $\{\varphi_k\}_{k=1}^\infty$ of the eigenfunctions, which form a Riesz's basis in H. Note that the operator A in general is an unbounded operator in H.

Our approach, similar to that used in [3], for the case $m = 1$ and in [11] for $m = 2$, is based on the consideration of the one-dimensional equation (1), i.e. when A is the operator of multiplication by a number a, $a \in \mathbb{C}$, $Au = au$ (see [4]).

In Section 2 we define the weighted Sobolev space W_α^m, describe the behavior of the functions from this space close to $t = 0$ (see [7], [8], [13]). We give the desciption of the domain of the definition $D(B)$ of the operator B and prove that for $1 - a \notin \sigma\mathbb{B}$ ($\sigma\mathbb{B}$ is the spectrum of the operator $\mathbb{B}$) the generalized solution of the Neumann problem for the one–dimensional equation (1) exists and is unique for every $f \in L_{2,-\alpha}$.

In Section 3 under some conditions on the spectrum of the operator A we prove unique solvability of the operator equation (1) for every $f \in L_{2,-\alpha}((0, b), H)$ and give the description of the spectrum for the corresponding operator $\mathbb{P} = t^{-\alpha} P$.

2. The One-dimensional Case

2.1. The space W_α^m

Denote by W_α^m the completion of $C^m[0,b]$ in the norm

$$||u||^2_{W^m_\alpha} = \int_0^b \left(t^\alpha \, |u^{(m)}(t)|^2 + t^\alpha |u(t)|^2\right) \, dt. \tag{2}$$

For the proofs of the Propositions 1, 2 and Remark 3 see [2] and [13].

Proposition 1 *For every $u \in W_\alpha^m$ close to $t=0$ we have*

$$|u^{(j)}(t)|^2 \leq (B_j + C_j t^{2m-2j-1-\alpha})||u||^2_{W^m_\alpha}, \tag{3}$$

where $\alpha \neq 1, 3, \ldots, 2m-1$, $j = 0, 1, \ldots, m-1$. For $\alpha = 2n+1$, $n = 0, 1, \ldots, m-1$ in (3) the factor $t^{2m-2j-1-\alpha}$ is to be replaced by $t^{2m-2j-2n-2}|\ln t|$, $j = 0, 1, \ldots, m-n-1$.

From Proposition 1 it follows that in the case $\alpha < 1$ (weak degeneracy) $u^{(j)}(0)$ exist for all $j = 0, 1, \ldots, m-1$, while for $\alpha \geq 1$ (strong degeneracy) not all $u^{(j)}(0)$ exist. More precisely, for $1 \leq \alpha < 2m-1$ the derivatives at zero $u^{(j)}(0)$ exist only for $j = 0, 1, \ldots, s_\alpha$, where $s_\alpha = m - 1 - [\frac{\alpha+1}{2}]$ (here $[a]$ is the integer part of a number a) and for $\alpha \geq 2m-1$ all $u^{(j)}(0)$, $j = 0, 1, \ldots, m-1$, in general may be infinite.

Proposition 2 *The embedding*

$$W_\alpha^m \subset L_{2,\alpha} \tag{4}$$

is compact for every $\alpha \geq 0$.

Remark 3 *The embedding*

$$W_\alpha^m \subset L_{2,\beta} \tag{5}$$

is compact for every $\alpha > 2m-1$ and $\beta > \alpha - 2m$.

Observe that in the case $\beta = \alpha - 2m$ and $\alpha \leq 2m-1$ the embedding (5) fails (see [8]). For $\alpha \leq 2m-1$ we only have the embedding $W_\alpha^m \subset L_{2,\gamma}, \gamma > -1$. However, for $\alpha > 2m-1$ we have the embedding $W_\alpha^m \subset L_{2,\alpha-2m}$, which can be proved by using of the Hardy inequality (see [6] and [8]) and this embedding is not compact. Indeed, it is easy to verify, that for the bounded in W_α^m sequence $u_n(t) = n^{-\frac{1}{2}} t^{\frac{2m-\alpha-1}{2}} (\ln t)^{-\frac{1}{2}-\frac{1}{2n}} \varphi(t)$, where $\varphi \in C^m[0,b]$, $\varphi(t) = 1$ for $t \in [0, \frac{\varepsilon}{2}], 0 < \varepsilon < \min\{1, b\}$ and $\varphi(t) = 0$ for $t \in [\varepsilon, b]$ doesn't exist the convergent in $L_{2,\alpha-2m}$ subsequence (see [5], [12]).

2.2 One–dimensional Equation

Now we consider the Neumann problem for the special case $a = 1$ of the one-dimensional equation (1)

$$Bu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = f, \quad f \in L_{2,-\alpha}. \tag{6}$$

Definition 4 *A function $u \in W^m_\alpha$ is called a generalized solution of the Neumann problem for the equation (6) if for every $v \in W^m_\alpha$ we have*

$$(t^\alpha u^{(m)}, v^{(m)}) + (t^\alpha u, v) = (f, v), \tag{7}$$

where $(\cdot\,,\cdot)$ is the scalar product in $L_2(0,b)$.

Proposition 5 *The generalized solution of the Neumann problem for the equation (6) exists and is unique for every $f \in L_{2,-\alpha}$.*

The uniqueness of the generalized solution immediately follows from Definition 4. To prove the existence we note that the linear functional $l_f(v) = (f, v)$ is continuous in W^m_α because

$$|l_f(v)| \le ||f||_{L_{2,-\alpha}} ||v||_{L_{2,\alpha}} \le ||f||_{L_{2,-\alpha}} |||v||_{W^m_\alpha}$$

and use Riesz's lemma on the representation of continuous functionals.

If the generalized solution is classical then from (7) after integration by parts we get

$$(-1)^m((t^\alpha u^{(m)})^{(m)}, v) + \sum_{j=0}^{m-1} (-1)^j \left((t^\alpha u^{(m)}(t))^{(j)} \overline{v}^{(m-j-1)}(t)\right)\Big|_0^b + (t^\alpha u, v) = (f, v).$$

Since the function $v \in W^m_\alpha$ is arbitrarily we conclude that the function $u(t)$ fulfills the following conditions (see [10])

$$(t^\alpha u^{(m)}(t))^{(j)}\big|_{t=0} = u^{(m+j)}(t)\big|_{t=b} = 0, \quad j = 0, 1, \dots, m-1. \tag{8}$$

For $\alpha = 0$ the conditions (8) are usual Neumann conditions, which are of Sturm type and, therefore, regular (see [9]).

Definition 6 *We say that $u \in W^m_\alpha$ belongs to $D(B)$, if the equality (7) is satisfied for some $f \in L_{2,-\alpha}$. In this case we will write $Bu = f$.*

According to Definition 6 we have an operator

$$B : D(B) \subset W^m_\alpha \subset L_{2,\alpha} \to L_{2,-\alpha}\,.$$

If $u \in W^m_\alpha$ we know that $u^{(j)}(0), j = 0, 1, \dots, m-1$ exist only for $\alpha < 2m - 2j - 1$ (see Proposition 1). But for the generalized solution of the equation (7) we can improve it and give the following description of $u \in D(B)$.

Proposition 7 *The domain of definition of the operator B consists of the functions $u \in W^m_\alpha$ for which $u^{(j)}(0)$ are finite for $0 \le \alpha < 2m-2j$ and $2m-1 \le \alpha < 4m-2j-1$. The value $u(0)$ is finite for $0 \le \alpha < 2m+1$. The values $u^{(j)}(0)$ cannot be specified arbitrarily, but are determined by the right-hand side of* (7).

Proof. To describe the domain of definitions $D(B)$ of the operator B first note, that $t^\alpha u \in L_{2,-\alpha}$ since $u \in L_{2,\alpha}$. Hence it is enough to study the behaviour of the solutions for the equation $(-1)^m(t^\alpha u^{(m)})^{(m)} = f, f \in L_{2,-\alpha}$ near to the point $t=0$. Let $\alpha \ge 2m-1$. For the solution $u(t)$ of this equation we have

$$u^{(m)}(t) = t^{-\alpha}P_{m-1}(t) + \frac{(-1)^m t^{-\alpha}}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)\,d\tau, \tag{9}$$

where $P_{m-1}(t) = a_0 + a_1 t + \cdots + a_{m-1}t^{m-1}$ is the polynomial of the degree $m-1$. Since $u \in W^m_\alpha$ we have that $P_{m-1}(t) = 0$. Hence we can write

$$|u^{(m)}(t)| = \left|\frac{(-1)^m t^{-\alpha}}{(m-1)!}\int_0^t (t-\tau)^{m-1}\tau^{\frac{\alpha}{2}}\tau^{-\frac{\alpha}{2}} f(\tau)\,d\tau\right| \le c||f||_{L_{2,-\alpha}} t^{\frac{2m-1-\alpha}{2}},$$

therefore, integrating $u^{(m)}(t) \quad (m-j)$-times, $j = 0,1,\ldots,m-1$, we get for some polynomial Q_{m-j-1} of the degree $m-j-1$

$$|u^{(j)}(t)| = \left|Q_{m-j-1}(t) + \frac{1}{(m-j-1)!}\int_0^t (t-\tau)^{m-j-1}u^{(m)}(t)\,d\tau\right| \le$$

$$\le c_j + d_j||f||_{L_{2,-\alpha}} t^{\frac{2m-1-\alpha}{2}+m-j} = c_j + d_j||f||_{L_{2,-\alpha}} t^{\frac{4m-2j-1-\alpha}{2}}.$$

Let now $2m-2j-1 \le \alpha < 2m-2j+1, j = 1,2,\ldots,m-1$. Then in the equality (9) $a_0 = a_1 = \ldots = a_{m-j-1} = 0$ since $u \in W^m_\alpha$. The second term in (9) we have already estimated and it exists for $\alpha < 2m-1$. Now it is enough to estimate only first term after integrating $u^{(m)}(t) \quad (m-j)$-times. The main term after integration remains $c_{m-j}t^{2m-2j-\alpha}$, therefore for the existence of $u^{(j)}(0)$ we get the condition $2m-2j-\alpha > 0$, i.e., $\alpha < 2m-2j$. Note that for other values of α the existence of the values $u^{(j)}(0)$ is proved in Proposition 1 (see [13]). Note also that the conditions in Proposition 7 are exact, i.e., if we take for example $\alpha = 2m-2j$, then the statement is false, the value $u^{(j)}(0)$ in general doesn't exist. □

To get an operator in the same space we set $g(t) = t^{-\alpha}f(t)$. It is evident that $g(t)$ belongs to $L_{2,\alpha}$ and $||f||_{L_{2,-\alpha}} = ||g||_{L_{2,\alpha}}$. Therefore, we get an operator $\mathbb{B} \equiv t^{-\alpha}B : D(\mathbb{B}) = D(B) \subset W^m_\alpha \subset L_{2,\alpha} \to L_{2,\alpha}$ with $\mathbb{B}u = g$.

Proposition 8 *The operator $\mathbb{B} : L_{2,\alpha} \to L_{2,\alpha}$ is positive and selfadjoint. Moreover, the inverse operator $\mathbb{B}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ is compact.*

Proof. The self–adjointness of the symmetric operator $\mathbb{B}$ (symmetry and positivity of the operator $\mathbb{B}$ follow from the Definition 6) is a consequence of the existence of the generalized solution for every $f \in L_{2,-\alpha}$ (see [4]). Now using (2) and the equality (7) with $v = u$ we get

$$||u||^2_{W^m_\alpha} = (f, u) \leq ||f||_{L_{2,-\alpha}} ||u||_{L_{2,\alpha}} \leq ||g||_{L_{2,\alpha}} ||u||_{W^m_\alpha},$$

and, therefore, we have

$$||u||_{L_{2,\alpha}} \leq ||\mathbb{B}u||_{L_{2,\alpha}}. \tag{10}$$

The compactness of the operator $\mathbb{B}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ now follows from the inequality (10) and Proposition 2. □

Corollary 9 *The operator* $\mathbb{B}$ *has a discrete spectrum, and the system of the corresponding eigenfunctions is dense in* $L_{2,\alpha}$.

This follows from the connection of the spectra of the operators $\mathbb{B}$ and $\mathbb{B}^{-1}$ and from the properties of compact selfadjoint operators (see [4]).

Note that if λ is an eigenvalue and $u(t)$ a corresponding eigenfunction of the operator $\mathbb{B}$ then we have

$$(-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = \lambda t^\alpha u. \tag{11}$$

It follows then from the inequality (10) and Definition 6 that $\lambda \geq 1$. Note that the number $\lambda = 1$ is an eigenvalue for the operator $\mathbb{B}$ with the multiplicity m since every polynomial of order $m - 1$ is an eigenfunction. Therefore, for the solvability of the equation

$$(-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\alpha}\,, \tag{12}$$

we get the following result:

Proposition 10 *The generalized solution of the Neumann problem for the equation* (12) *exists if and only if* $(f, P_{m-1}(t)) = 0$ *for any polynomial* $P_{m-1}(t)$ *of order* $m - 1$.

Here we have used both $(g,\, P_{m-1}(t))_\alpha = (f,\, P_{m-1}(t))$ since $t^\alpha g(t) = f(t)$ ($(\cdot,\cdot)_\alpha$ is the scalar product in $L_{2,\alpha}$) and the definition of the operator $\mathbb{B}$. Note that the generalized solution of the Neumann problem for the equation (12) is unique up to an arbitrary additive polynomial of order $m - 1$.

Now we can consider the general case of the one-dimensional equation (1)

$$Pu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + a t^\alpha u = f, \quad f \in L_{2,-\alpha}, \tag{13}$$

because the number $1 - a$ can be regarded as a spectral parameter for the operator $\mathbb{B}$. Therefore, we can state that if $1 - a \notin \sigma\mathbb{B}$ then the equation (13) is uniquely solvable for every $f \in L_{2,-\alpha}$.

3. The Operator Equation

In this section we consider the operator version of the equation (1)

$$Pu \equiv (-1)^m D_t^m (t^\alpha D_t^m) u + t^\alpha A u = f, \quad f \in L_{2,-\alpha}((0,b), H), \quad \alpha \geq 0. \quad (14)$$

Suppose that the operator $A : H \to H$ has a complete system of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$, $A\varphi_k = a_k \varphi_k, k \in \mathbb{N}$, forming a Riesz's basis in H (see [4]), i.e., for every $x \in H$ we have $x = \sum_{k=1}^{\infty} x_k \varphi_k$, and there are some positive constants c_1 and c_2 such that

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \leq ||x||^2 \leq c_2 \sum_{k=1}^{\infty} |x_k|^2. \quad (15)$$

Hence for every $u \in L_{2,\alpha}((0,b), H)$, $f \in L_{2,-\alpha}((0,b), H)$ we have

$$u = \sum_{k=1}^{\infty} u_k(t) \varphi_k, \quad f = \sum_{k=1}^{\infty} f_k(t) \varphi_k, \quad k \in \mathbb{N}. \quad (16)$$

Therefore, the operator equation (14) can be decomposed into an infinite chain of ordinary differential equations

$$P_k u_k \equiv (-1)^m (t^\alpha u_k^{(m)})^{(m)} + a_k t^\alpha u_k = f_k, \quad f_k \in L_{2,-\alpha}, \quad k \in \mathbb{N}. \quad (17)$$

For the equations (17) we can define the generalized solutions $u_k(t), k \in \mathbb{N}$, of the Neumann problem (see Section 2).

Definition 11 *A function $u \in L_{2,\alpha}((0,b), H)$ is called a generalized solution of the Neumann problem for the equation* (14) *if the functions $u_k(t), k \in \mathbb{N}$, in the representation* (16) *are generalized solutions of the Neumann problem for the equations* (17).

Proposition 12 *The operator equation* (14) *is uniquely solvable for every $f \in L_{2,-\alpha}((0,b), H)$ if and only if the equations* (17) *are uniquely solvable for every $f_k \in L_{2,-\alpha}$, $k \in \mathbb{N}$, and the inequalities*

$$||u_k||_{L_{2,\alpha}} \leq c||f_k||_{L_{2,-\alpha}} \quad (18)$$

are satisfied uniformly with respect to $k \in \mathbb{N}$.

For the proof of Proposition 12 see [4].

Let the numbers $\lambda_1 = 1 < \lambda_2 < \cdots < \lambda_k < \cdots$, $\lambda_k \to +\infty$ when $k \to \infty$, are the eigenvalues of the operator $\mathbb{B}$ (see Section 2). Suppose that

$$\rho(1 - a_k, \lambda_m) > \varepsilon, \quad k, m \in \mathbb{N}, \quad (19)$$

where $\varepsilon > 0$ and ρ is the distance in the complex plane.

Theorem 13 *Under the condition* (19) *the generalized solution of the Neumann problem for the operator equation* (14) *exists and is unique for every* $f \in L_{2,-\alpha}((0,b),H)$.

First note that under the condition (19) the equations (17) are uniquely solvable for every $f_k \in L_{2,-\alpha}$, $k \in \mathbb{N}$ and the inequalities (18) are satisfied. Now the proof of Theorem 13 follows from Proposition 12.

Let $g = t^{-\alpha} f$, $f \in L_{2,-\alpha}((0,b),H)$. Then $g \in L_{2,\alpha}((0,b),H)$ and we define an operator

$$\mathbb{P} \equiv t^{-\alpha} P : D(\mathbb{P}) = D(P) \subset L_{2,\alpha}((0,b),H) \to L_{2,\alpha}((0,b),H),$$

with $\mathbb{P}u = g$ in $L_{2,\alpha}((0,b),H)$. It follows from the condition (19) that for the generalized solution of the Neumann problem we have

$$||u||_{L_{2,\alpha}((0,b),H)} \leq c||g||_{L_{2,\alpha}((0,b),H)}. \tag{20}$$

The operator $\mathbb{P}^{-1} : L_{2,\alpha}((0,b),H) \to L_{2,\alpha}((0,b),H)$ in general is not compact in contrast to Proposition 8 (it will be compact only in the case when the space H is finite-dimensional). If the operator $A : H \to H$ is selfadjoint we can describe the spectrum of the operator $\mathbb{P}$.

Proposition 14 *The spectrum of the operator* $\mathbb{P}$ *is equal to the closure of the direct sum of the spectra* $\sigma\mathbb{B}$ *and* $\sigma(A - I)$, *i.e.,*

$$\sigma\mathbb{P} = \overline{\sigma\mathbb{B} + \sigma(A - I)} \equiv \overline{\{\lambda_1 + \lambda_2 - 1 : \lambda_1 \in \sigma\mathbb{B},\ \lambda_2 \in \sigma A\}}.$$

The proof of Proposition 14 immediately follows from the equality

$$\mathbb{P} = \mathbb{B} \otimes I_H + I_{L_{2,\alpha}} \otimes (A - I)$$

(here $\otimes$ means the tensor product of the operators). Note that here we use the assertion, that if $\lambda \in \sigma T$ for the selfadjoint operator T in some separable Hilbert space $T : X \to X$, then there is some sequence $x_n \in D(T), n \in \mathbb{N}, ||x_n|| = 1$ such that $(T - \lambda)x_n \to 0, n \to \infty$ (see [1], [14]).

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