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Uniform Convergence of Double Fourier-Legendre series of Functions of Bounded Generalized Variation

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The Uniform convergence of double Fourier-Legendre series of function of bounded Harmonic variation and bounded partial Λ-variation are investigated.

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1. Classes of Functions of Bounded Generalized Variation

In 1881 Jordan [14] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation ets., see [5, 18, 23, 24]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [13].

Let f be a real function of two variables. Given intervals $\Delta = (a, b)$, $J = (c, d)$ and points x, y from $I := [-1, 1]$ we denote

$$f(\Delta, y) := f(b, y) - f(a, y), \qquad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(\Delta, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{\Delta_i\}$ be a collection of nonoverlapping intervals from I ordered in an arbitrary way and let Ω be the set of all such collections E. Denote by Ω_n the set of all collections of n nonoverlapping intervals $\Delta_k \subset I$.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $I^2 := [-1, 1]^2$ we denote

$$\Lambda V_1(f; I^2) = \sup_{y} \sup_{E \in \Omega} \sum_{i} \frac{|f(\Delta_i, y)|}{\lambda_i} \quad (E = \{\Delta_i\}),$$

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$$\Lambda V_2(f;I^2)=\sup_x\sup_{F\in\Omega}\sum_j\frac{|f(x,J_j)|}{\lambda_j}\qquad (F=\{J_j\}),$$

$$\Lambda V_{1,2}(f;I^2)=\sup_{F,\,E\in\Omega}\sum_i\sum_j\frac{|f(\Delta_i,J_j)|}{\lambda_i\lambda_j}.$$

Definition 1.1: We say that the function f has bounded Λ-variation on I^2 and write $f\in\Lambda BV$, if

$$\Lambda V(f;I^2):=\Lambda V_1(f;I^2)+\Lambda V_2(f;I^2)+\Lambda V_{1,2}(f;I^2)<\infty.$$

We say that the function f has bounded partial Λ-variation and write $f\in P\Lambda BV$ if

$$P\Lambda V(f;I^2):=\Lambda V_1(f;I^2)+\Lambda V_2(f;I^2)<\infty.$$

Definition 1.2: We say that the function f is continuous in $\left(\Lambda^1,\Lambda^2\right)$-variation on I^2 and write $f\in C\left(\Lambda^1,\Lambda^2\right)V\left(I^2\right)$, if

$$\lim_{n\to\infty}\Lambda^1_nV_1\left(f;I^2\right)=\lim_{n\to\infty}\Lambda^2_nV_2\left(f;I^2\right)=0$$

and

$$\lim_{n\to\infty}\left(\Lambda^1_n,\Lambda^2\right)V_{1,2}\left(f;I^2\right)=\lim_{n\to\infty}\left(\Lambda^1,\Lambda^2_n\right)V_{1,2}\left(f;I^2\right)=0,$$

where $\Lambda^i_n:=\left\{\lambda^i_k\right\}_{k=n}^{\infty}=\left\{\lambda^i_{k+n}\right\}_{k=0}^{\infty},i=1,2.$

If $\lambda_n\equiv 1$ (or if $0<c<\lambda_n<C<\infty,\ n=1,2,\ldots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n\to\infty$ and since the intervals in $E=\{\Delta_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1<\lambda_1\le\lambda_2\le\ldots,\qquad \lim_{n\to\infty}\lambda_n=\infty. \tag{1}$$

In the case when $\lambda_n=n,\ n=1,2\ldots$ we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ (HBV, $PHBV$, $HV(f)$, ets).

The notion of Λ-variation was introduced by Waterman [24] in the one dimensional case, by Sahakian [21] in the two dimensional case. The notion of bounded partial variation (PBV) was introduced by Goginava [11] and the notion of bounded partial Λ-variation, by Goginava and Sahakian [12].

The statements of the following theorem is known.

Theorem D *[Dragoshanski [9]]* $HBV=CHV$.

Definition 1.3: Let Φ-be a strictly increasing continuous function on $[0,+\infty)$ with $\Phi(0)=0$. We say that the function f has bounded partial Φ-variation on I^2

and write $f \in PBV_\Phi$, if

$$V_\Phi^{(1)}(f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^{n} \Phi\left(\left|f\left(I_i, y\right)\right|\right) < \infty, \quad n = 1, 2, ...,$$

$$V_\Phi^{(2)}(f) := \sup_x \sup_{\{J_j\} \in \Omega_m} \sum_{j=1}^{m} \Phi\left(\left|f\left(x, J_j\right)\right|\right) < \infty, \quad m = 1, 2, ...$$

In the case when $\Phi(u) = u^p$, $p \geq 1$, the notion of bounded partial p-variation (class PBV_p) was introduced in [10].

In [12] it is proved that the following theorem is true.

Theorem GS *Let* $\Lambda = \{\lambda_n\}$ *and* $\lambda_n / n \geq \lambda_{n+1} / (n+1) > 0$, $n = 1, 2, ...$.
1) If

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^2} < \infty,$$

then $P\Lambda BV \subset HBV$.
2) If, in addition, for some $\delta > 0$

$$\frac{\lambda_n}{n} = O\left(\frac{\lambda_{n^{[1+\delta]}}}{n^{[1+\delta]}}\right) \quad as \quad n \to \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^2} = \infty,$$

then $P\Lambda BV \not\subset HBV$.

Corollary 1.4: $PBV \subset HBV$ *and* $PHBV \not\subset HBV$.

Definition 1.5 see [11:] The partial modulus of variation of a function f are the functions $v_1(n, f)$ and $v_2(m, f)$, defined by

$$v_1(n, f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^{n} \left|f\left(I_i, y\right)\right|, \quad n = 1, 2, \ldots,$$

$$v_2(m, f) := \sup_x \sup_{\{J_k\} \in \Omega_m} \sum_{i=1}^{m} \left|f\left(x, J_k\right)\right|, \quad m = 1, 2, \ldots.$$

For functions of one variable the concept of the modulus variation was introduced by Chanturia [5].

The following theorem is proved by Goginava and Sahakian [12].

Theorem GS2 *If $f \in B$ is bounded on I^2 and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n,f)}}{n^{3/2}} < \infty, \quad j = 1, 2,$$

then $f \in HBV$.

$C\left(I^2\right)$ is the space of continuous functions on I^2 with the uniform norm

$$\|f\|_C := \max_{(x,y)\in I^2} |f(x,y)|.$$

The partial moduli of continuity of a function $f \in C\left(I^2\right)$ in the C-norm are defined by

$$\omega_1(f;\delta) := \max\left\{|f(x,y) - f(s,y)|, x, y, s \in I, |x-s| \le \delta\right\},$$

$$\omega_2(f;\delta) := \max\left\{|f(x,y) - f(x,t)|, x, y, t \in I, |y-t| \le \delta\right\},$$

while the mixed modulus of continuity is defined as follows:

$$\begin{aligned}\omega_{1,2}(f;\delta_1,\delta_2) : &= \max\left\{|f(x,y) - f(s,y) - f(x,t) + f(s,t)|,\right.\\ &\left. x, y, s, t \in I, |x-s| \le \delta_1, |y-t| \le \delta_2\right\}.\end{aligned}$$

2. Fourier-Legendre Series

Let $p_n(x)$ be the Legendre orthonormal polynomial of degree n. If f is an integrable function on I^2, then Fourier-Legendre series of f is the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widehat{f}(n,m)\, p_n(x)\, p_n(y),$$

where

$$\widehat{f}(n,m) := \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, p_n(s)\, p_m(t)\, ds dt$$

is the (n,m)th Fourier coefficient of the function f.

The rectangular partial sums of double Fourier-Legendre series are defined by

$$S_{MN} f(x,y) := \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \widehat{f}(n,m)\, p_n(x)\, p_m(y).$$

It is easy to show that

$$S_{MN}f(x,y)=\int_{-1}^{1}\int_{-1}^{1}f(s,t)\,K_n(x,s)\,K_m(y,t)\,dsdt, \tag{2}$$

where

$$K_n(x,s):=\sum_{k=0}^{n-1}p_k(s)\,p_k(x)\,.$$

The Chrestoffel-Darboux formula is well-know (see ([22]))

$$K_n(x,t)=\frac{\gamma_{n-1}}{\gamma_n}\frac{p_{n-1}(t)\,p_n(x)-p_{n-1}(x)\,p_n(t)}{x-t}. \tag{3}$$

Since

$$\frac{\gamma_{n-1}}{\gamma_n}\leq 1$$

and

$$|p_n(x)|\leq\frac{c}{(1-x^2)^{1/4}},x\in(-1,1) \tag{4}$$

from (3) we have

$$|K_n(x,t)|\leq\frac{c}{|x-t|\,(1-x^2)^{1/4}\,(1-t^2)^{1/4}}. \tag{5}$$

In [4, 19] it is proved that the following estimations hold

$$\left|\int_{-1}^{s}K_n(x,t)\,dt\right|\leq\frac{c}{n\,(x-s)\,(1-x^2)^{1/4}}\quad(-1\leq s<x<1)\ , \tag{6}$$

$$\left|\int_{s}^{1}K_n(x,t)\,dt\right|\leq\frac{c}{n\,(s-x)\,(1-x^2)^{1/4}}\quad(-1<x<s\leq 1)\,, \tag{7}$$

$$\int_{x-\frac{1+x}{n}}^{x}|K_n(x,t)|\,dt\leq\frac{c\,(1+x)}{(1-x^2)^{1/2}}\qquad(-1<x<1)\,, \tag{8}$$

$$\int_{x}^{x+\frac{1-x}{n}} |K_n(x,t)|\, dt \leq \frac{c(1-x)}{(1-x^2)^{1/2}} \qquad (-1 < x < 1). \tag{9}$$

3. Convergence of double Fourier-Legendre series

The well known Dirichlet-Jordan theorem (see [25]) states that the trigonometric Fourier series of a function $f(x)$, $x \in [0, 2\pi)$ of bounded variation converges at every point x to the value $[f(x+0) + f(x-0)]/2$.

Hardy [13] generalized the Dirichlet-Jordan theorem to the double trigonometric Fourier series. He proved that if the function $f(x,y)$ has bounded variation in the sense of Hardy ($f \in BV$), then the double trigonometric Fourier series of the continuous function f converges uniformly on $[0, 2\pi]^2$. The author [11] has proved that in Hardy's theorem there is no need to require the boundedness of $V_{1,2}(f)$; moreover, it is proved that if f is a continuous function and has bounded partial variation ($f \in PBV$) then its double trigonometric Fourier series converges uniformly on $[0, 2\pi]^2$.

Convergence of rectangular and spherical partial sums of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [21], Dyachenko [6–8], Bakhvalov [1], Sablin [20].

For the one-dimensional Fourier-Legendre series the convergence of partial sums of functions Harmonic bounded variation and other bounded generalized variation were studied by Agakhanov, Natanson [2], Bojanic [4], Belenkiĭ [3], Kvernadze [15–17], Powierska [19].

In this paper we prove that the following are true.

Theorem 3.1: *Let $\varepsilon > 0$ and f be a function of $CHV\left(I^2\right) \bigcap C\left(I^2\right)$. Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$.*

Theorem D and Theorem 3.1 imply

Theorem 3.2: *Let $\varepsilon > 0$ and f be a function of $HBV\left(I^2\right) \bigcap C\left(I^2\right)$. Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$.*

Theorem GS1 and Theorem 3.2 imply

Theorem 3.3: *Let $f \in P\Lambda BV(I^2) \bigcap C\left(I^2\right)$ with*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{j^2} < \infty, \quad \frac{\lambda_j}{j} \downarrow 0.$$

Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$,$\varepsilon > 0$.

Corollary 3.4: *If $f \in P\left\{\frac{n}{\log^{1+\delta}(n+1)}\right\} BV\left(I^2\right) \bigcap C\left(I^2\right)$ for some $\delta > 0$. Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$.*

Theorem GS2 and Theorem 3.2 imply

Theorem 3.5: *Let $f \in C\left(I^2\right)$ and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j\left(n,f\right)}}{n^{3/2}} < \infty \qquad j = 1, 2.$$

Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$,$\varepsilon > 0$.

Corollary 3.6: *Let $f \in f \in C\left(I^2\right)$ and $v_1\left(k,f\right) = O\left(k^{\alpha}\right), v_2\left(k,f\right) = O\left(k^{\beta}\right), 0 < \alpha, \beta < 1$. Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$,$\varepsilon > 0$.*

Corollary 3.7: *Let $f \in PBV_p \bigcap C\left(I^2\right), p \geq 1$. Then the double Fourier-Legendre series of the function f uniformly converges to f on $[-1+\varepsilon, 1-\varepsilon]^2$,$\varepsilon > 0$.*

4. Proofs of Main Results

Proof: [Proof of Theorem 3.1] Denote

$$s_j := x + \frac{j\left(1-x\right)}{n}, j = 1, 2, ..., n, x \in \left(-1, 1\right), \tag{10}$$

$$t_i := y - \frac{i\left(1+y\right)}{m}, i = 1, 2, ..., m, y \in \left(-1, 1\right) \tag{11}$$

$$g\left(s,t\right) := f\left(s,t\right) - f\left(x,y\right). \tag{12}$$

Then from (2) we can write

$$\begin{aligned} & S_{mn}f\left(x,y\right) - f\left(x,y\right) \qquad (13) \\ &= \int_{-1}^{1}\int_{-1}^{1} g\left(s,t\right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt \\ &= \left(\int_{x}^{1}\int_{-1}^{y} + \int_{-1}^{x}\int_{-1}^{y} + \int_{-1}^{x}\int_{y}^{1} + \int_{x}^{1}\int_{y}^{1}\right)\left(g\left(s,t\right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt\right) \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

$$\begin{aligned} & I_1 \qquad (14) \\ &= \left(\int_{x}^{s_1}\int_{-1}^{t_1} + \int_{x}^{s_1}\int_{t_1}^{y} + \int_{s_1}^{1}\int_{-1}^{t_1} + \int_{s_1}^{1}\int_{t_1}^{y}\right)\left(g\left(s,t\right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt\right) \\ &= II_1 + II_2 + II_3 + II_4. \end{aligned}$$

For II_4 we have

$$II_4 = \int_{t_1}^{y} \left(\sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \left(g(s,t) - g(s_j,t) \right) K_n(x,s)\, ds \right) K_m(y,t)\, dt \tag{15}$$
$$+ \int_{t_1}^{y} \left(\int_{s_{n-1}}^{1} \left(g(s,t) - g(s_{n-1},t) \right) K_n(x,s)\, ds \right) K_m(y,t)\, dt$$
$$+ \int_{t_1}^{y} \left(\sum_{j=1}^{n-1} g(s_j,t) \int_{s_j}^{s_{j+1}} K_n(x,s)\, ds \right) K_m(y,t)\, dt$$
$$= II_{41} + II_{42} + II_{43}.$$

From (4) and (5) we have

$$|II_{42}| \le 2\,\|f\|_C \int_{t_1}^{y} \sum_{j=0}^{m-1} |p_j(t)\, p_j(y)|\, dt \int_{s_{n-1}}^{1} |K_n(x,s)|\, ds \tag{16}$$
$$\le c\,\|f\|_C \int_{y-\frac{1+y}{m}}^{y} \frac{m dt}{(1-t^2)^{1/4}(1-y^2)^{1/4}}$$
$$\times \int_{s_{n-1}}^{1} \frac{ds}{(s-x)(1-x^2)^{1/4}(1-s^2)^{1/4}}$$
$$\le c(\varepsilon)\,\|f\|_C \int_{x+\frac{n-1}{n}(1-x)}^{1} \frac{ds}{(1-s)^{1/4}}$$
$$\le \frac{c(\varepsilon)\,\|f\|_C}{n^{3/4}} = o(1) \quad \text{as} \quad n,m \to \infty$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.
From (5), (10) and (11) we obtain

$$|II_{41}| \tag{17}$$
$$\le \int_{t_1}^{y} \left(\sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \frac{|g(s,t) - g(s_j,t)|}{(s-x)(1-x^2)^{1/4}(1-s^2)^{1/4}} ds \right) |K_m(y,t)|\, dt$$
$$\le c(\varepsilon)\, m \int_{t_1}^{y} \left(\sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \frac{|g(s,t) - g(s_j,t)|}{(s_j-x)(1-s_{j+1})^{1/4}} ds \right) dt$$

$$\leq c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left(\sum_{j=1}^{n-2} \frac{1}{j(n-j)^{1/4}} \int_{s_j}^{s_{j+1}} |g(s,t) - g(s_j,t)| \, ds \right) dt$$

$$= c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left(\int_0^{\frac{1-x}{n}} \sum_{j=1}^{n-2} \frac{|g(s+s_j,t) - g(s_j,t)|}{j(n-j)^{1/4}} ds \right) dt$$

$$= c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left(\int_0^{\frac{1-x}{n}} \sum_{1 \leq j < n/2} \frac{|g(s+s_j,t) - g(s_j,t)|}{j(n-j)^{1/4}} ds \right) dt$$

$$+ c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left(\int_0^{\frac{1-x}{n}} \sum_{n/2 \leq j < n-1} \frac{|g(s+s_j,t) - g(s_j,t)|}{j(n-j)^{1/4}} ds \right) dt$$

$$\leq c(\varepsilon) n m \int_{t_1}^{y} \left(\int_0^{\frac{1-x}{n}} \sum_{1 \leq j < n/2} \frac{|g(s+s_j,t) - g(s_j,t)|}{j} ds \right) dt$$

$$+ c(\varepsilon) n^{1/4} m \int_{t_1}^{y} \left(\int_0^{\frac{1-x}{n}} \sum_{n/2 \leq j < n-1} \frac{|g(s+s_j,t) - g(s_j,t)|}{(n-j)^{1/4}} ds \right) dt.$$

It is easy to show that

$$\sum_{1 \leq j < n/2} \frac{|g(s+s_j,t) - g(s_j,t)|}{j} \tag{18}$$

$$\leq \min_{1 \leq k < n} \left\{ \sum_{1 \leq j < k} \frac{|g(s+s_j,t) - g(s_j,t)|}{j} + \sum_{k \leq j < n} \frac{|g(s+s_j,t) - g(s_j,t)|}{j} \right\}$$

$$\leq c(\varepsilon) \min_{1 \leq k < n} \left\{ \omega_1 \left(f; \frac{1}{n} \right) \log(k+1) + \{j+k\} V_1 \left(f; I^2 \right) \right\}$$

$$= o(1) \quad \text{as} \quad n \to \infty$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

On the other hand,

$$\frac{1}{n^{3/4}} \sum_{n/2 \leq j < n-1} \frac{|g(s+s_j,t) - g(s_j,t)|}{(n-j)^{1/4}} \tag{19}$$

$$\leq \sum_{n/2 \leq j < n-1} \frac{|g(s+s_j,t) - g(s_j,t)|}{n-j}$$

$$\leq c \min_{1\leq k<n} \left\{ \omega_1 \left(f; \frac{1}{n} \right) \log (k+1) + \{j+k\} V_1 \left(f; I^2 \right) \right\}$$
$$= o(1) \quad \text{as} \quad n \to \infty$$

uniformly with respect to $(x, y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Combining (17)-(19), we obtain that

$$II_{41} = o(1) \quad \text{as} \quad n \to \infty \tag{20}$$

uniformly with respect to $(x, y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Applying the Abel transformation we obtain

$$\begin{aligned} & II_{43} \qquad (21) \\ &= \int_{t_1}^{y} \left(g(s_1, t) \sum_{k=1}^{n-1} \int_{s_k}^{s_{k+1}} K_n(x, s)\, ds \right) K_m(y, t)\, dt \\ &\quad + \int_{t_1}^{y} \left(\sum_{j=1}^{n-2} \left(g(s_{j+1}, t) - g(s_j, t) \right) \sum_{k=j+1}^{n-1} \int_{s_k}^{s_{k+1}} K_n(x, s)\, ds \right) K_m(y, t)\, dt \\ &= \int_{t_1}^{y} \left(g(s_1, t) \int_{s_1}^{1} K_n(x, s)\, ds \right) K_m(y, t)\, dt \\ &\quad + \int_{t_1}^{y} \left(\sum_{j=1}^{n-2} \left(g(s_{j+1}, t) - g(s_j, t) \right) \int_{s_{j+1}}^{1} K_n(x, s)\, ds \right) K_m(y, t)\, dt \\ &= II_{431} + II_{432}. \end{aligned}$$

It is easy to show that

$$\begin{aligned} |II_{431}| &\leq \frac{c(\varepsilon)\, m}{n(s_1 - x)} \int_{t_1}^{y} |f(s_1, t) - f(x, y)|\, dt \qquad (22) \\ &\leq c(\varepsilon)\, m \int_{t_1}^{y} |f(s_1, t) - f(s_1, y)|\, dt \\ &\quad + c(\varepsilon)\, m \int_{t_1}^{y} |f(s_1, y) - f(x, y)|\, dt \\ &\leq c(\varepsilon) \left\{ \omega_1 \left(f, \frac{1}{n} \right) + \omega_2 \left(f, \frac{1}{m} \right) \right\} = o(1) \quad \text{as} \quad n, m \to \infty \end{aligned}$$

uniformly with respect to $(x, y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

From (7), (18) and (19) we obtain

$$|II_{432}| \leq c(\varepsilon) \int_{t_1}^{y} \sum_{j=1}^{n-2} \frac{|g(s_{j+1},t) - g(s_j,t)|}{(s_{j+1}-x)\,n} |K_m(y,t)|\,dt \tag{23}$$

$$\leq c(\varepsilon) \sup_{t\in[t_1,y]} \sum_{j=1}^{n-2} \frac{|g(s_{j+1},t) - g(s_j,t)|}{j}$$

$$= o(1) \quad \text{as} \quad n,m \to \infty$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

From (21)-(23) we have

$$II_{43} = o(1) \quad \text{as} \quad n,m \to \infty \tag{24}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Combining (15), (20), (16) and (24) we conclude that

$$II_4 = o(1) \quad \text{as} \quad n,m \to \infty \tag{25}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Analogously, we can prove that

$$II_1 = o(1) \quad \text{as} \quad n,m \to \infty \tag{26}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

For II_2 we can write

$$|II_2| \leq \int_{x}^{s_1}\int_{t_1}^{y} |f(s,t) - f(x,y)|\,|K_n(x,s)|\,|K_m(y,t)|\,dsdt \tag{27}$$

$$\leq c(\varepsilon)\left\{\omega_1\left(f,\frac{1}{n}\right) + \omega_2\left(f,\frac{1}{m}\right)\right\} = o(1) \quad \text{as} \quad n,m \to \infty$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

We can write

$$II_3 \tag{28}$$

$$= \sum_{j=1}^{n-1}\sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}}\int_{t_{i+1}}^{t_i} g(s,t)\,K_n(x,s)\,K_m(y,t)\,dsdt$$

$$= \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_{i+1}}^{t_i} \left(g\left(s,t\right) - g\left(s_j,t\right) - g\left(s,t_i\right) + g\left(s_j,t_i\right) \right)$$
$$\times K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$+ \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_{i+1}}^{t_i} \left(g\left(s_j,t\right) - g\left(s_j,t_i\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$+ \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_{i+1}}^{t_i} \left(g\left(s,t_i\right) - g\left(s_j,t_i\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$+ \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} g\left(s_j,t_i\right) \int_{s_j}^{s_{j+1}} \int_{t_{i+1}}^{t_i} K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$= III_1 + III_2 + III_3 + III_4.$$

For III_3 we have

$$III_3 \tag{29}$$
$$= \sum_{j=1}^{n-2} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_{i+1}}^{t_i} \left(g\left(s,t_i\right) - g\left(s_j,t_i\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$+ \sum_{i=1}^{m-1} \int_{s_{n-1}}^{1} \int_{t_{i+1}}^{t_i} \left(g\left(s,t_i\right) - g\left(s_{n-1},t_i\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$= III_{31} + III_{32}.$$

Applying the Abels transformation we get

$$III_{31} \tag{30}$$
$$= \sum_{j=1}^{n-2} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_{i+1}}^{t_i} \left(f\left(s,t_1\right) - f\left(s_j,t_1\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$+ \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \sum_{k=i+1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_{k+1}}^{t_k} \left(f\left(s,t_{i+1}\right) - f\left(s_j,t_{i+1}\right) \right.$$
$$\left. - f\left(s,t_i\right) + f\left(s_j,t_i\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$
$$= \sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \int_{-1}^{t_1} \left(f\left(s,t_1\right) - f\left(s_j,t_1\right) \right) K_n\left(x,s\right) K_m\left(y,t\right) dsdt$$

$$+\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\int\limits_{s_j}^{s_{j+1}}\int\limits_{-1}^{t_{i+1}}\left(f\left(s,t_{i+1}\right)-f\left(s_j,t_{i+1}\right)\right.$$
$$\left.-f\left(s,t_i\right)+f\left(s_j,t_i\right)K_n\left(x,s\right)\right)K_m\left(y,t\right)dsdt$$
$$=III_{311}+III_{312}.$$

From (5), (6), (18) and (19) we obtain

$$\begin{aligned}
&\left|III_{311}\right| &(31)\\
&\leq c\left(\varepsilon\right)\left|\int\limits_{-1}^{t_1}K_m\left(y,t\right)dt\right|\sum_{j=1}^{n-2}\int\limits_{s_j}^{s_{j+1}}\frac{\left|f\left(s,t_1\right)-f\left(s_j,t_1\right)\right|}{\left(s_j-x\right)\left(1-s_{j+1}\right)^{1/4}\left(1+s_j\right)^{1/4}}ds\\
&\leq\frac{c\left(\varepsilon\right)n^{5/4}}{m\left(y-t_1\right)}\sum_{j=1}^{n-2}\int\limits_{s_j}^{s_{j+1}}\frac{\left|f\left(s,t_1\right)-f\left(s_j,t_1\right)\right|}{j\left(n-j\right)^{1/4}}ds\\
&=c\left(\varepsilon\right)n^{5/4}\int\limits_{0}^{\frac{1-x}{n}}\sum_{j=1}^{n-2}\frac{\left|f\left(s+s_j,t_1\right)-f\left(s_j,t_1\right)\right|}{j\left(n-j\right)^{1/4}}ds\\
&=o\left(1\right)\quad\text{as}\ \ n,m\to\infty
\end{aligned}$$

uniformly with respect to $(x,y)\in\left[-1+\varepsilon,1-\varepsilon\right]^2$,

$$\begin{aligned}
&\left|III_{312}\right| &(32)\\
&\leq\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\left|\int\limits_{-1}^{t_{i+1}}K_m\left(y,t\right)dt\right|\int\limits_{s_j}^{s_{j+1}}\left|f\left(s,t_{i+1}\right)-f\left(s_j,t_{i+1}\right)\right.\\
&\quad\left.-f\left(s,t_i\right)+f\left(s_j,t_i\right)K_n\left(x,s\right)\right|ds\\
&\leq c\left(\varepsilon\right)\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\frac{1}{m\left(y-t_{i+1}\right)}\int\limits_{s_j}^{s_{j+1}}\left|f\left(s,t_{i+1}\right)-f\left(s_j,t_{i+1}\right)\right.\\
&\quad\left.-f\left(s,t_i\right)+f\left(s_j,t_i\right)\right|\frac{ds}{\left(s_j-x\right)\left(1-s_{j+1}\right)^{1/4}}\\
&\leq c\left(\varepsilon\right)\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\frac{1}{i}\frac{n^{5/4}}{j\left(n-j\right)^{1/4}}\int\limits_{s_j}^{s_{j+1}}\left|f\left(s,t_{i+1}\right)-f\left(s_j,t_{i+1}\right)\right.\\
&\quad\left.-f\left(s,t_i\right)+f\left(s_j,t_i\right)\right|ds\\
&=c\left(\varepsilon\right)\int\limits_{0}^{\frac{1-x}{n}}\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\frac{1}{i}\frac{n^{5/4}}{j\left(n-j\right)^{1/4}}\left|f\left(s+s_j,t_{i+1}\right)-f\left(s_j,t_{i+1}\right)\right.
\end{aligned}$$

$$-f(s+s_j,t_i)+f(s_j,t_i)|\,ds$$

$$=c(\varepsilon)\int\limits_0^{\frac{1-x}{n}}\sum_{1\le j<n/2}\sum_{i=1}^{m-2}\frac{n^{5/4}}{ji(n-j)^{1/4}}\left|f(s+s_j,t_{i+1})-f(s_j,t_{i+1})\right.$$

$$-f(s+s_j,t_i)+f(s_j,t_i)|\,ds$$

$$+c(\varepsilon)\int\limits_0^{\frac{1-x}{n}}\sum_{n/2\le j<n-1}\sum_{i=1}^{m-2}\frac{n^{5/4}}{ji(n-j)^{1/4}}\left|f(s+s_j,t_{i+1})-f(s_j,t_{i+1})\right.$$

$$-f(s+s_j,t_i)+f(s_j,t_i)|\,ds$$

$$\le c(\varepsilon)\,n\int\limits_0^{\frac{1-x}{n}}\sum_{1\le j<n/2}\sum_{i=1}^{m-2}\frac{1}{ji}\left|f(s+s_j,t_{i+1})-f(s_j,t_{i+1})\right.$$

$$-f(s+s_j,t_i)+f(s_j,t_i)|\,ds$$

$$+c(\varepsilon)\,n\int\limits_0^{\frac{1-x}{n}}\sum_{n/2\le j<n-1}\sum_{i=1}^{m-2}\frac{1}{(n-j)\,i}\left|f(s+s_j,t_{i+1})-f(s_j,t_{i+1})\right.$$

$$-f(s+s_j,t_i)+f(s_j,t_i)|\,ds$$

$$\le\min_{1\le k<n}\min_{1\le l<m}\left\{\omega_{1,2}\left(f;\frac{1}{n},\frac{1}{m}\right)\log(k+1)\log(l+1)\right.$$

$$\left.+\{i+k\}\{j\}V_{1,2}\left(f;I^2\right)+\{i\}\{j+l\}V_{1,2}\left(f;I^2\right)\right\}$$

$$=o(1)\quad\text{as}\quad n,m\to\infty$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

Combining (30), (31) and (32) we have

$$III_{31}=o(1)\quad\text{as}\quad n,m\to\infty \tag{33}$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

Applying the Abel's transformation we obtain

$$III_{32} \tag{34}$$

$$=\sum_{i=1}^{m-1}\int\limits_{s_{n-1}}^{1}\int\limits_{t_{i+1}}^{t_i}\left(f(s,t_i)-f(s_{n-1},t_i)\right)K_n(x,s)\,K_m(y,t)\,dsdt$$

$$=\sum_{i=1}^{m-1}\int\limits_{s_{n-1}}^{1}\int\limits_{t_{i+1}}^{t_i}\left(f(s,t_1)-f(s_{n-1},t_1)\right)K_n(x,s)\,K_m(y,t)\,dsdt$$

$$+\sum_{i=1}^{m-2}\sum_{k=i+1}^{m-1}\int\limits_{s_{n-1}}^{1}\int\limits_{t_{k+1}}^{t_k}\left(f(s,t_{i+1})-f(s_{n-1},t_{i+1})\right)$$

$$-f(s,t_i)+f(s_{n-1},t_i))\,K_n(x,s)\,K_m(y,t)\,dsdt$$

$$=\int_{s_{n-1}}^{1}\int_{-1}^{t_{m-1}}(f(s,t_1)-f(s_{n-1},t_1))\,K_n(x,s)\,K_m(y,t)\,dsdt$$

$$+\sum_{i=1}^{m-2}\int_{s_{n-1}}^{1}\int_{-1}^{t_{i+1}}(f(s,t_{i+1})-f(s_{n-1},t_{i+1})$$

$$-f(s,t_i)+f(s_{n-1},t_i))\,K_n(x,s)\,K_m(y,t)\,dsdt$$

$$=III_{321}+III_{322}.$$

Since

$$\int_{s_{n-1}}^{1}|K_n(x,s)|\,ds\le c(\varepsilon)\int_{s_{n-1}}^{1}\frac{ds}{(s-x)(1-s)^{1/4}}$$

$$\le c(\varepsilon)\int_{s_{n-1}}^{1}\frac{ds}{(1-s)^{1/4}}\le\frac{c(\varepsilon)}{n^{3/4}}$$

and

$$\int_{-1}^{t_{m-1}}|K_m(y,t)|\,dt\le\frac{c(\varepsilon)}{m^{3/4}}$$

for III_{321} we can write

$$|III_{321}|\le\frac{c(\varepsilon)\,\|f\|_C}{(nm)^{3/4}}=o(1)\quad\text{as}\quad n,m\to\infty \tag{35}$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

On the other hand,

$$|III_{322}|\le\frac{c(\varepsilon)}{n^{3/4}}\sup_s\sum_{i=1}^{m-2}|f(s,t_{i+1})-f(s,t_i)|\left|\int_{-1}^{t_{i+1}}K_m(y,t)\,dt\right| \tag{36}$$

$$\le\frac{c(\varepsilon)}{n^{3/4}}\sup_s\sum_{i=1}^{m-2}\frac{|f(s,t_{i+1})-f(s,t_i)|}{m(t_{i+1}-x)}$$

$$=\frac{c(\varepsilon)}{n^{3/4}}\sup_s\sum_{i=1}^{m-2}\frac{|f(s,t_{i+1})-f(s,t_i)|}{i}$$

$$\le\frac{c(\varepsilon)}{n^{3/4}}HV_2\left(f,I^2\right)=o(1)\quad\text{as}\quad n,m\to\infty$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

From (34), (35) and (36) we have

$$III_{32}=o\left(1\right)\quad \text{as}\ \ n,m\to\infty \tag{37}$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

Combining (29), (33) and (37), we conclude that

$$III_{3}=o\left(1\right)\quad \text{as}\ \ n,m\to\infty \tag{38}$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

Analogously we can prove that

$$III_{2}=o\left(1\right)\quad \text{as}\ \ n,m\to\infty \tag{39}$$

uniformly with respect to $(x,y)\in[-1+\varepsilon,1-\varepsilon]^2$.

From (5) we have

$$\begin{aligned}
&\left|III_1\right| \\
&\leq c\left(\varepsilon\right)\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}\int\limits_{s_j}^{s_{j+1}}\int\limits_{t_{i+1}}^{t_i}\left|f\left(s,t\right)-f\left(s_j,t\right)-f\left(s,t_i\right)+f\left(s_j,t_i\right)\right| \\
&\quad\times\frac{1}{s-x}\frac{1}{y-t}\frac{1}{\left(1-s\right)^{1/4}}\frac{1}{\left(1+t\right)^{1/4}}dsdt \\
&\leq\left(nm\right)^{5/4}\int\limits_0^{\frac{1-x}{n}}\int\limits_0^{\frac{1-y}{m}}\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}\frac{1}{j\left(n-j\right)^{1/4}}\frac{1}{i\left(m-i\right)^{1/4}} \\
&\quad\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|dsdt.
\end{aligned}\tag{40}$$

We can write

$$\begin{aligned}
&\sqrt[4]{nm}\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}\frac{1}{j\left(n-j\right)^{1/4}}\frac{1}{i\left(m-i\right)^{1/4}} \\
&\quad\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right| \\
&\leq\sum_{1\leq j<n/2}\ \sum_{1\leq i<m/2}\frac{1}{ji} \\
&\quad\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right| \\
&\quad+\frac{1}{m^{3/4}}\sum_{1\leq j<n/2}\ \sum_{m/2\leq i<m-1}\frac{1}{j\left(m-i\right)^{1/4}} \\
&\quad\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|
\end{aligned}\tag{41}$$

$$
\begin{aligned}
&+\frac{1}{n^{3/4}}\sum_{n/2\leq j<n-1}\sum_{1\leq i<m/2}\frac{1}{(n-j)^{1/4}\,i}\\
&\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|\\
&+\frac{1}{(nm)^{3/4}}\sum_{n/2\leq j<n-1}\sum_{m/2\leq i<m-1}\frac{1}{(n-j)^{1/4}\,(m-j)^{1/4}}\\
&\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|\\
\leq&\sum_{1\leq j<n/2}\sum_{1\leq i<m/2}\frac{1}{ji}\\
&\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|\\
&+\sum_{1\leq j<n/2}\sum_{m/2\leq i<m-1}\frac{1}{j\,(m-i)}\\
&\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|\\
&+\sum_{n/2\leq j<n-1}\sum_{1\leq i<m/2}\frac{1}{(n-j)\,i}\\
&\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|\\
&+\sum_{n/2\leq j<n-1}\sum_{m/2\leq i<m-1}\frac{1}{(n-j)\,(m-j)}\\
&\times\left|f\left(s+s_j,t+t_i\right)-f\left(s_j,t+t_i\right)-f\left(s+s_j,t_i\right)+f\left(s_j,t_i\right)\right|\\
=&\,IV_1+IV_2+IV_3+IV_4.
\end{aligned}
$$

It is easy to show that

$$
\begin{aligned}
IV_1 \leq& \min_{1\leq l<n}\min_{1\leq r<m}\left\{\omega_{12}\left(f;\frac{1}{n},\frac{1}{m}\right)\log\left(l+1\right)\log\left(r+1\right)\right.\\
&\left.+\left\{i+l\right\}\left\{j\right\}V_{1,2}\left(f;I^2\right)+\left\{i\right\}\left\{j+r\right\}V_{1,2}\left(f;I^2\right)\right\}\\
=&\,o\left(1\right)\quad\text{as}\ \ n,m\rightarrow\infty
\end{aligned}
\tag{42}
$$

uniformly with respect to $(x,y)\in\left[-1+\varepsilon,1-\varepsilon\right]^2$.

Analogously, we can prove that

$$
IV_i=o\left(1\right)\quad\text{as}\ \ n,m\rightarrow\infty,i=2,3,4 \tag{43}
$$

uniformly with respect to $(x,y)\in\left[-1+\varepsilon,1-\varepsilon\right]^2$.

Combining (40), (41), (42) and (43) we get

$$
III_1=o\left(1\right)\quad\text{as}\ \ n,m\rightarrow\infty \tag{44}
$$

uniformly with respect to $(x,y)\in\left[-1+\varepsilon,1-\varepsilon\right]^2$.

Finally, we estimate III_4. By Abel's transformation we have

$$\begin{aligned}
&III_4 \qquad (45)\\
&=\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\left(f\left(s_j,t_i\right)-f\left(s_{j+1},t_i\right)-f\left(s_j,t_{i+1}\right)+f\left(s_{j+1},t_{i+1}\right)\right)\\
&\times\sum_{k=j+1}^{n-1}\sum_{l=i+1}^{m-1}\int\limits_{s_k}^{s_{k+1}}\int\limits_{t_{l+1}}^{t_l}K_n\left(x,s\right)K_m\left(y,t\right)dsdt\\
&+\sum_{i=1}^{m-2}\left(f\left(s_1,t_{i+1}\right)-f\left(s_1,t_i\right)\right)\sum_{l=i+1}^{m-1}\int\limits_{s_0}^{s_1}\int\limits_{t_{l+1}}^{t_l}K_n\left(x,s\right)K_m\left(y,t\right)dsdt\\
&+\sum_{j=1}^{n-2}\left(f\left(s_{j+1},t_1\right)-f\left(s_j,t_1\right)\right)\sum_{k=j+1}^{n-1}\int\limits_{s_k}^{s_{k+1}}\int\limits_{t_1}^{t_0}K_n\left(x,s\right)K_m\left(y,t\right)dsdt\\
&+g\left(s_1,t_1\right)\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}\int\limits_{s_j}^{s_{j+1}}\int\limits_{t_{i+1}}^{t_i}K_n\left(x,s\right)K_m\left(y,t\right)dsdt\\
&=III_{41}+III_{42}+III_{43}+III_{44}.
\end{aligned}$$

From (6), (7), (18), (19) and (41) we have

$$\begin{aligned}
&\left|III_{41}\right| \qquad (46)\\
&\leq\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\left|f\left(s_j,t_i\right)-f\left(s_{j+1},t_i\right)-f\left(s_j,t_{i+1}\right)+f\left(s_{j+1},t_{i+1}\right)\right|\\
&\times\left|\int\limits_{s_{j+1}}^{1}\int\limits_{-1}^{t_i}K_n\left(x,s\right)K_m\left(y,t\right)dsdt\right|\\
&\leq\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\left|f\left(s_j,t_i\right)-f\left(s_{j+1},t_i\right)-f\left(s_j,t_{i+1}\right)+f\left(s_{j+1},t_{i+1}\right)\right|\\
&\times\frac{1}{n\left(s_{j+1}-x\right)}\frac{1}{m\left(y-t_i\right)}\\
&\leq\sum_{j=1}^{n-2}\sum_{i=1}^{m-2}\frac{\left|f\left(s_j,t_i\right)-f\left(s_{j+1},t_i\right)-f\left(s_j,t_{i+1}\right)+f\left(s_{j+1},t_{i+1}\right)\right|}{ij}\\
&=o\left(1\right)\quad\text{as}\quad n,m\to\infty
\end{aligned}$$

uniformly with respect to $(x,y)\in\left[-1+\varepsilon,1-\varepsilon\right]^2$.

$$\left|III_{42}\right| \leq \sum_{i=1}^{m-2} \left|f\left(s_1,t_{i+1}\right)-f\left(s_1,t_i\right)\right| \tag{47}$$

$$\times \int_{x}^{x+\frac{1-x}{n}} \left|K_n\left(x,s\right)\right| ds \left|\int_{-1}^{t_{i+1}} K_m\left(y,t\right) dt\right|$$

$$\leq c\left(\varepsilon\right) \sum_{i=1}^{m-2} \frac{\left|f\left(s_1,t_{i+1}\right)-f\left(s_1,t_i\right)\right|}{i}$$

$$= o\left(1\right) \quad \text{as} \ \ n,m \to \infty$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Analogously, we can prove that

$$III_{43} = o\left(1\right) \quad \text{as} \ \ n,m \to \infty \tag{48}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$,

$$\left|III_{44}\right| \leq \left|f\left(s_1,t_1\right)-f\left(x,y\right)\right| \left|\int_{s_1}^{1} K_m\left(y,t\right) dt\right| \left|\int_{-1}^{t_1} K_n\left(x,s\right) ds\right| \tag{49}$$

$$\leq c\left(\varepsilon\right)\left(\omega_1\left(f;\frac{1}{n}\right)+\omega_2\left(f;\frac{1}{m}\right)\right)$$

$$= o\left(1\right) \quad \text{as} \ \ n,m \to \infty$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

From (45)-(48) we have

$$III_4 = o\left(1\right) \quad \text{as} \ \ n,m \to \infty \tag{50}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

By (28), (38), (39), (44) and (50) we obtain

$$II_3 = o\left(1\right) \quad \text{as} \ \ n,m \to \infty \tag{51}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

From (14), (25), (26), (27) and (51) we conclude that

$$I_1 = o\left(1\right) \quad \text{as} \ \ n,m \to \infty \tag{52}$$

uniformly with respect to $(x,y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Analogously we can prove that

$$I_i = o\left(1\right) \quad \text{as} \ \ n,m \to \infty, i=2,3,4 \tag{53}$$

uniformly with respect to $(x, y) \in [-1+\varepsilon, 1-\varepsilon]^2$.

Combining (13), (52) and (53) we complete the proof of Theorem 3.1. □

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