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Toplict Graph of a Tree

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In this communications, the concept of Toplict graph of a tree is introduced. We present characterization of graphs whose Toplict graph of a tree is planar, maximal outerplanar, minimally nonouterplanar. Further, Also we establish a characterization of graphs whose Toplict graph of a tree is Eularian and Hamiltonian.

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1. Introduction

The concept of pathos of a graph G was introduced by Harary [3], as a collection of minimum number of line disjoint open paths whose union is G. The path number of a graph G is the number of paths in pathos. The path number of a tree T is equal to k where 2k is the number of odd degree vertices of T. Also the end vertices of each path of any pathos of a tree are of odd degree.

The Toplict graph of a tree T denoted by $T_n(T)$ is defined as the graph whose vertex set is the union of the set of edges, set of cutvertices and set of paths of pathos of T in which two vertices are adjacent if and only if the corresponding edges of T are adjacent, edges are incident to the cutvertex, the edge lies on the corresponding path p_i of pathos and two pathos have a common vertex. Since the system of path of pathos for a tree T is not unique, the corresponding Toplict graph of a tree T is either not unique.

The edgedegree of an edge uv of a tree T is the sum of the degrees of u and v. The pathoslength is the number of edges which lie on a particular path P_i of pathos of T. A pendent pathos is a path P_i of pathos having unit length which corresponds to a pendent edge in T. A pathosvertex is a vertex in $T_L(T)$ corresponding to the path P_i of pathos in T. A graph is said to be minimally nonouterplanar if $i(G) = 1$.

All graphs considered here are finite, undirected and simple. We refer [3] for unexplained terminology and notation..

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2. Preliminaries

The following will be useful in the proof of our results.

Theorem 2.1 : *[3] If G is a (p,q) graph whose vertices have degree d_i then L(G) has q vertices and $q_L$edges, where $q_L = -q + \frac{1}{2}\sum d_i^2$ edges.*

Theorem 2.2 : *[3] A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 2.3 : *[3] A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.*

Theorem 2.4 : *[5] A graph G is nonempty path if and only if it is a connected graph with $p \geqq 2$ vertices and $\sum d_i^2 - 4p + 6 = 0$.*

Theorem 2.5 : *[5] The line graph L(G) of a graph is planar if and only if G is planar, $\Delta(G) \leq 4$and if $degv = 4$, for a vertex v of G, then v is a cutvertex.*

Theorem 2.6 : *[3] The line graph L(G) of a graph G is outerplanar if and only if degree of each vertex of G is at most three and every vertex of degree three is a cutvertex.*

Theorem 2.7 : *[5] Every maximal outerplanar graph G with p vertices has $(2p-3)$ edges.*

3. Toplict graph of a tree

Proposition 3.1: *The edgedegree of an edge uv in a tree is odd if the degree of one vertex is of odd degree.*

Proposition 3.2: *If the edge degree of an edge uv in a tree T is even (or odd) and u and v are cutvertices then the corresponding vertex in $\mathrm{T}_n(T)$ is of even (odd)degree.*

Proposition 3.3: *The degree of the pathosvertex in $\mathrm{T}_n(T)$ is equal to the pathoslength of the corresponding path p_i of pathos of T plus one.*

Proposition 3.4: *If the edge degree of an edge in a tree T is even (or odd) and edge is an pendent edge then the corresponding vertex in $\mathrm{T}_n(T)$ is of odd (or even) degree.*

In the following theorem we obtain the number of vertices and edges in $T_n(T)$.

Theorem 3.5 : *If T is a tree with p vertices, q edges whose vertices have degree d_i and cutvertices c have degree c_j then the Toplict graph of a tree* $\mathrm{T}_n(T)$ *has* $(q+k+c)$ *vertices and* $\frac{1}{2}\sum d_i^2 + [n_i(n_i-1)/2] + \sum c_j$ *edges where n_i be the degree of the star v_i and k be the path number.*

Proof: By definition of $T_n(T)$, the number of vertices is the sum of the number of edges, number of cutvetices and the number of pathos of T. Hence $T_n(T)$ has (p+k+c) vertices. The number of edges in $T_n(T)$ is the sum of the number of edges in L(T), the number of edges which lie on the paths p_i of pathos of T which is q, the number of edges incident with cutvertices in T and the number of vertices n_i which lies on the pathos p_i which is $[n_i(n_i-1)/2]$. Hence the number of edges in $T_L(T)$ is $-q+\frac{1}{2}\sum d_i^2+q+[n_i(n_i-1)/2]+\sum c_j = = \frac{1}{2}\sum d_i^2+[n_i(n_i-1)/2]+\sum c_j$. □

Theorem 3.6 : *For any tree $T \neq K_2$,* $\mathrm{T}_n(T)$ *is non separable.*

Proof: Suppose $T_n(T)$ is non separable. If T has $m \geq 3$ cutvertices and n pendent pathos, we have the following cases.

Case 1. For $m=1, T=K_{1,n}$, we have the following the subcases of case1.

subcase 1.1 If p is even then n = 0, the number of cutvertices in $T_n(T)=0$.

Subcase 1.2 If p is odd then n = 0 the pendent pathos is adjacent to the remaining pathos. Clearly the number of cutvertices in $T_n(T)$=0.

Case 2. Let $m>1$, with n pendent pathos. In L(T) each block is a complete subgraph $\langle K_p \rangle$ of L(T) and $T_n(T)$ and this does not increase the number of cutvertex. Since at least two pathos have a vertex in common, so $T_n(T)$ does not contain any pendent vertex. Hence it is nonseparable.

□

Theorem 3.7 : *For any tree T,* $\mathrm{T}_n(T)$ *is planar if and only if* $\Delta(T) \leq 3$.

Proof: Suppose $T_n(T)$ is planar. Assume that $\Delta(T) \geq 4$. If there exists a vertex v of degree 4 in T, then by theorem 2.5, L(T) has K_4 as a subgraph. Also by the definition of $T_n(T)$ the vertex v is adjacent to all the vertices of K_4 and this gives K_5 as subgraph and two edges lie on one path of pathos and both pathos have a vertex v in common. Clearly $T_n(T)$ has $\langle K_5 \rangle$ as a subgraph which is nonplanar, a contradiction.

For sufficiency, suppose every vertex of T lies on at most three edges. By the definition of lict graph it is planar. Each block of lict graph is either K_3 or K_4. The pathosvertex is adjacent to atmost two vertices of each block of lict graph.The edges joining these blocks from the pathosvertices are adjacent to at most two vertices of each block of L(T). Also the adjacency between the pathos does not loose the planarity. This gives planar $T_n(T)$. □

Theorem 3.8 : *For any tree T,* $\mathrm{T}_n(T)$ *is outerplanar if and only if T is a path.*

Proof : Suppose $T_n(T)$ is outerplanar. Assume tha T has a vertex v of degree 3. The edges incident to v and the cutvertex v form K_4 as a subgraph in $T_n(T)$. Hence $T_n(T)$ is nonouterplanar, a contradiction.

Conversely, suppose T is a path P_i of length $t \geq 1$. For $t = 1$, the result is obvious. For $t > 1$, by definition each block of lict graph is $K_3$and has (t -1) blocks. Also T has exactly one path of pathos and the pathosvertex is adjacent to atmost two vertices of each block of lict graph. The pathosvertex together with each block form (t-1) number of $K_4 - x$ subgraphs in $T_n(T)$. Hence $T_n(T)$ is outerplanar. □

Theorem 3.9 : *For any tree T,* $\mathrm{T}_n(T)$ *is maximal outerplanar if and only if T is a path.*

Proof : Suppose $T_n(T)$ is a maximal outerplanar. Then by theorem 3.5 $T_n(T)$ is non separable and hence T is connected.

If $T_n(T)$ is K_2, then T is so. Let T be any connected tree with $p \geq 2$ vertices, q edges,c cutvertices and path number k. Clearly $T_n(T)$ has $(q + k + c)$ vertices and $q_L = \frac{1}{2}\sum d_i^2 + [n_i(n_i - 1)/2] + \sum c_j$ edges. Since $T_n(T)$ is maximal outerplanar, by theorem 2.7, $T_n(T)$ has $2(q + k + c) - 3$ edges. Hence $\frac{1}{2}\sum d_i^2 + [n_i(n_i - 1)/2] + \sum c_j = 2(q + k + c) - 3$. For any path T, $k = 1$, $n_i = 0$ and $\sum c_j = 2c$. We have $\frac{1}{2}\sum d_i^2 + 0 + \sum c_j = 2(q + k + c) - 3. \Rightarrow \frac{1}{2}\sum d_i^2 + 2c = 2q + 2k + 2c - 3 = 2q - 1.$ $\Rightarrow \frac{1}{2}\sum d_i^2 = 2(p - 1) - 1 = 2p - 3. \Rightarrow \sum d_i^2 = 4p - 6$ is true. $\Rightarrow \sum d_i^2 - 4p + 6 = 0.$ By theorem 2.4, it follows that T is a non empty path. Hence necessity is proved. For sufficiency, suppose T is a path. We have considered two cases.

Case(1). Suppose T is K_2. Then $T_n(T)$ is K_2. Hence it is maximal outerplanar.

Case(2). Suppose T is a nonempty path. We prove that $T_n(T)$ maximal outerplanar by an induction on the number of vertices $n \geq 3$ of T. Clearly $T_n(P_3)$, which is maximal outerplanar. By definition, $T_n(T)$ of a path with 3 vertices is $K_4 - x$, which is maximal outerplanar. As the inductive hypothesis, let the $T_n(T)$ of a non empty path T with n vertices be maximal outerplanar. We now show that the $T_n(T)$ of a path T' with (n + 1) vertices is maximal outerplanar. First we prove that it is outerplanar.

Let the vertex and edge sequence of the path T' be $v_1e_1v_2e_2...v_{n-1}e_{n-1}e_nv_{n+1}$. Without loss of generality, $T' - v_{n+1} = T$. By inductive hypothesis, $T_n(T)$ is maximal outerplanar. Now the vertex v_{n+1} is one vertex more in $T'_n(T)$ than in $T_n(T)$. Also there are only four edges $(e_{n-1}, e_n), (e_{n-1}, c_{n-1}), (c_{n-1}, e_n)$ and (e_n, p_i) more in $T_n(T)$. Clearly the induced subgraph on the vertices e_{n-1}, e_n, c_{n-1}, R is not K_4 . Hence $T_n(T)$ is outerplanar. Since $T_n(T)$ is maximal outerplanar, it has $2(q + c + 1) - 3$ edges. The outerplanar graph $T_n(T)$ has $2(q + c + 1) - 3 + 4 = 2[(q + 1) + (c + 1) + 1] - 3$ edges. Hence $T_n(T)$ is maximal outerplanar.

□

Theorem 3.10: *For any tree T,* $\mathrm{T}_n(T)$ *is not minimally nonouterplanar*

Proof: Proof follows from the above theorem. □

Theorem 3.11: *For any tree T,* $\mathrm{T}_n(T)$ *is always noneulerian.*

Proof: We have the following cases.

Case 1. Suppose every edge of a tree T is of edgedegree odd. By proposion 3.2, T contains alternative even and odd degree vertices. Since every path of pathos starts and ends at odd degree vertices, the path must pass through at least one vertex of even degree. By definition of $T_n(T)$, degree of each vertex in $T_n(T)$, except pathos vertex is even. Since the corresponding pathos are adjacent. Hence $T_n(T)$ is noneulerian.

Case 2. Suppose at least one edge of edgedegree even. Then $T_n(T)$ contains at least one vertex of odd degree. Hence $T_n(T)$ is non eulerian. □

Theorem 3.12: *For any tree T,* $\mathrm{T}_n(T)$ *is always Hamiltonian.*

Proof: We have the following cases.

Case 1. If T is a path then it has exactly one path of pathos. Let $V[L(T)] = e_1, e_2, ...e_n$. In $T_n(T)$, the pathosvertex w is adjacent to $e_1, e_2...e_n$. Hence $V[T_n(T)] = e_1, e_2...e_n, c_1, c_2...c_j \cup w$ form a closed path $w, e_1, e_2...e_n, c_1, c_2...c_j \cup w$ containing all the vertices of $T_n(T)$, clearly $T_n(T)$ is hamiltonian.

Case 2. T has cutvertex of even degree and is not a path. Again we consider the following sub cases.

Subcase 2.1. T has exactly one vertex of even degree and is a star with n vertices. Then $L(T) = K_n$, which contains a hamiltonian cycle $e_1, e_2 ... e_n, e_1$. For K_n it has $n/2$ paths of pathos with pathosvertex as $p_1, p_2, ... p_{n/2}$. By definition of $T_n(T)$, each pathosvertex is adjacent to exactly two vertices of L(T). Clearly $V[T_n(T)] = e_1, c_1, e_2, c_2 ... e_{n-1}, c_{n-2}, e_n, w \cup p_1, p_2, ... p_{n/2}$. Then there exists a cycle containing all the vertices of $T_L(T)$ and is a hamiltonian cycle. Hence $T_n(T)$ is hamiltonian.

Subcase 2.2. T has more than one cutvertex of even degree. Then in L(T), each block is complete and every cutvertex lies on exactly two blocks of L(T). Let $V[L(T)] = e_1, e_2 ... e_n$. Since T has p_i pathosvertices,$i > 1$, then V[L(T)] contains $e_1, e_2, ... e_n, \cup p_1, p_2, p_{n/2}$ vertices. But each p_i is adjacent to exactly two vertices of e_n and it forms a cycle $p_1, e_1, c_1, e_2, c_2,\ e_3, e_4, p_2, e_{n-1}, e_{n-1}, c_n, p_1$ containing all vertices of $T_n(T)$. Hence $T_n(T)$ is hamiltonian.

Case 3. Suppose T has at least one odd degree vertex. If T has exactly one cutvertex which is of odd degree, then $G = K_{1,n}$ and $L(T) = K_n$ and a number of path of pathos is $\frac{n+1}{2}$ in which there exists at least one path of pathos p_i, $1 \leq i \leq \frac{n+1}{2}$ incident to v and an end vertex of T. In $T_n(T)$, each pathosvertex is adjacent to exactly two vertices of K_n. Clearly there exist one vertex of K_n joined by a pathosvertex $p_j, j < i$ and the paths p_i, p_j are adjacent if both have a vertex v in common. Clearly $e_1, e_2, e_3, ... p_i, p_j, e_1$,form a cycle contains all vertices of $T_n(T)$. Hence $T_n(T)$ is hamiltonian.

Case 4. Suppose T has more than one cutvertex of odd degree. Since every path of pathos starts and ends at odd degree vertices. In drawing different paths of pathos there exists at least one path of pathos p_i which starts from $\Delta(T)$ and ends at an end vertex of T. Then in $T_n(T)$, $p_i \in V[T_n(T)]$, let $e_1, e_2, ... e_i$ be the number of edges incident with vertices of $\Delta(T)$. Then $e_1, e_2, ... e_n$ form a complete subgraph. In $T_n(T)$, p_i is adjacent with exactly one vertex of e_i and the vertex p_j which is a common vertex with p_i. Hence $p_1, e_1, c_1, e_2, c_2, e_3, e_4, p_2, e_{n-1}, e_{n-1}, c_n, p_1$ form a cycle containing all vertices of $T_n(T)$. Hence $T_L(T)$ is hamiltonian. $\square$

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