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Hedging of the European Option of the Exotic Type with a Nonsmooth Payoff Function

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We consider an Exotic European Option in the case of Black-Scholes financial market model, whose payoff function is a certain combination of payoff functions of the Binary and Asian options and investigate the hedging problem. To achieve this aim, the Clark-Ocone stochastic integral representation formula for the corresponding path-dependent Wiener functional with the explicit form of integrand is given also, which is based on the generalization of the Clark-Ocone's formula, obtained by us earlier.

Keywords: European Option, Payoff function, Hedging problem, Clark's integral representation formula, Stochastic (Malliavin) derivative, Clark-Ocone's formula.

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1. Introduction

The payoff functions of derivative securities with more complicated forms than standard European or American call and put options are known as Exotic Options. One of the Exotic Options of this kind is the so-called Binary Option. It is an option with discontinuous payoff function. The simplest examples of the Binary Options are call and put options "cash or nothing". The payoff function of the call option has the form $BC_T = QI_{\{S_T>K\}}$, and for the put option $BC_T = QI_{\{S_T<K\}}$, where K is the strike price at the time of execution T. The Binary Option "an asset or nothing" is also commonly used. There are the same conditions as in "cash or nothing" option, but the difference is that the owner of the call option receives price of the asset S_T instead of amount Q. The Standard European Call Option $((S_T - K)^+)$ is equivalent to a long position (the bought asset) in the "asset or nothing" option and short position (the sold asset) in the "cash or nothing" option, when $Q = K$.

Moreover, so-called Asian Options are also type of Exotic Option. The payoff function of this option depends on average value of the price of an asset during the certain period of option life time. The payoff function of the Asian Option by

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definition has the following representation: $C_T^A = (A_S(T_0, T) - K)^+$, where

$$A_S(T_0, T) = \frac{1}{T - T_0} \int_{T_0}^{T} S_t dt$$

is arithmetic mean of the prices of asset on the time interval $[T_0, T]$, K is a strike and $S = (S_t)$ $(0 \le t \le T)$ is a geometrical Brownian motion. The main difficulty in pricing and hedging the Asian Option is that the random variable $A_S(T_0, T)$ is not lognormal distributed and therefore, it is rather difficult to obtain explicit formulas of pricing of this option.

Beginning with the works of Harrison and Pliska ([1], [2]) which showed that the stochastic integral representation theorem of Wiener functional (also known as martingale representation theorem) and the Girsanov change of probability measure are the "keys" to understanding option pricing in the celebrated Black and Scholes model, these methodologies have been applied with considerable success to questions of various problems of modern financial mathematics. Instrumental in obtaining these representations is an extension of the familar Clark formula ([3]-[5]). The purpose of this paper is to derive the representation formula for the optimal portfolios associated with option pricing for some Exotic Type European Option with stochastically nonsmooth payoff function.

Let W_t $(t \in [0, T])$ be a standard Wiener process and let $\Im_t^W$ be a natural filtration generated by this Wiener process. If F is a square integrable $\Im_t^W$-measurable random variable, then there exists a unique $\{\Im_t^W\}$-adapted square integrable in $L_2([0, T])$ random process ψ_t such that (see [3])

$$F = \boldsymbol{E}F + \int_0^T \psi_t dW_t.$$

Karatzas and Ocone showed in [6] how to use the Ocone-Haussmann-Clark formula in financial mathematics, in particular, for constructing hedging strategies in the complete financial markets driven by a Wiener process. Since that time interest to Malliavin calculus has been significantly increasing. Therefore developing of the theory has intensively begun together with looking for the new sphere of its applications ([7]-[9]). Among them the applications in mathematical statistics are especially important (regularity of density, hypothesis testing).

At the same time, finding explicit expression for ψ_t is a very difficult problem. In this direction, one general result is known, called Ocone-Clark formula (see [5]), according to which $\psi_t = \boldsymbol{E}(D_t F | \Im_t^W)$, where D_t is the so called Malliavin stochastic derivative. But, on the one hand, here the stochastically smoothness of considered functional is required and on the other hand, even in case of smoothness, calculations of Malliavin derivative and conditional mathematical expectation are rather difficult.

Absolutely different method for finding ψ_t was offered by Shyriaev, Yor and Graversen ([10], [11]). This method is based on using Ito's (generalized) formula and Levy's theorem for Levy's martingale $m_t = \boldsymbol{E}(F | \Im_t^W)$ associated to F. Our earlier approach (see Jaoshvili, Purtukhia [12]) within the classical Ito's calculus allows to construct ψ_t explicitly by using both the standard L_2 theory and the theory of weighted Sobolev spaces, in the case when the functional F has not stochastic derivative (in particular, the class of functionals considered by us includes, for example, the functional $F = I_{\{W_T > c\}}$ which is not stochastically differentiable).

We have developed some methods of obtaining the stochastic integral representation of nonsmooth (in the Malliavin sense) Wiener functionals and their applications to the problems of hedging of European Options ([12]-[17]). In particular, in [13] we consider the path-dependent Wiener functional

$$F = (W_T - K)^- I_{\{\min\limits_{0\le t\le T} W_t \le c\}}$$

which isn't stochastically smooth. For this functional the stochastic integral representation formula with an explicit form of integrand is obtained. Note that this functional is a typical example of payoff function for so called European barrier[1] and lookback[2] Options. Hence, obtained there stochastic integral representation formula could be used for calculating the explicit hedging portfolio of such barrier and lookback option.

It turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation ([14]). In particular, we (with prof. O. Glonti) generalized the Clark-Ocone formula in case, when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable. The method of finding this integrand under this condition was established. It is well-known that if random variable is stochastically differentiable in the Malliavin sense, then its conditional mathematical expectation is differentiable too ([18]). In particular, if $F \in D_{2,1}$, then $\boldsymbol{E}(F|\Im_s^w) \in D_{2,1}$ and $D_t[\boldsymbol{E}(F|\Im_s^W)] = \boldsymbol{E}(D_t F|\Im_s^W) I_{[0,s]}(t)$. On the other hand, it is possible that conditional expectation can be smooth even if the random variable is not stochastically smooth ([14]). For example, it is well-known that $I_{\{w_T \le c\}} \notin D_{2,1}$ (indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one ([18])), but for all $t \in [0, T)$:

$$\boldsymbol{E}[I_{\{W_T \le c\}}|\Im_t^W] = \varPhi\left(\frac{c - W_t}{\sqrt{T-t}}\right) \in D_{2,1}.$$

On the other hand, of course, there are also nonsmooth functionals whose conditional mathematical expectation is not stochastically differentiable too. In particular, we consider in [16] the functional of the integral type $\int_0^T u_s(\omega)ds$ with the nonsmooth integrand $u_s(\omega)$. It is well-known that if $u_s(\omega)$ is not differentiable in the Malliavin sense, then the Lebesgue average (with respect to ds) is not differentiable in the Malliavin sense either (see [16], Theorem 2). On the other hand, in this case even the conditional mathematical expectation of the mentioned functional is not smooth, because we have:

$$\boldsymbol{E}\Big[\int_0^T u_s(\omega)ds|\Im_t^W\Big] = \int_0^t u_s(\omega)ds + \int_t^T \boldsymbol{E}[u_s(\omega)|\Im_t^W]ds,$$

[1]The barrier option is either nullified, activated or exercised when the underlying asset price breaches a barrier during the life of the option.
[2]The payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option.

where the first summand is not differentiable, but the second summand is differentiable in the Malliavin sense (if $u_s(\cdot) \in D_{2,1}$ for almost all s and $u_\cdot(\omega)$ is Lebesgue integrable for a.a. ω, then $\int_t^T u_s(\omega)ds \in D_{2,1}$ (see [19])). It should be noted that such type of integral functionals have been considered in our previous works ([15]-[17]). In [17] the method of hedging of option based on the using the local time of the risky asset price S was developed, but this approach isn't applicable here. So, firstly, we derived the Clark stochastic integral representation of local time. Then using the relation between payoff of option and local time (the Trotter-Meyer Theorem), based on the stochastic type Fubini theorem, we obtain the Clark integral representation for the integral type payoff function $\int_0^T I_{\{a \le S_t \le b\}} S_t^2 dt$ of European option. Finally, we solved the corresponding hedging problem in the case of Black-Scholes model with zero interest rate. In [15] the method for hedging of European Option with payoff function $\int_0^T I_{\{c_1 \le S_t \le c_2\}} dt$ in the case of Bachelier market model was elaborated.

In the present work we consider an Exotic Option which is a certain combination of the Binary and Asian Options. The hedging problem for this type of options is investigated. In particular, we study the European Option with payoff function $\int_0^T S_t I_{\{c_1 \le S_t \le c_2\}} dt$, where S_t is a geometrical Brownian motion and $c_1 < c_2$ are some real numbers. For this purpose the Clark stochastic integral representation for such a payoff function with the explicit form of integrand is obtained.

2. Auxiliary results

Let the standard Wiener process $W = (W_t)$, $t \in [0,T]$ on the probability space $(\Omega, \Im, P)$ be given, and let $(\Im_t^W)$, $t \in [0,T]$, be the natural filtration generated by the Wiener process W.

Let $p(u, t, W_u, A)$ be the transition probability of the Wiener process W, i.e. $P[W_t \in A | W_u] = p(u, t, W_u, A)$, where $0 \le u \le t$, A is a Borel subset of R and

$$p(u,t,x,A) = \frac{1}{\sqrt{2\pi(t-u)}} \int_A \exp\Big\{ -\frac{(y-x)^2}{2(t-u)} \Big\} dy.$$

For the computation of conditional mathematical expectation below we use the following statement:

Proposition 2.1: *For any bounded or positive measurable function f we have the relation*

$$\boldsymbol{E}[f(W_t)|W_u] = \int_R f(y) p(u,t,W_u,dy) \quad (P-a.s.).$$

We will denote (see, [18]) domain of stochastic (Malliavin) derivative operator D by $D_{2,1}$. That means, $D_{2,1}$ is equal to the adherence of the class of smooth random variables S [1] with respect to the norm

[1]Here S denotes the class of a random variables which has the form $F = f(W_{t_1}, ..., W_{t_n})$, $f \in C_p^\infty(R^n)$, $t_i \in [0,T]$, $n \ge 1$, where $C_p^\infty(R^n)$ is the set of all infinitely continuously differentiable functions $f : R^n \to R$ such that f and all of its partial derivatives have polynomial growth. In addition, we will note here if $F \in S$,

$$||F||_{2,1} = ||F||_{L_2(\Omega)} + |||DF|||_{L_2(\Omega;L_2([0,T]))}.$$

The following rule of stochastic differentiation of the composite function is valid

Proposition 2.2: *Let $\psi : R^m \to R^1$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, ..., F^m)$ is a random vector whose components belong to the space $D_{2,1}$. Then $\psi(F) \in D_{2,1}$, and*

$$D_t\psi(F) = \sum_{i=1}^{m} \frac{\partial}{\partial x^i}\psi(F)D_tF^i.$$

(see, [18], Proposition 1.2.3.).

Theorem 2.3: *Suppose that $g_t = \boldsymbol{E}[F|\Im_t^W]$ is Malliavin differentiable ($g_t(\cdot) \in D_{2,1}$) for almost all $t \in [0,T)$. Then we have the stochastic integral representation*

$$g_T = F = \boldsymbol{E}F + \int_0^T \upsilon_u dW_u \quad (P - a.s.),$$

where

$$\upsilon_u := \lim_{t\uparrow T} \boldsymbol{E}[D_u g_t|\Im_u^W] \quad in\ the \quad L_2([0,T]\times\Omega)$$

(see, [14], Theorem 1]).

Denote

$$S_t = \exp\{\sigma W_t + (\mu - \sigma^2/2)t\}.$$

Theorem 2.4: *For any real number $c > 0$ and $\tau \in (0,T]$ the random variable $S_\tau I_{\{S_\tau \le c\}}$ has the following stochastic integral representation*

$$S_\tau I_{\{S_\tau\le c\}} = \exp\{\mu\tau\}\Phi\Big(\frac{\ln c - \mu\tau - \sigma^2\tau/2}{\sigma\sqrt{\tau}}\Big) - \int_0^\tau \frac{c}{\sqrt{\tau-u}}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau-u}}\Big)dW_u$$

$$+\sigma\int_0^\tau \Big[\exp\Big\{\frac{2\mu\tau + 2\sigma W_u - \sigma^2 u}{2}\Big\}\Phi\Big(\frac{\ln c - \nu\tau - \sigma W_u - \sigma^2(\tau-u)}{\sigma\sqrt{\tau-u}}\Big)\Big]dW_u, \quad (1)$$

where $\nu = \mu - \sigma^2/2$, Φ is the standard normal distribution function and φ is its density function.

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then

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}((W_{t_1}, ..., W_{t_n}))I_{[0,t_i]}(t).$$

Proof: According to Proposition 2.1, using the standard technique of integration (in particular, highlighting the full square in the argument of the exponential function) and the well-known property of the normal distribution and its density function, it is not difficult to see that

$$g_t^\tau := \boldsymbol{E}[S_\tau I_{\{S_\tau \le c\}}|\Im_t] = \boldsymbol{E}[\exp\{\sigma W_\tau + \nu\tau\} I_{\{W_\tau \le (\ln c - \nu\tau)/\sigma\}}|\Im_t]$$

$$= \frac{1}{\sqrt{2\pi(\tau - t)}} \int_{-\infty}^{\infty} I_{\{x \le (\ln c - \nu\tau)/\sigma\}} \exp\{\sigma x + \nu\tau\} \exp\Big\{ -\frac{(x - W_t)^2}{2(\tau - t)}\Big\} dx$$

$$= \frac{1}{\sqrt{2\pi(\tau - t)}} \int_{-\infty}^{(\ln c - \nu\tau)/\sigma} \exp\Big\{ -\frac{x^2 - 2[W_t + \sigma(\tau - t)]x + W_t^2 - 2\nu\tau(\tau - t)}{2(\tau - t)}\Big\} dx$$

$$= \frac{1}{\sqrt{2\pi(\tau - t)}}$$

$$\times \int_{-\infty}^{(\ln c - \nu\tau)/\sigma} \exp\Big\{ -\frac{[x - W_t - \sigma(\tau - t)]^2 - [W_t + \sigma(\tau - t)]^2 + W_t^2 - 2\nu\tau(\tau - t)}{2(\tau - t)}\Big\} dx$$

$$= \exp\Big\{ \frac{2\nu\tau + 2\sigma W_t + \sigma^2(\tau - t)}{2}\Big\}$$

$$\times \frac{1}{\sqrt{2\pi(\tau - t)}} \int_{-\infty}^{(\ln c - \nu\tau)/\sigma} \exp\Big\{ -\frac{[x - W_t - \sigma(\tau - t)]^2}{2(\tau - t)}\Big\} dx$$

$$= \exp\Big\{ \frac{2\mu\tau + 2\sigma W_t - \sigma^2 t}{2}\Big\} \varPhi\Big(\frac{\ln c - \nu\tau - \sigma W_t - \sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}}\Big). \qquad (2)$$

Therefore, according to Proposition 2.2, the random variable $g_t^\tau = \boldsymbol{E}[S_\tau I_{\{S_\tau \le c\}}|\Im_t]$ is Malliavin differentiable, i.e. $g_t^\tau \in D_{2,1}$ for all $t \in [0, T)$. Hence, due to Theorem 2.4, we have the following stochastic integral representation:

$$S_\tau I_{\{S_\tau \le c\}} = \boldsymbol{E}[S_\tau I_{\{S_\tau \le c\}}] + \int_0^\tau \upsilon_u dW_u \quad (P - a.s.), \qquad (3)$$

where

$$\upsilon_u := \lim_{t \uparrow T} \boldsymbol{E}[D_u g_t^\tau | \Im_u^W] \quad \text{in the} \quad L_2([0,T] \times \Omega). \tag{4}$$

It is obvious that

$$\boldsymbol{E}[S_\tau I_{\{S_\tau \le c\}}] = g_0^\tau = \exp\{\mu\tau\} \Phi(\frac{\ln c - \mu\tau - \sigma^2\tau/2}{\sigma\sqrt{\tau}}). \tag{5}$$

Farther, due to Proposition 2.2, from relation (2) we can write

$$D_u g_t^\tau = \sigma \exp\Big\{\frac{2\mu\tau + 2\sigma W_t - \sigma^2 t}{2}\Big\} \Phi\Big(\frac{\ln c - \nu\tau - \sigma W_t - \sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}}\Big) I_{[0,t]}(u)$$

$$-\frac{1}{\sqrt{\tau - t}} \exp\Big\{\frac{2\mu\tau + 2\sigma W_t - \sigma^2 t}{2}\Big\} \varphi\Big(\frac{\ln c - \nu\tau - \sigma W_t - \sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}}\Big) I_{[0,t]}(u)$$

$$:= \exp\Big\{\frac{2\mu\tau - \sigma^2 t}{2}\Big\}\Big\{\sigma J_1(t,\tau,W_t) - \frac{1}{\sqrt{\tau - t}} J_2(t,\tau,W_t)\Big\} I_{[0,t]}(u). \tag{6}$$

Using Proposition 2.1, we obtain

$$\boldsymbol{E}[J_1(t,\tau,W_t)|\Im_u] := \boldsymbol{E}\Big[\exp\{\sigma W_t\}\Phi\Big(\frac{\ln c - \nu\tau - \sigma W_t - \sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}}\Big)\Big|\Im_u\Big]$$

$$= \frac{1}{\sqrt{2\pi(t-u)}} \int_{-\infty}^{\infty} \Phi\Big(\frac{\ln c - \nu\tau - \sigma x - \sigma^2(\tau - t)}{\sigma\sqrt{\tau - t}}\Big) \exp\Big\{\sigma x - \frac{(x - W_u)^2}{2(t-u)}\Big\} dx.$$

Therefore, according to the relation

$$\lim_{t \uparrow \tau} \Phi\Big(\frac{y}{\sqrt{\tau - t}}\Big) = \begin{cases} 0, & y < 0; \\ 0.5, & y = 0; \\ 1, & y > 0, \end{cases}$$

using the Lebesgue dominated convergence theorem and the standard technique of integration, we conclude that

$$\lim_{t \uparrow \tau} \boldsymbol{E}\Big[\exp\Big\{\frac{2\mu\tau - \sigma^2 t}{2}\Big\} \sigma J_1(t,\tau,W_t) I_{[0,t]}(u) | \Im_u\Big]$$

$$= \frac{\sigma \exp\{\nu\tau\}}{\sqrt{2\pi(\tau - u)}} \int_{-\infty}^{\infty} I_{\{\ln c - \nu\tau - \sigma x > 0\}} \exp\Big\{\sigma x - \frac{(x - W_u)^2}{2(\tau - u)}\Big\} dx I_{[0,\tau]}(u)$$

$$= \frac{\sigma \exp\{\nu\tau\}}{\sqrt{2\pi(\tau-u)}} \exp\Big\{\frac{2\sigma W_u + \sigma^2(\tau-u)}{2}\Big\}$$

$$\times \int_{-\infty}^{(\ln c - \nu\tau)/\sigma} \exp\Big\{-\frac{[x - W_u - \sigma(\tau-u)]^2}{2(\tau-u)}\Big\} dx I_{[0,\tau]}(u)$$

$$= \sigma \exp\Big\{\frac{2\mu\tau + 2\sigma W_u - \sigma^2 u}{2}\Big\} \varPhi\Big(\frac{\ln c - \nu\tau - \sigma W_u - \sigma^2(\tau-u)}{\sigma\sqrt{\tau-u}}\Big) I_{[0,\tau]}(u). \qquad (7)$$

On the other hand, due to Proposition 2.1, we have

$$\boldsymbol{E}[J_2(t,\tau,W_t)|\Im_u] := \boldsymbol{E}[\exp\{\sigma W_t\}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_t - \sigma^2(\tau-t)}{\sigma\sqrt{\tau-t}}\Big)|\Im_u]$$

$$= \frac{1}{\sqrt{2\pi(t-u)}} \int_{-\infty}^{\infty} \varphi\Big(\frac{\ln c - \nu\tau - \sigma x - \sigma^2(\tau-t)}{\sigma\sqrt{\tau-t}}\Big) \exp\Big\{\sigma x - \frac{(x-W_u)^2}{2(t-u)}\Big\} dx$$

$$= \frac{1}{2\pi\sqrt{t-u}} \int_{-\infty}^{\infty} \exp\Big\{-\frac{[\ln c - \nu\tau - \sigma x - \sigma^2(\tau-t)]^2}{2\sigma^2(\tau-t)} - \frac{(x-W_u)^2 - 2\sigma(t-u)x}{2(t-u)}\Big\} dx$$

$$= \frac{1}{2\pi\sqrt{t-u}} \exp\Big\{-\frac{\alpha^2(t-u) + W_u^2(\tau-t)}{2(\tau-t)(t-u)}\Big\}$$

$$\times \int_{-\infty}^{\infty} \exp\Big\{-\frac{(\tau-u)x^2 + 2[\alpha(t-u) - W_u - \sigma(t-u)](\tau-t)x}{2(\tau-t)(t-u)}\Big\} dx,$$

where

$$\alpha = \sigma(\tau-t) - \frac{\ln c - \nu\tau}{\sigma}.$$

Farther, using again the standard technique of integration and the well-known property of the normal distribution density function, we easily ascertain that

$$\boldsymbol{E}[J_2(t,\tau,W_t)|\Im_u] = \frac{1}{2\pi\sqrt{t-u}}$$

$$\times \int_{-\infty}^{\infty} \exp\Big\{ - \frac{x^2 - 2\frac{(\ln c - \nu\tau)(t-u) + \sigma(\tau - t)W_u}{\sigma(\tau-u)} x + \frac{t-u}{\tau-u}\alpha^2 + \frac{\tau-t}{\tau-u} W_u^2}{2\frac{(\tau-t)(t-u)}{\tau-u}} \Big\} dx$$

$$= \frac{1}{2\pi\sqrt{t-u}} \int_{-\infty}^{\infty} \exp\Big\{ - \frac{[x - \frac{(\ln c - \nu\tau)(t-u) + \sigma(\tau-t)W_u}{\sigma(\tau-u)}]^2}{2\frac{(\tau-t)(t-u)}{\tau-u}} \Big\} dx$$

$$\times \exp\Big\{ - \frac{\frac{t-u}{\tau-u}\alpha^2 + \frac{\tau-t}{\tau-u}W_u^2 - [\frac{(\ln c - \nu\tau)(t-u) + \sigma(\tau-t)W_u}{\sigma(\tau-u)}]^2}{2\frac{(\tau-t)(t-u)}{\tau-u}} \Big\}$$

$$= \frac{1}{2\pi\sqrt{t-u}} \sqrt{2\pi \frac{(\tau-t)(t-u)}{\tau-u}}$$

$$\times \exp\Big\{ - \frac{\sigma^2(\tau-t)(W_u - \frac{\ln c - \nu\tau}{\sigma})^2 - (\tau-u)(\ln c - \nu\tau)^2 + \sigma^2(\tau-u)\alpha^2}{2\sigma^2(\tau-u)(\tau-t)} \Big\}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\tau-t}{\tau-u}} \exp\Big\{ - \frac{(W_u - \frac{\ln c - \nu\tau}{\sigma})^2 + (\tau-u)[\sigma^2(\tau-t) - 2(\ln c - \nu\tau)]}{2(\tau-u)} \Big\}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\tau-t}{\tau-u}} \exp\Big\{ - \frac{(W_u - \frac{\ln c - \nu\tau}{\sigma})^2}{2(\tau-u)} \Big\} \exp\Big\{ \ln c - \nu\tau - \frac{\sigma^2}{2}(\tau-t) \Big\}$$

$$= \sqrt{\frac{\tau-t}{\tau-u}} \exp\Big\{ \ln c - \nu\tau - \frac{\sigma^2}{2}(\tau-t) \Big\} \varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau-u}} \Big).$$

Therefore, we can conclude that

$$\lim_{t\uparrow\tau} \boldsymbol{E}\Big[\exp\Big\{ \frac{2\mu\tau - \sigma^2 t}{2} \Big\} \frac{1}{\sqrt{\tau-t}} J_2(t,\tau,W_t) I_{[0,t]}(u) | \Im_u \Big]$$

$$= \frac{1}{\sqrt{\tau-t}} \varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau-u}} \Big)$$

$$\times \lim_{t\uparrow\tau} \Big[\exp\Big\{ \frac{2\mu\tau - \sigma^2 t}{2} \Big\} \exp\Big\{ \ln c - \nu\tau - \frac{\sigma^2}{2}(\tau-t) \Big\} I_{[0,t]}(u) \Big]$$

$$= \frac{1}{\sqrt{\tau - t}}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau - u}}\Big)\exp\{\nu\tau\}\exp\{\ln c - \nu\tau\}I_{[0,\tau]}(u)$$

$$= \frac{c}{\sqrt{\tau - t}}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau - u}}\Big)I_{[0,\tau]}(u). \tag{8}$$

Combining now relations (4), (6), (7) and (8), we obtain that

$$\upsilon_u^\tau = -\frac{c}{\sqrt{\tau - t}}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau - u}}\Big)I_{[0,\tau]}(u)$$

$$+\sigma\exp\Big\{\frac{2\mu\tau + 2\sigma W_u - \sigma^2 u}{2}\Big\}\varPhi\Big(\frac{\ln c - \nu\tau - \sigma W_u - \sigma^2(\tau - u)}{\sigma\sqrt{\tau - u}}\Big)I_{[0,\tau]}(u).$$

The last relation together with (3), (4) and (5) complete the proof of the theorem. □

Corollary 2.5: *In the case $\mu = \sigma^2/2$ for any real number $c > 0$ and $\tau \in (0, T]$ the random variable $S_\tau I_{\{S_\tau \le c\}}$ has the following stochastic integral representation*

$$S_\tau I_{\{S_\tau \le c\}} = \exp\{\sigma^2\tau/2\}\varPhi\Big(\frac{\ln c - \sigma^2\tau}{\sigma\sqrt{\tau}}\Big) - \int_0^\tau \frac{c}{\sqrt{\tau - u}}\varphi\Big(\frac{\ln c - \sigma W_u}{\sigma\sqrt{\tau - u}}\Big)dW_u$$

$$+\sigma\int_0^\tau \Big[\exp\Big\{\frac{2\sigma W_u + \sigma^2(\tau - u)}{2}\Big\}\varPhi\Big(\frac{\ln c - \sigma W_u - \sigma^2(\tau - u)}{\sigma\sqrt{\tau - u}}\Big)\Big]dW_u. \tag{9}$$

Theorem 2.6: *For any real positive numbers $c_1 < c_2$ the integral type functional $\int_0^T S_t I_{\{c_1 \le S_t \le c_2\}}dt$ admits the following stochastic integral representation*

$$\int_0^T S_t I_{\{c_1 \le S_t \le c_2\}}dt = \Big[\int_0^T \exp\{\mu t\}\varPhi\Big(\frac{\ln c - \mu t - \sigma^2 t/2}{\sigma\sqrt{t}}\Big)dt\Big]\Big|_{c=c_1}^{c_2} \tag{10}$$

$$-\int_0^T \Big\{\Big[\int_u^T \frac{c}{\sqrt{t - u}}\varphi\Big(\frac{\ln c - \nu t - \sigma W_u}{\sigma\sqrt{t - u}}\Big)dt\Big]\Big|_{c=c_1}^{c_2}\Big\}dW_u$$

$$+\sigma\int_0^T \Big\{\Big[\int_u^T \exp\Big\{\frac{2\mu t + 2\sigma W_u - \sigma^2 u}{2}\Big\}\varPhi\Big(\frac{\ln c - \nu t - \sigma W_u - \sigma^2(t - u)}{\sigma\sqrt{t - u}}\Big)dt\Big]\Big|_{c=c_1}^{c_2}\Big\}dW_u.$$

Proof: Integrating both parts of relation (1) with respect to $d\tau$, using the stochastic type Fubini theorem (see [20], Corollary of Lemma IV.2.4), it is not difficult to see that the following stochastic integral representation is fulfilled

$$\int_0^T S_\tau I_{\{c_1 \le S_\tau \le c_2\}} d\tau = \int_0^T \exp\{\mu\tau\}\Phi\Big(\frac{\ln c - \mu\tau - \sigma^2\tau/2}{\sigma\sqrt{\tau}}\Big)d\tau$$

$$-\int_0^T\int_0^\tau \frac{c}{\sqrt{\tau-u}}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau-u}}\Big)dW_u d\tau \tag{11}$$

$$+\sigma\int_0^T\int_0^\tau \exp\Big\{\frac{2\mu\tau + 2\sigma W_u - \sigma^2 u}{2}\Big\}\Phi\Big(\frac{\ln c - \nu\tau - \sigma W_u - \sigma^2(\tau-u)}{\sigma\sqrt{\tau-u}}\Big)dW_u d\tau$$

$$= \int_0^T \exp\{\mu\tau\}\Phi\Big(\frac{\ln c - \mu\tau - \sigma^2\tau/2}{\sigma\sqrt{\tau}}\Big)d\tau - \int_0^T\Big\{\int_u^T \frac{c}{\sqrt{\tau-u}}\varphi\Big(\frac{\ln c - \nu\tau - \sigma W_u}{\sigma\sqrt{\tau-u}}\Big)d\tau\Big\}dW_u$$

$$+\sigma\int_0^T\Big\{\int_u^T \exp\Big\{\frac{2\mu\tau + 2\sigma W_u - \sigma^2 u}{2}\Big\}\Phi\Big(\frac{\ln c - \nu\tau - \sigma W_u - \sigma^2(\tau-u)}{\sigma\sqrt{\tau-u}}\Big)dt\Big\}dW_u.$$

On the other hand, we have

$$\int_0^T S_t I_{\{c_1 \le S_t \le c_2\}} dt = \int_0^T S_t I_{\{S_t \le c_2\}} dt - \int_0^T S_t I_{\{S_t < c_1\}} dt.$$

This relation together with (11) completes the proof of the theorem. □

3. Main results

Let a Wiener process $w = (w_t)$, $t \in [0, T]$ on the probability space $(\Omega, \Im, P)$ be given, and let $(\Im_t^W)$, $t \in [0, T]$ be the natural filtration generated by the Wiener process W. Consider the Black-Scholes model with risk-free asset price evolution described by

$$dB_t = rB_t dt, \;\; B_0 = 1, \tag{12}$$

where $r \ge 0$ is the interest rate and risky asset price evolution

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \;\; S_0 = 1, \tag{13}$$

where $\mu \in R$ is appreciation rate and $\sigma > 0$ is volatility coefficient.

Denote

$$Z_T = exp\Big\{ -\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 T\Big\}$$

and let $\widetilde{P}_T$ be the measure on $(\Omega, \Im_T^W)$ such that

$$d\widetilde{P}_T = Z_T dP.$$

From Girsanov's Theorem it follows that under this measure (martingale risk neutral measure)

$$\widetilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$$

is the standard Wiener process and

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t, \ \ S_0 = 1,$$

or

$$S_t = exp\{\sigma \widetilde{W}_t + (r - \sigma^2/2)t\}. \tag{14}$$

Consider now the problem of "replication" the European Option of Exotic type with the payoff of integral type

$$F = \int_0^T I_{\{c_1 \leq S_t \leq c_2\}} S_t dt, \tag{15}$$

where c_1 and c_2 are some positive constants, $c_1 < c_2$. It means that one needs to find a trading strategy (β_t, γ_t), $t \in [0, T]$, such that the capital process

$$X_t = \beta_t B_t + \gamma_t S_t, \ \ X_T = F, \tag{16}$$

under the self-financing condition

$$dX_t = \beta_t dB_t + \gamma_t dS_t. \tag{17}$$

From relations (1.3), (1.5) and (1.6) we obtain

$$F = X_T = X_0 + \int_0^T r(\beta_t B_t + \gamma_t S_t) dt + \int_0^T \sigma \gamma_t S_t d\widetilde{W}_t. \tag{18}$$

Our goal to find the trading strategy $(\beta, \gamma) = (\beta_t, \gamma_t)$, $t \in [0, T]$.

Taking r and $\widetilde{W}$ instead of μ and W respectively in Theorem 2.6, we can obtain the next result.

Theorem 3.1: *In the scheme (12), (13), for any real positive numbers $c_1 < c_2$, the following stochastic integral representation for the functional F from (15) is true*

$$\int_0^T S_t I_{\{c_1 \le S_t \le c_2\}} dt = \Big[\int_0^T \exp\{rt\} \Phi(\frac{\ln c - rt - \sigma^2 t/2}{\sigma\sqrt{t}}) dt \Big] \Big|_{c=c_1}^{c_2} \tag{19}$$

$$- \int_0^T \Big\{ \Big[\int_u^T \frac{c}{\sqrt{t-u}} \varphi(\frac{\ln c - \widetilde{\nu} t - \sigma \widetilde{W}_u}{\sigma\sqrt{t-u}}) dt \Big] \Big|_{c=c_1}^{c_2} \Big\} d\widetilde{W}_u$$

$$+ \sigma \int_0^T \Big\{ \Big[\int_u^T \exp\Big\{ \frac{2rt + 2\sigma\widetilde{W}_u - \sigma^2 u}{2} \Big\} \Phi\Big(\frac{\ln c - \widetilde{\nu} t - \sigma\widetilde{W}_u - \sigma^2(t-u)}{\sigma\sqrt{t-u}} \Big) dt \Big] \Big|_{c=c_1}^{c_2} \Big\} d\widetilde{W}_u,$$

where $\widetilde{\nu} = r - \sigma^2/2$.

If the interest rate is zero we obtain the next result.

Corollary 3.2: *If $r = 0$, then we have the following stochastic integral representation*

$$\int_0^T S_t I_{\{c_1 \le S_t \le c_2\}} dt = \Big[\int_0^T \Phi\Big(\frac{\ln c - \sigma^2 t/2}{\sigma\sqrt{t}} \Big) dt \Big] \Big|_{c=c_1}^{c_2}$$

$$- \int_0^T \Big\{ \Big[\int_u^T \frac{c}{\sqrt{t-u}} \varphi\Big(\frac{\ln c + \sigma^2 t/2 - \sigma\widetilde{W}_u}{\sigma\sqrt{t-u}} \Big) dt \Big] \Big|_{c=c_1}^{c_2} \Big\} d\widetilde{W}_u$$

$$+ \sigma \int_0^T \Big\{ \Big[\int_u^T \exp\Big\{ \frac{2\sigma\widetilde{W}_u - \sigma^2 u}{2} \Big\} \Phi\Big(\frac{\ln c + \sigma^2 t/2 - \sigma\widetilde{W}_u - \sigma^2(t-u)}{\sigma\sqrt{t-u}} \Big) dt \Big] \Big|_{c=c_1}^{c_2} \Big\} d\widetilde{W}_u.$$

The result of Theorem 3.1 gives us a possibility to find the component γ_t of the hedging strategy $\pi = (\beta_t, \gamma_t)$, $t \in [0, T]$, which is defined by integrand of representation (19) and is equal to

$$\gamma_t = -\frac{1}{\sigma S_t} \Big[\int_t^T \frac{c}{\sqrt{v-u}} \varphi\Big(\frac{\ln c - \widetilde{\nu} v - \sigma\widetilde{W}_u}{\sigma\sqrt{v-u}} \Big) dv \Big] \Big|_{c=c_1}^{c_2} \tag{20}$$

$$+\frac{1}{S_t}\Big[\int\limits_t^T \exp\Big\{\frac{2rv+2\sigma\widetilde{W}_u-\sigma^2 u}{2}\Big\}\Phi\Big(\frac{\ln c-\widetilde{\nu}v-\sigma\widetilde{W}_u-\sigma^2(v-u)}{\sigma\sqrt{v-u}}\Big)dv\Big]\Big|_{c=c_1}^{c_2}.$$

Now, using the result of Theorem 3.1, we can find the capital process

$$X_t=\widetilde{\boldsymbol{E}}[F|\Im_t^{\widetilde{W}}]=\Big[\int\limits_0^T \exp\{rv\}\Phi\Big(\frac{\ln c-rv-\sigma^2 v/2}{\sigma\sqrt{v}}\Big)dv\Big]\Big|_{c=c_1}^{c_2} \tag{21}$$

$$-\int\limits_0^t\Big\{\Big[\int\limits_u^T \frac{c}{\sqrt{v-u}}\varphi\Big(\frac{\ln c-\widetilde{\nu}v-\sigma\widetilde{W}_u}{\sigma\sqrt{v-u}}\Big)dv\Big]\Big|_{c=c_1}^{c_2}\Big\}d\widetilde{W}_u+\sigma$$

$$\times\int\limits_0^t\Big\{\Big[\int\limits_u^T \exp\Big\{\frac{2rv+2\sigma\widetilde{W}_u-\sigma^2 u}{2}\Big\}\Phi\Big(\frac{\ln c-\widetilde{\nu}v-\sigma\widetilde{W}_u-\sigma^2(v-u)}{\sigma\sqrt{v-u}}\Big)dv\Big]\Big|_{c=c_1}^{c_2}\Big\}d\widetilde{W}_u,$$

where the symbol $\widetilde{\boldsymbol{E}}$ denotes the mathematical expectation under the measure $\widetilde{P}$.

The second component β_t of hedging strategy π can be found as follows

$$\beta_t=\frac{1}{B_t}(X_t-\gamma_t S_t). \tag{22}$$

Therefore, the hedging strategy $\pi=(\beta_t,\gamma_t)$, $t\in[0,T]$ in the problem of "replication" of Exotic type European Option with payoff F given by (15) in case of Black-Scholes financial market model, is defined by relations (20), (21) and (22) and the price $\widetilde{C}$ of this option

$$\widetilde{C}=\Big[\int\limits_0^T \exp\{rt\}\Phi\Big(\frac{\ln c-rt-\sigma^2 t/2}{\sigma\sqrt{t}}\Big)dt\Big]\Big|_{c=c_1}^{c_2}.$$

Corollary 3.3: *In the case $r=\sigma^2/2$, the hedging strategy $\pi=(\beta_t,\gamma_t)$ and the price of the considered option are defined correspondingly by the following relations*

$$\gamma_t=-\frac{1}{\sigma S_t}\Big[\int\limits_t^T \frac{c}{\sqrt{v-u}}\varphi\Big(\frac{\ln c-\sigma\widetilde{W}_u}{\sigma\sqrt{v-u}}\Big)dv\Big]\Big|_{c=c_1}^{c_2}$$

$$+\frac{1}{S_t}\Big[\int\limits_t^T \exp\Big\{\frac{2\sigma\widetilde{W}_u+\sigma^2(v-u)}{2}\Big\}\Phi\Big(\frac{\ln c-\sigma\widetilde{W}_u-\sigma^2(v-u)}{\sigma\sqrt{v-u}}\Big)dv\Big]\Big|_{c=c_1}^{c_2},$$

$$\beta_t = \frac{1}{B_t}(X_t - \gamma_t S_t),$$

$$\widetilde{C} = \Big[\int_0^T \exp\{rt\}\Phi\Big(\frac{\ln c - rt - \sigma^2 t/2}{\sigma\sqrt{t}}\Big)dt\Big]\Big|_{c=c_1}^{c_2}.$$

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