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Necessary Conditions of Optimality for the Optimal Control Problem with Several Delays and the Discontinuous Initial Condition

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The nonlinear optimal control problem with several constant delays in the phase coordinates and controls is considered. The necessary conditions of optimality are obtained for the initial and final moments, for delays having in the phase coordinates and the initial vector, for the initial function and control.

Keywords: Optimal control problem with delay, Necessary conditions of optimality, Discontinuous initial condition.

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Let $O \subset \mathbb{R}^n$ be an open set and let $U \subset \mathbb{R}^r$ be a convex compact set. Let $h_{i2} > h_{i1} > 0, i = \overline{1,s}$ and let $\theta_k > \cdots > \theta_1 > 0$ be given numbers and n-dimensional function $f(t, x, x_1, ..., x_s, u, u_1, ..., u_k), (t, x, x_1, ..., x_s, u, u_1, ..., u_k) \in I \times O^{1+s} \times U^{1+k}$ satisfies the following conditions: for almost all fixed $t \in I = [a, b]$ the function $f(t, \cdot) : I \times O^{1+s} \times U^{1+k} \to \mathbb{R}^n$ is continuous and continuously differentiable in $(x, x_1, ..., x_s, u, u_1, ..., u_k) \in O^{1+s} \times U^{1+k}$; for each fixed $(x, x_1, ..., x_s, u, u_1, ..., u_k) \in O^{1+s} \times U^{1+k}$, the function $f(t, x, x_1, ..., x_s, u, u_1, ..., u_k)$ and the matrices $f_x(t, \cdot), f_{x_i}(t, \cdot), i = \overline{1,s}$ and $f_u(t, \cdot), f_{u_i}(t, \cdot), i = \overline{1,k}$ are measurable on I ; for any compact set $K \subset O$ there exists a function $m_K(t) \in L_1(I, [0, \infty))$ such that

$$| f(t, x, x_1, ..., x_s, u, u_1, ..., u_k) | + | f_x(t, x, \cdot) | + \textstyle\sum_{i=1}^{s} | f_{x_i}(t, x, \cdot) |$$
$$+ | f_u(t, x, \cdot) | + \textstyle\sum_{i=1}^{k} | f_{u_i}(t, x, \cdot) | \leq m_K(t)$$

for all $(x, x_1, ..., x_s, u, u_1, ..., u_k) \in K^{1+s} \times U^{1+k}$ and for almost all $t \in I$.

Furthermore, let Φ be the set of continuous functions $\varphi(t) \in N, t \in I_1 = [\hat{\tau}, b]$, where $\hat{\tau} = a - \max\{h_{12}, ..., h_{s2}\}, N \subset O$ is a convex compact set; Ω is the set of measurable functions $u(t) \in U, t \in I_2 = [a - \theta_k, b]$; $X_0 \subset O$ is a convex compact set.

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To each element $v = (t_0, t_1, \tau_1, ..., \tau_s, x_0, \varphi, u) \in A = I \times I \times [h_{11}, h_{12}] \times ... \times [h_{s1}, h_{s2}] \times X_0 \times \Phi \times \Omega$ on the interval $[t_0, t_1]$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_s), u(t), u(t-\theta_1), ..., u(t-\theta_k)), \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), \ x(t_0) = x_0. \tag{2}$$

The condition (2) is called discontinuous because, in general, $x(t_0) \neq \varphi(t_0)$.

Definition 1: Let $\nu = (t_0, t_1, \tau_1, ..., \tau_s, \ x_0, \varphi, u) \in A$. A function $x(t) = x(t; \nu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$ is called a solution of equation (1) with the discontinuous initial condition (2), or the solution corresponding to ν and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let the scalar-valued functions $q^i(t_0, t_1, \tau_1, ..., \tau_s, x_0, x_1), \ i = \overline{0, l}$, be continuously differentiable on $I^2 \times [h_{11}, h_{12}] \times ... \times [h_{s1}, h_{s2}] \times O^2$.

Definition 2: An element $\nu = (t_0, t_1, \tau_1, ..., \tau_s, x_0, \varphi, u) \in A$ is said to be admissible if the corresponding solution $x(t) = x(t; \nu)$ satisfies the boundary conditions

$$q^i(t_0, t_1, \tau_1, ..., \tau_s, x_0, x(t_1)) = 0, \ i = \overline{1, l}. \tag{3}$$

Denote by A_0 the set of admissible elements.

Definition 3: An element $\nu_0 = (t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, x_{00}, \varphi_0, u_0) \in A_0$ is said to be locally optimal if there exist a number $\delta_0 > 0$ and a compact set $K_0 \subset O$ such that for an arbitrary element $\nu \in A_0$ satisfying the condition

$$|t_{00} - t_0| + |t_{10} - t_1| + \sum_{i=1}^{s} |\tau_{i0} - \tau_i| + |x_{00} - x_0| + \| \varphi_0 - \varphi \|_{I_1} + \| u_0 - u \|_{I_2} \leq \delta_0$$

the inequality

$$q^0(t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, x_{00}, x_0(t_{10})) \leq q^0(t_0, t_1, \tau_1, ..., \tau_s, x_0, x(t_1)) \tag{4}$$

holds. Here

$$\| \varphi_0 - \varphi \|_{I_1} = \max_{t \in I_1} |\varphi_0(t) - \varphi(t)|, \ \| u_0 - u \|_{I_2} = \sup_{t \in I_2} |u_0(t) - u(t)|.$$

The problem (1)-(4) is called an optimal control problem with the discontinuous initial condition.

Theorem 4: *Let ν_0 be an optimal element with $t_{00}, t_{10} \in (a, b)$ and the following conditions hold:*

1) $\tau_{s0} > ... > \tau_{10}$ and $t_{00} + \tau_{s0} < t_{10}$, with $\tau_{i0} \in (h_{i1}, h_{i+10}), i = \overline{1, s-1}$;

2) the function $\varphi_0(t)$ is absolutely continuous and $\dot{\varphi}_0(t)$ is bounded;

3) the function $f_0(w) = f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k))$, where $w = (t, x, x_1, ..., x_s) \in I \times O^{1+s}$ is bounded on $I \times O^{1+s}$;

4) there exists the finite limit

$$\lim_{w\to w_0} f_0(w) = f^-, w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), ..., \varphi_0(t_{00} - \tau_{s0}))$;

5) there exist the finite limits

$$\lim_{(w_{1i},w_{2i})\to(w^0_{1i},w^0_{2i})} [f_0(w_{1i}) - f_0(w_{2i})] = f_i,$$

where $w_{1i}, w_{2i} \in (a, b) \times O^{1+s}, i = \overline{1, s},$

$$w^0_{1i} = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), ..., x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$

$$x_{00}, x_0(t_{00} + \tau_{i0} - \tau_{i+10}), ..., x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big),$$

$$w^0_{2i} = \Big(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), ..., x_0(t_{00} + \tau_{i0} - \tau_{i-10}),$$

$$\varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), ..., x_0(t_{00} + \tau_{i0} - \tau_{s0})\Big);$$

6) there exists the finite limit

$$\lim_{w\to w_{s+1}} f_0(w) = f^-_{s+1}, w \in (t_{00}, t_{10}] \times O^{1+s},$$

$$w_{s+1} = (t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_{10}), ..., x_s(t_{10} - \tau_{s0})).$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, *with* $\pi_0 \leq 0$, *and a solution* $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ *of the equation*

$$\dot\psi(t) = -\psi(t)f_{0x}[t] - \sum_{i=1}^{s} \psi(t + \tau_{i0})f_{0x_i}[t + \tau_{i0}], t \in [t_{00}, t_{10}], \psi(t) = 0, t > t_{10}, \quad (5)$$

where $f_{0x}[t] = f_{0x}(t, x_0(t), x_0(t - \tau_{10}), ..., x_0(t - \tau_{s0}))$, *such that the following conditions hold:*

7)the conditions for the moments t_{00} *and* t_{10} :

$$\pi Q_{0t_0} \geq \psi(t_{00})f^- + \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0})f_i, \quad \pi Q_{0t_1} \geq -\psi(t_{10})f^-_{s+1},$$

where

$$Q = (q^0, ..., q^l)^T, Q_0 = Q(t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, x_{00}, x_0(t_{10})), Q_{0t_0} = \frac{\partial}{\partial t_0} Q_0;$$

8) the conditions for the delays $\tau_{i0}, i=\overline{1,s}$,

$$\pi Q_{0\tau_{i0}} = \psi(t_{00}+\tau_{i0})f_i + \int_{t_{00}}^{t_{00}+\tau_{i0}} \psi(t) f_{0x_i}[t]\dot{\varphi}_0(t-\tau_{i0})dt$$

$$+\int_{t_{00}+\tau_{i0}}^{t_{10}} \psi(t) f_{0x_i}[t]\dot{x}_0(t-\tau_{i0})dt, i=\overline{1,s};$$

9) the conditions for the vector x_{00},

$$(\pi Q_{0x_0}+\psi(t_{00}))x_{00} = \max_{x_0\in X_0}(\pi Q_{0x_0}+\psi(t_{00}))x_0;$$

10) the linearized integral maximum principle for the initial function $\varphi_0(t)$,

$$\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi_0(t)dt = \max_{\varphi(t)\in\Phi}\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi(t)dt;$$

11) the linearized integral maximum principle for the control function $u_0(t)$,

$$\int_{t_{00}}^{t_{10}} \psi(t)\Big[f_{0u}[t]u_0(t)+\sum_{i=1}^{k} f_{0u_i}[t]u_0(t-\theta_{i0})\Big]dt$$

$$=\max_{u(t)\in\Omega}\int_{t_{00}}^{t_{10}} \psi(t)\Big[f_{0u}[t]u(t)+\sum_{i=1}^{k} f_{0u_i}[t]u(t-\theta_{i0})\Big]dt$$

12) the condition for the function $\psi(t)$

$$\psi(t_{10}) = \pi Q_{0x_1}.$$

Theorem 5: *Let* ν_0 *be an optimal element with* $t_{00}, t_{10}\in(a,b)$ *and the conditions 1),2),3),5) of Theorem 4 hold. Moreover, there exist the finite limits*

$$\lim_{w\to w_0} f_0(w) = f^+, w\in[t_{00},t_{10})\times O^{1+s},$$

$$\lim_{w\to w_{s+1}} f_0(w) = f^+_{s+1}, w\in[t_{10},b)\times O^{1+s},$$

Then there exists a vector $\pi=(\pi_0,...,\pi_l)\neq 0$, *with* $\pi_0\le 0$, *and a solution* $\psi(t) = (\psi_1(t),...,\psi_n(t))$ *of equation (5) such that conditions 8)-12) hold. Moreover,*

$$\pi Q_{0t_0} \le \psi(t_{00})f^+ + \sum_{i=1}^{s}\psi(t_{00}+\tau_{i0})f_i, \quad \pi Q_{0t_1}\le -\psi(t_{10})f^+_{s+1}.$$

Theorem 6 : *Let ν_0 be an optimal element with $t_{00}, t_{10} \in (a,b)$ and the conditions of Theorems 4 and 5 hold. Moreover,*

$$f^- = f^+ := f, \;\; f^-_{s+1} = f^+_{s+1} := f_{s+1}.$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of equation (5) such that the conditions 8)-12) hold. Moreover,

$$\pi Q_{0t_0} = \psi(t_{00})f + \sum_{i=1}^{s} \psi(t_{00} + \tau_{i0})f_i, \;\; \pi Q_{0t_1} = -\psi(t_{10})f_{s+1}.$$

It is clear that, if the function $f(t, x, x_1, ..., x_s, u, u_1, ..., u_k)$ is continuous and the functions $u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_s)$ are continuous at the points $t_{00}, t_{00}-\tau_{i0}, i = \overline{1, s};\; t_{00} + \tau_{i0}, \overline{1, s};\; t_{10}, t_{10} - \tau_{i0}, i = \overline{1, s}$. Then we have

$$f = f(t_{00}, x_{00}, \varphi_0(t_{00} - \tau_{10}), ..., \varphi_0(t_{00} - \tau_{s0}), u_0(t_{00}), u_0(t_{00} - \theta_1), ..., u_0(t_{00} - \theta_s)),$$

$$f_{s+1} = f(t_{10}, x_0(t_{10}), x_0(t_{10}-\tau_{10}), ..., x_0(t_{10}-\tau_{s0}), u_0(t_{10}), u_0(t_{10}-\theta_1), ..., u_0(t_{10}-\theta_s)),$$

$$f_i = f_0(t_{00} + \tau_{i0}, x_0(t_{00} + \tau_{i0}), x_0(t_{00} + \tau_{i0} - \tau_{10}), ..., x_0(t_{00} + \tau_{i0} - \tau_{i-10}), x_{00},$$

$$x_0(t_{00}+\tau_{i0}-\tau_{i+10}), ..., x_0(t_{00}+\tau_{i0}-\tau_{s0})) - f_0(t_{00}+\tau_{i0}, x_0(t_{00}+\tau_{i0}), x_0(t_{00}+\tau_{i0}-\tau_{10}),$$

$$..., x_0(t_{00} + \tau_{i0} - \tau_{i-10}), \varphi_0(t_{00}), x_0(t_{00} + \tau_{i0} - \tau_{i+10}), ..., x_0(t_{00} + \tau_{i0} - \tau_{s0})).$$

On the basis of variation formulas [1] Theorems 4-6 are proved by the scheme given in [2,3].

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