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On Consistent Criteria of Hypotheses Testing for Non-separable Complete Metric Space

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In this paper, we define consistent criteria of hypotheses testing for a non-separable complete metric space, such that the probability of any kind of error is zero for a given criterion. The necessary and sufficient conditions for the existence of such criteria are given.

Keywords: Statistical structure, Orthogonal structure, Separable structure, Strongly separable structure, Hypotheses testing, Consistent criterion.

AMS Subject Classification: 62H05, 62H12.

1. Introduction

Statistics of random processes is used in various fields of science and technology (for example, in theoretical physics, genetics, economics, radio physics, ...). When using random processes as models of real phenomena, the question of determining the probabilistic characteristics of the process arises. To determine these characteristics statistical methods should be used. Among the problems of statistics, a class of problems is distinguished in which the number of observations is unique.

Despite the uniqueness of observation, in many cases, one can authentically determine the values of unknown distribution parameters or reliably choose one of an infinite number of competing hypotheses about the exact form of the distribution. In the case when a parameter or hypothesis is determined by one observation reliably, it is said that for it there exists a consistent estimate of parameter or a consistent criterion for hypothesis testing. This article is devoted to the question of the existence of consistent criteria for hypothesis testing and the method of their finding. Previously, our work focused on the property of strong separability of statistical structures, introduced by A. Skorokhod, which was associated with the existence of consistent criteria for hypothesis testing. Recall that a statistical criterion is any measurable mapping from the set of all possible sample values to the set of hypotheses. It is said that an error of the h-th type of the δ criterion occurs, if the criterion rejected the main hypothesis of H_h. The following probability $\{\alpha_h(\delta) = \mu_h(\{x : \delta(x) = h\})\}$ is called the probability of an error of the h-th kind for a given criterion δ. Examples 1.1 and 1.2 show a general trend - when we decrease one of the probabilities of an error, the other, as a rule, increases.

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Example 1.1 Consider the case when there are two simple hypotheses h_1 and h_2 about the distribution of population and criterion $\delta : R^n \to \{h_1, h_2\}$ such that $\delta(x) \equiv h_1$, then the probability of an error of the first kind is zero $\alpha_1 = \mu_{h_1}(\{x : \delta(x) \neq h_1\}) = 0$, and the probability of an error of the second kind is equal to one $\alpha_2 = \mu_{h_2}(\{x : \delta(x) \neq h_2\}) = 1$.

Example 1.2 There are observations from the normal distribution in R with variation one and different means $a \in R$ and two simple hypotheses $H_1 = \{a = 0\}$ and $H_2 = \{a = 1\}$. Consider the following criteria:

$$\delta(x) = \begin{cases} H_1, & if\ x \leq c; \\ H_2, & if\ x > c, \end{cases}$$

for some $c \in R$. It is obvious that with the increase of the number c the probability of an error of the first type decreases, and the probability of an error of the second kind increases.

Remark 1: Let the statistical structure $\{E, S, \mu_h, h \in H\}$ admit a consistent criterion δ for hypothesis testing, then the probability of an error of all types is equal to zero for the criterion δ.

The purpose of this work is the further study of statistical structures that allow consistent criteria for hypothesis testing and obtaining their characteristic properties.

By (ZFC) we denote the formal system of Zermelo-Fraenkel with the addition of axiom of choice (AC), i.e. (ZFC)=(ZF)&(AC). By (ZFC)&(CH) we denote the theory with the addition of a continuum hypothesis (CH): $2^{\chi_0} = \chi_1$, where χ_1 denotes the first uncountable cardinal number, and by (ZFC)&(MA) we denote the theory with the addition of Martin's axiom (MA). It is known that in the theory (ZFC)&(CH) Martin's axiom (MA) is automatically satisfied. It is well known that Martin's axiom (MA) is much weaker than the continuum hypothesis (CH). Moreover, the negation of the continuum hypothesis (¬CH) is compatible with Martin's axiom (see [4], [5]).

In the general theory of hypotheses testing there often arises a problem of transition from orthogonal statistical structure to the corresponding strongly statistical structure. A. Skorokhod (see [2]) proved that if the continuum hypothesis is true, then an arbitrary weakly separable statistical structure, whose cardinality is not greater than the cardinality of the continuum, is strongly separable. The validity of the inverse relation was established in [7], [8]. In particular, it was shown there that if an arbitrary weakly separable statistical structure, whose cardinality is less or equal than the cardinality of the continuum, is strongly separable. Z. Zerakidze (see [11], [12], [14]) proved: 1) In (ZF) theory for the countable statistical structure the notions of weak separability, separability, strong separability and orthogonality are equivalent; 2) In (ZFC)&(MA) theory Borel weakly separable statistical structure, whose cardinality is not greater than the cardinality of the continuum, is strongly separable; 3) In (ZFC) theory orthogonal statistical structure, whose cardinality is 2^{2^c}, is weakly separable.

2. The consistent criteria for hypotheses testing

Let (E, S) be a measurable space with a given family of probability measures: $\{\mu_i, i \in I\}$.

The following definitions are taken from [1] - [19].

Definition 2.1: An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

Definition 2.2: A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) (O) if a family of probability measures $\{\mu_i, i \in I\}$ constists of pairwise singular measures (i.e. $\mu_i \perp \mu_j$, $\forall i \neq j$).

Example 2.3 Let $E = [0, 1]$, S be a Borel σ-algebra of subsets of $[0, 1]$. Let $\mu_1(B) = 2l(B \cap [0, \frac{1}{2}])$, $B \in S$; $\mu_2(B) = 2l(B \cap [\frac{1}{2}, 1])$, $B \in S$ and $\mu_3(B) = 3l(B \cap [0, \frac{1}{3}])$, $B \in S$, where l is Lebesgue measure on S. Then $\mu_1 \perp \mu_2$ and $\mu_2 \perp \mu_3$, but μ_1 is not orthogonal to μ_3.

Example 2.4 A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable (WS) if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the realations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j \end{cases} \qquad (i, j \in I).$$

Definition 2.5: A statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable (S) if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the realations are fulfilled:

$$1)\ \mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j \end{cases} \qquad (i, j \in I);$$

$$2)\ \forall i, j \in I: \quad card(X_i \cap X_j) < c, \quad if\ i \neq j,$$

where c denotes the continuum power.

Definition 2.6: A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable (SS) if there exist a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that the realations are fulfilled:

$$\mu_i(X_i) = 1, \quad \forall\, i \in I.$$

Example 2.7 Let $E = R \times R$ (where $R = (-\infty, +\infty)$) and let $S = B(R \times R)$ be a Borel σ-algebra of subsets of $R \times R$. Let's take the S-measurable sets

$$X_h = \{-\infty < x < +\infty,\ y = h,\ h \in (0, +\infty)\}$$

and assume that

$$\mu_h(A) = \int_A \frac{1}{\sqrt{2\pi}h} e^{-\frac{x^2}{2h^2}}$$

are linear Gaussian measures on X_h, $h \in (0, +\infty)$. Then the statistical structure $\{R \times R,\ S,\ \mu_h,\ h \in (0, +\infty)\}$ is strongly separable.

Example 2.8 Let $E = R \times R$ and let $S = B(R \times R)$ be a Borel σ-algebra of subsets of $R \times R$. Let's take the S-measurable sets

$$X_h = \begin{cases} -\infty < x < +\infty,\ y = h,\ if\ h \in R; \\ x = h,\ -\infty < y < +\infty,\ if\ h \in R \end{cases}$$

and assume that μ_h are linear Gaussian measures on X_h, $h \in R$. Then the statistical structure $\{R \times R,\ S,\ \mu_h,\ h \in R\}$ is separable, but not strongly separable.

Example 2.9 Let $E = R \times R \times R$, let S be a Borel σ-algebra on E. Let's take the S-measurable sets

$$X_h = \begin{cases} -\infty < x < +\infty,\ -\infty < y < +\infty,\ z = h,\ if\ \ h \in R; \\ x = h,\ -\infty < y < +\infty,\ -\infty < z < +\infty,\ if\ \ h \in R; \\ -\infty < x < +\infty,\ y = h,\ -\infty < z < +\infty,\ if\ \ h \in R \end{cases}$$

and assume that μ_h are plane Gaussian measures on X_h. Then the statistical structure $\{R \times R \times R,\ S,\ \mu_h,\ h \in R\}$ is weakly separable, but not separable.

Example 2.10 Let $E = R \times R$, S be a Borel σ-algebra of subsets of $R \times R$. Let's take the S-measurable sets

$$X_h = \begin{cases} -\infty < x < +\infty,\ y = h,\ if\ \ h \in R \setminus \{0\}; \\ -\infty < x < +\infty,\ -\infty < y < +\infty,\ if\ \ h = 0 \end{cases}$$

and assume that $\mu_h,\ h \in R \setminus \{0\}$, are linear Gaussian measures on X_h and μ_0 is a plane Gaussian measure on $R \times R$. Then the statistical structure $\{R \times R,\ S,\ \mu_h,\ h \in R\}$ is orthogonal, but not weakly separable.

Remark 2 : From strong separability there follows separability, from separability there follows weak separability and from weak separability there follows orthogonality but not vice versa, i.e.

$$(SS) \Rightarrow (S) \Rightarrow (WS) \Rightarrow (0).$$

Remark 3 : On an arbitrary set E of continuum power one can define an orthogonal statistical structure having the maximal possible power equal to 2^{2^c}, a weakly separable statistical structure having the maximal possible power equal to 2^c, and a strongly separable statistical structure with the maximum possible power equal to c, where c is continuum power (see [13] - [19]).

Lemma 2.11: *If the statistical structure $\{E, S, \mu_h, h \in H\}$ is weakly separable then it is orthogonal.*

Proof: If the statistical structure $\{E, S, \mu_h, h \in H\}$ is weakly separable then there exist S-measurable sets $\{X_h,\ h \in H\}$ such that

$$\mu_h(X_{h'}) = \begin{cases} 1, & if\ h = h^{'}; \\ 0, & if\ h \neq h^{'} \end{cases} \qquad (h, h^{'} \in H).$$

Since $\mu_h(X_h) = 1$ and $\mu_{h'}(X_h) = 0$ for $h^{'} \neq h$, we have $\mu_h(E \setminus X_h) = 0$. Hence, the measures μ_h and $\mu_{h'}$, $h \neq h^{'}$, are orthogonal. □

The notion and corresponding construction of consistent criteria for hypotheses testing was introduced and studied by Z. Zerakidze (see [14], [17]).

Definition 2.12: We consider the concept of the hypothesis as any assumption that determines the form of the distribution of population.

Let H be the set of hypotheses and let $B(H)$ be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 2.13: A statistical criterion is any measurable mapping $\delta : (E, S) \longrightarrow (H, B(H))$.

Definition 2.14: We will say that the statistical structure $\{E, S, \mu_h, h \in H\}$ admits a consistent criterion for hypothesis testing (CC) if there exists at least one measurable mapping $\delta : (E, S) \longrightarrow (H, B(H))$, such that

$$\mu_h(\{x : \delta(x) = h\}) = 1, \ \ \forall\, h \in H.$$

Definition 2.15: The probability $\alpha_h(\delta) = \mu_h(x : \{\delta(x) \neq h\})$ is called the probability of error of the h-th type for a given criterion δ.

Definition 2.16: We will say that the statistical structure $\{E, S, \mu_h, h \in H\}$ admits a consistent criterion for hypothesis testing of any parametric function (PC) if for any real bounded measurable function $g : (H, B(H)) \longrightarrow (R, B(R))$ there exists at least one measurable function $f : (E, S) \longrightarrow (R, B(R))$ such that

$$\mu_h(\{x : f(x) = g(h)\}) = 1, \ \ \forall\, h \in H.$$

Definition 2.17: We will say that the statistical structure $\{E, S, \mu_h, h \in H\}$ admits an unbiased criterion for hypothesis testing (UC) if for any real bounded measurable function $g : (H, B(H)) \longrightarrow (R, B(R))$ there exists at least one measurable function $f : (E, S) \longrightarrow (R, B(R))$ such that

$$\int_E f(x)\mu_h(dx) = g(h), \ \ \forall\, h \in H.$$

In the example below, we give the construction of a strongly separable statistical structure that does not admit a consistent criterion for hypotheses testing.

Example 2.18 Let $E = [0, 1] \times [0, 1]$, let $B([0, 1] \times [0, 1])$ be a Borel σ-algebra of subsets of $E = [0, 1]\times[0, 1]$. As a set of hypotheses consider the set $H = [0, 1]\cup[2, 3]$.

Let us take the $B(E = [0,1] \times [0,1])$-measurable sets

$$X_h = \begin{cases} 0 \le x \le 1, \ y = h, \ \ if \ \ h \in [0,1]; \\ x = h-2, \ 0 \le y \le 1, \ \ if \ h \in [2,3] \end{cases}$$

and denote by μ_h, $h \in [0,1] \cup [2,3]$, linear Lebesgue measures on X_h. Then the statistical structure $\{[0,1]\times[0,1], B([0,1]\times[0,1]), \mu_h, h \in [0,1]\cup[2,3]\}$ is a separable statistical structure. Suppose that it admits a consistent criterion for hypotheses testing

$$\delta : ([0,1] \times [0,1], B([0,1] \times [0,1])) \longrightarrow (H, B(H)),$$

with $\mu_h(\{x : \delta(x) = h\}) = 1$, $\forall h \in [0,1] \cup [2,3]$. Let's introduce sets $A_1 = \{x : \delta(x) \in [0,1]\}$ and $A_2 = \{x : \delta(x) \in [2,3]\}$. It is clear that A_1 and A_2 are $B([0,1] \times [0,1])$-measurable sets and we have $\mu_h(A_1 \cap \{[0,1] \times \{h\}\}) = 1$, $\forall h \in [0,1]$ and $\mu_h(A_2 \cap \{\{h-2\} \times [0,1]\}) = 1$, $\forall h \in [2,3]$. Further, according to the Fubini theorem we conclude that $l(A_1) = 1$ and $l(A_2) = 1$ (where l is the Lebesgue plane measure). From here, taking into account that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = [0,1] \times [0,1]$, we verify that $l([0,1] \times [0,1]) = 2$, which contradicts the fact that $l([0,1] \times [0,1]) = 1$. Hence, this statistical structure does not admit a consistent criterion for hypotheses testing.

3. The consistent criterion for hypotheses testing in the Hilbert space of measures

Let $\{\mu_h, h \in H\}$ be probability measures defined on the measurable space (E, S). For each $h \in H$ denote by $\overline{\mu}_h$ the completion of the measure μ_h, and denote by $dom(\overline{\mu}_h)$ the σ-algebra of all $\overline{\mu}_h$-measurable subsets of E. Let

$$S_1 = \cap_{h \in H} dom(\overline{\mu}_h).$$

Definition 3.1: A statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is called strongly separable if there exists the family of S_1-measurable sets $\{Z_h, h \in H\}$ such that the relations are fulfilled:
1) $\overline{\mu}_h(Z_h) = 1 \ \ \forall h \in H$;
2) $Z_{h_1} \cap Z_{h_2} = \emptyset \ \ \forall h_1 \neq h_2; \ \ h_1, h_2 \in H$;
3) $\cup_{h \in H} Z_h = E$.

Definition 3.2: We will say that the orthogonal statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ admits a consistent criterion for hypothesis testing if there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (H, B(H))$, such that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = 1 \ \ \forall \, h \in H.$$

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 3.3: A linear subset $M_H \subset M^\sigma$ is called a Hilbert space of measures if:

1) One can introduce on M_H a scalar product (μ,ν) $(\mu,\nu \in H$ so that M_H is a Hilbert space and for every mutually singular measures μ and ν $(\mu,\nu \in H)$ the scalar product $(\mu,\nu)=0$;

2) If $\nu \in M_H$ and $|f(x)| \leq 1$, then

$$\nu_f(A) = \int_A f(x)\nu(dx) \in M_H,$$

where f is a S_1-measurable real function and $(\nu_f,\nu_f) \leq (\nu,\nu)$;

3) If $\nu_n \in M_H$, $\nu_n \geq 0$, $\nu_n(E) < \infty$, $n = 1, 2, ...$ and $\nu_n \downarrow 0$, then for any $\nu \in M_H$:

$$\lim_{n\to\infty} (\nu_n.\nu) = 0.$$

Remark 4: The notion and corresponding construction of the Hilbert space of measures was introduced and studied by Z. Zerakidze (see [15]).

The following theorem has also been proved in this paper [15].

Theorem 3.4: *If M_H is a Hilbert space of measures, then it is represented as a direct sum of the Hilbert spaces $H_2(\overline{\mu}_h)$, that is*

$$M_H = \oplus_{h\in H} H_2(\overline{\mu}_h),$$

where $H_2(\overline{\mu}_h)$ is the family of measures

$$\nu(A) = \int_A f(x)\overline{\mu}_h(dx),\ A \in S_1,$$

such that

$$\int_E |f(x)|^2 \overline{\mu}_h(dx) < +\infty$$

and

$$||\nu||_{H_2(\overline{\mu}_h)} = \Big(\int_E |f(x)|^2 \overline{\mu}_h(dx)\Big)^{1/2}.$$

Denote by $F = F(M_H)$ the set of real functions f for which

$$\int_E f(x)\overline{\mu}_h(dx)$$

is defined for all $\overline{\mu}_h \in M_H$.

Theorem 3.5: *Let*

$$M_H = \oplus_{h\in H} H_2(\overline{\mu}_h)$$

be a Hilbert space of measures, let E be a complete metric space, whose topological weights are not measurable in a wider sense. Let S_1 be a Borel σ-algebra on E. In order for the Borel orthogonal statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ to admit a *consistent criterion for hypotheses testing in the theory of* (ZFC) & (MA) *it is necessary and sufficient that the correspondence* $f \longleftrightarrow \psi_f$ *defined by the equality*

$$\int_E f(x)\nu(dx) = (\psi_f, \nu) \ \ \forall h \in M_H$$

was one-to-one (here l_f *is a linear continuous functional on* M_B, $f \in F(M_B)$*).*

Proof: Sufficiency. Since for each $f \in F(M_H)$ and $\overline{\mu}_h \in M_H$ the integral

$$\int_E f(x)\overline{\mu}_h(dx)$$

is defined, then there exists a countable subset H_f in H for which

$$\int_E f(x)\overline{\mu}_h(dx) = 0, \ if \ h \notin H_f; \ \ \sum_{h \in H_f} \int_E |f(x)|^2 \overline{\mu}_h(dx) < \infty$$

and for any countable subset $\widetilde{H} \subset H$ and for the measure

$$\nu(C) = \sum_{h \in \widetilde{H}} \int_C g_h(x)\overline{\mu}_h(dx)$$

we have

$$\int_E f(x)\nu(dx) = \sum_{h \in H_f \cap \widetilde{H}} \int_E f(x) g_h(x) \overline{\mu}_h(dx).$$

Since the correspondence $f \longleftrightarrow \psi_f$ is defined by the equality

$$\int_E f(x)\nu(dx) = (\psi_f, \nu), \ \ \nu \in M_H$$

is one-to-one, according to Theorem 2 from [15], we conclude that the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is weakly separable. Consequently, there is a family of S_1-measurable sets X_h, $h \in H$, for which the following condition is satisfied:

$$\overline{\mu}_{h_1}(X_{h_2}) = \begin{cases} 1, & if \ h_1 = h_2; \\ 0, & if \ h_1 \neq h_2. \end{cases}$$

Recall now the following lemmas.

Lemma 3.6: *(see [12]). Let* (E, ϱ) *be a complete separable metric space and let* μ *be a Borel probability measure defined on* $(E, B(E))$*. Let* $\{X_h\}_{h \in H}$, $cardH \le c$, *be a family of* $B(E)$*-measurable sets and* $\mu(X_h) = 0 \ \forall h \in H$*. Then in the theory*

(ZFC) & (MA):

$$\mu^*(\cup_{h\in H}X_h) = 0.$$

Lemma 3.7: *(see [4]). Let (E, ϱ) be a complete metric space, whose topological weights are not measurable in a wider sense than in the theory (ZFC) & (MA) and let μ be an arbitrary Borel measure defined on $(E, B(E))$. Then there exists a closed separable subspace $E(\mu) \subset E$, such that $\mu(E(\mu)) = 1$ and $\mu(E \setminus E(\mu)) = 0$.*

Then we easily ascertain that the following is true

$$(\forall h)\ (\forall \{X_h\}_{h\in H})\ ((cardH \le c)\&\forall h\ (h \in H \Longrightarrow \mu(X_h) = 0)) \Longrightarrow$$

$$\Longrightarrow \mu^*(\cup_{h\in H}X_h) = \mu^*[(\cup_{h\in H}X_h) \cap E(\mu)] + \mu^*[(\cup_{h\in H}X_h) \cap (E \setminus E(\mu))] = 0.$$

Further, we represent $\{\overline{\mu}_h, h \in H\}$, as an inductive sequence $\{\overline{\mu}_h < \omega_1\}$, where ω_1 denotes the first ordinal number of the power of the set H.

We define ω_1 sequence Z_h of parts of the space E such that the following relations hold:

1) Z_h is a Borel subset of E $\forall h < \omega_1$;
2) $Z_h \subset X_h$ $\forall h < \omega_1$;
3) $Z_h \cap Z_{h'} = \emptyset$ for all $h < \omega_1,\ h' < \omega_1,\ h \ne h'$;
4) $\overline{\mu}_h(Z_h) = 1$ $\forall h < \omega_1$.

Suppose that $Z_{h_0} = X_{h_0}$. Suppose further that the partial sequence $\{Z_{h'}\}_{h'<h}$ is already defined for $h < \omega_1$. It is clear that $\mu^*(\cup_{h'<h}Z_{h'}) = 0$ (see [11]). Thus there exists a Borel subset Y_h of the space E such that the following relations are valid:

$$\cup_{h'<h}Z_{h'} \subset Y_h \ \ and\ \ \mu^*(Y_h) = 0.$$

Assuming that $Z_h = X_h \setminus Y_h$, we construct the ω_1 sequence $\{Z_h\}_{h<\omega_1}$ of disjunctive measurable subsets of the space E. Therefore $\overline{\mu}_h(Z_h) = 1$ for all $h < \omega_1$ and the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, $cardH = c$, is strongly separable because there exists a family of elements of the σ-algebra $S_1 = \cap_{h\in H} dom(\overline{\mu}_h)$ such that:

1) $\overline{\mu}_h(Z_h) = 1$ $\forall h \in H$;
2) $Z_h \cap Z_{h'} = \emptyset$ for all different h and h' from H;
3) $\cup_{h\in H}Z_h = E$.

For $x \in E$, we put $\delta(x) = h$, where h is the unique hypothesis from the set H for which $x \in Z_h$. The existence of such a unique hypothesis from H can be proved using conditions 2), 3).

Now let $Y \in B(H)$. Then $\{x : \delta(x) \in Y\} = \cup_{h\in Y}Z_h$. We must show that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_h)$ for each $h \in H$.

If $h_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{h\in Y}Z_h = Z_{h_0} \cup (\cup_{h\in Y\setminus\{h_0\}}Z_h).$$

On the one hand, from the validity of the condition 1), 2), 3) it follows that

$$Z_{h_0} \in S_1 = \cap_{h\in H} dom(\overline{\mu}_h) \subseteq dom(\overline{\mu}_{h_0}).$$

On the other hand, the validity of the condition

$$\cup_{h\in Y\setminus\{h_0\}} Z_h \subseteq (E \setminus Z_{h_0})$$

implies that

$$\overline{\mu}_{h_0}(\cup_{h\in Y\setminus\{h_0\}} Z_h) = 0.$$

The last equality yields $\cup_{h\in Y\setminus\{h_0\}} Z_h \in dom(\overline{\mu}_{h_0})$.
Since $dom(\overline{\mu}_{h_0})$ is a σ-algebra, we deduce that

$$\{x : \delta(x) \in Y\} = Z_{h_0} \cup (\cup_{h\in Y\setminus\{h_0\}} Z_h) \in dom(\overline{\mu}_{h_0}).$$

If $h_0 \notin Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{h\in Y} Z_h \subseteq (E \setminus Z_{h_0})$$

and we conclude that $\overline{\mu}_{h_0}\{x : \delta(x) \in Y\} = 0$. The last relation implies that

$$\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{h_0}).$$

Thus we have shown the validity of the relation

$$\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{h_0})$$

for an arbitrary $h_0 \in H$. Hence,

$$\{x : \delta(x) \in Y\} \in \cap_{h\in H} dom(\overline{\mu}_h) = S_1.$$

We have shown that the map $\delta : (E, S_1) \longrightarrow (H, B(H))$ is a measurable map. Since $B(H)$ contains all singletons of H we ascertain that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = \overline{\mu}_h(Z_h) = 1, \ \ \forall h \in H.$$

Necessity. The existence of a consistent criterion for hypotheses testing $\delta : (E, S_1) \longrightarrow (H, B(H))$ implies that $\overline{\mu}_h(\{x : \delta(x) = h\}) = 1 \ \ \forall h \in H$. Setting $X_h = \{x : \delta(x) = h\}$ for $h \in H$ we get:
1) $\overline{\mu}_h(X_h) = 1 \ \ \forall h \in H$;
2) $X_{h_1} \cap X_{h_2} = \emptyset$ for all different parameters h_1 and h_2 from H;
3) $\cup_{h\in H} X_h = \{x : \delta(x) \in H\} = E$.
Therefore the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is strongly separable, hence, there exist S_1-measurable sets X_h $(h \in H)$, such that

$$\overline{\mu}_h(X_{h'}) = \begin{cases} 1, & if\ h = h'; \\ 0, & if\ h \neq h'. \end{cases}$$

We associate the measure $\overline{\mu}_{h_i}$ with the function $I_{X_{h_i}}(x) \in F(M_H)$. Then we have

$$\int\limits_E I_{X_{h_i}}(x)\overline{\mu}_{h_i}(dx) = \int\limits_E I_{X_{h_i}}(x)I_{X_{h_i}}(x)\overline{\mu}_{h_i}(dx) = (\overline{\mu}_{h_i}, \overline{\mu}_{h_i}).$$

If we associate now the measure $\overline{\mu}_{h_1} \in H_2(\overline{\mu}_{h_i})$ with the function $f_{h_1}(x) = f_1(x)I_{X_{h_1}}(x) \in F(M_H)$ then for all $\overline{\mu}_{h_2} \in M_H(\overline{\mu}_{h_i})$ we can write

$$\int\limits_E f_{h_1}(x)\overline{\mu}_{h_2}(dx) = \int\limits_E f_1(x)I_{X_{h_1}}(x)\overline{\mu}_{h_2}(dx) = \int\limits_E f_1(x)f_2(x)I_{X_{h_1}}(x)\overline{\mu}_{h_i}(dx)$$

$$= \int\limits_E f_{h_1}(x)f_2(x)\overline{\mu}_{h_i}(dx) = (\overline{\mu}_{h_1}, \overline{\mu}_{h_2}),$$

where $f_2(x) = \overline{\mu}_{h_2}(dx)/\overline{\mu}_{h_i}(dx)$

Further, we associate the measure

$$\nu(C) = \sum_{i \in I_1 \subset H} \int\limits_C g_i(x)\overline{\mu}_{h_i}(dx) \in M_H$$

with the function

$$f(x) = \sum_{i \in I_1} g_i(x)I_{X_{h_i}}(x) \in F(M_B).$$

Then for the measure

$$\nu_1(C) = \sum_{i \in I_2 \subset H} \int\limits_C g_i^1(x)\overline{\mu}_{h_i}(dx) \in M_H$$

we have

$$\int\limits_E f(x)\nu_1(dx) = \int\limits_E \sum_{i \in I_1 \cap I_2} g_i(x)g_i^1(x)\overline{\mu}_{h_i}(dx)$$

$$= \sum_{i \in I_1 \cap I_2} \int\limits_E g_i(x)g_i^1(x)\overline{\mu}_{h_i}(dx) = (\nu, \nu_1).$$

It follows from the proved theorem that the above correspondence connects some function $f \in F(M_B)$ into correspondence with each linear continuous functional l_f. If in $F(M_B)$ we identify functions that coincide with respect to measures $\{\overline{\mu}_h,\ h \in H\}$, then the correspondence will be bijective. □

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