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Approximation Properties of Partial Sums of Fourier Series

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In this paper we find a class of functions for which the Lebesgue estimate can be improved.

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Let $C([0,2\pi])$ denote the space of continuous functions f with period 2π. If $f \in C([0,2\pi])$ then the function

$$\omega_p(\delta, f) = \sup_x \sup_{|h|\le\delta} |\delta_p(x;h,f)|, \ \ \omega_1(\delta,f) = \omega(\delta,f).$$

is called the modulus of continuity of the function f where

$$\Delta_1(x;h,f) = f(x+h) - f(x),$$
$$\Delta_{p+1}(x;h,f) = \Delta_p(x+h;h,f) - \Delta_p(x;h,f).$$

Denote by $\operatorname{Lip}\alpha$ the class of functions $f \in C([0,2\pi])$ for which $\omega(\delta,f) \le c(f)\delta^\alpha$ and let $S_n(f,x)$ be the n-th partial sum of the trigonometric Fourier series of the function f.

The Lebesgue estimate (see [3, p. 116] or [1, Ch. 1]) is well known

$$\|f - S_n(f)\|_C \le c\,\omega\Big(\frac{1}{n}, f\Big)\log(x+2).$$

Generalizations of this estimation were introduced by Chanturia [2], Oskolkov [8], Karchava [6]. The questions devoted to estimation of the uniform deviation of f from its partial Fourier sums with respect to the Walsh, Vilenkin (bounded and unbounded cases) systems were discussed by Fine [4], Onnewer [7], Tevzadze [9], Gát [5].

In the paper [6] we improved this estimation for some subclasses of the class $C([0,2\pi])$. In particular we proved the following theorems.

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Theorem 1: *If $f \in \operatorname{Lip}\alpha$, $0 < \alpha < 1$ and has a finite number of monotonocity intervals, then*

$$\|f - S_n(f)\|_C \leq \frac{c(f;\alpha)}{n^\alpha}\,.$$

Theorem 2: *If $C([0, 2\pi])$ and has a finite number of convexity or concavity intervals, then*

$$\|f - S_n(f)\|_C \leq c(f)\omega\Big(\frac{1}{n}\,; f\Big).$$

We showed that analogous estimates are valid for functions of several variables. For simplicity, let us show this for functions of two variables.

Let $x_2 = x + (i-1)h_1$, $y_2 = y + (i-1)h_2$ for functions $f = f(x;y)$. Assume

$$\Delta_1^1(x;y;h;f) = f(x+h;y) - f(x;y),$$
$$\Delta_{p+1}^1(x;y;h;f) = \Delta_p^1(x+h;y) - \Delta_p^1(x;y),$$
$$\Delta_1^2(x;y;h;f) = f(x;y+h) - f(x;y),$$
$$\Delta_{p+1}^2(x;y;h;f) = \Delta_p^2(x+h) - \Delta_p^2(x;y),$$
$$\omega_p^1(\delta;f) = \sup_{x;y}\sup_{|h|\leq\delta}|\Delta_p^1(x;y;h;f)|,$$
$$\omega_p^2(\delta;f) = \sup_{x;y}\sup_{|h|\leq\delta}|\Delta_p^2(x;y;h;f)|,$$
$$\sigma_p^1(i;x;y;h;f) = \sum_{q=1}^{i}(-1)^q\Delta_p^1(x_q;y;h;f) \equiv \sigma_p^1(i;f) \equiv \sigma^1(i;f),$$
$$\sigma_p^2(i;x;y;h;f) = \sum_{q=1}^{i}(-1)^q\Delta_p^2(x;y_q;h;f) \equiv \sigma_p^2(i;f) \equiv \sigma^2(i;f),$$
$$\sigma_p^1(i;\sigma_q^2(i;f)) = \sigma_q^2(j,\sigma_p^1(i;f)) = \sigma_{p,q}^{1,2}(i;j;f)$$

i.e. the operation $\sigma_{p,q}^{1,2}$ is obtained by successive application of the operations σ^1 and σ^2.

Let $S_{n,m}(f)$ be partial sums of the double trigonometric Fourier series of functions $f(x;y)$.

We prove the following estimation.

Theorem 3:

$$\|f - S_{n,m}(f)\|_C \leq c(f;p;q)\sup_{x;y}\bigg(\sum_{i=1}^{n}\frac{|\sigma_p^1(i;f)|}{i^{2-\varepsilon}} + \sum_{i=1}^{m}\frac{|\sigma_p^2(i;f)|}{i^{2-\varepsilon}}\bigg).$$

This theorem gives rise to two corollaries which generalize Theorems 1 and 2.

Corollary 4: *If the function $f(x;y)$ has a finite number of monotonicity intervals with respect to separate variables and $\omega_1^1(\delta;f) \leq c(f)\delta^\alpha$, $\omega_1^2(\delta;f) \leq c(f)\delta^\beta$,*

$0 < \gamma < 1$, $0 < \beta < 1$, *then*

$$\|f - S_{n,m}\|_C \le c(f)\Big(\frac{1}{n^\alpha} + \frac{1}{m^\beta}\Big).$$

Proof: In Theorem 3 we assume that $p = q = 1$. Then

$$|\sigma_p^1(i; f)| \le c(f)\omega\Big(\frac{i\pi}{n}\,; f\Big) \le c(f)\Big(\frac{i\pi}{n}\Big)^\alpha,$$

$$\sup_{x;y} \sum_{i=1}^n \frac{|\sigma_1^1(i; f)|}{i^2} \le \sum_{i=1}^n \frac{c(f)(\frac{i\pi}{n})^\alpha}{i^2} \le \frac{c(f)}{n^\alpha} \sum_{i=1}^n \frac{1}{i^{2-\alpha}} \le \frac{c(f)}{n^\alpha}.$$

Analogously,

$$\sup_{x;y} \sum_{i=1}^n \frac{|\sigma_1^2(i; f)|}{i^2} \le \frac{c(f)}{n^\beta}.$$

□

Corollary 5: *If the function $f(x; y)$ has a finite number of convexity or concavity intervals with respect to separate variables, then*

$$\|f - S_{n,m}\|_C \le c(f)\bigg(\omega_1^1\Big(\frac{1}{n}\,; f\Big) + \omega_1^2\Big(\frac{1}{m}\,; f\Big)\bigg).$$

Proof: In Theorem 3 we assume that $p = 1$, $q = 2$. Then

$$\Delta_2^1(x; y; h; f) \ge 0, \quad \Delta_2^2(x; y; h; f) \ge 0,$$

$$|\sigma_2^1(i; f)| = \Big|\sum_{q=1}^i (-1)^q \Delta_2^1(x_q; y; h; f)\Big|$$

$$= \Big|\sum_{q=1}^i \Delta_2^1(x_q; y; h; f)\Big| \le c(f)\omega_1^1\Big(\frac{1}{n}\,; f\Big),$$

$$\sum_{i=1}^n \frac{\Delta_2^1(i; f)}{i^{2-\varepsilon}} \le c(f)\omega_1^1(h; f) \sum_{i=1}^n \frac{1}{i^{2-\varepsilon}} \le c(f)\omega_1^1\Big(\frac{1}{n}\,; f\Big).$$

Analogously,

$$\sum_{i=1}^n \frac{\Delta_2^2(i; f)}{i^{2-\varepsilon}} \le c(f)\omega_1^2\Big(\frac{1}{m}\,; f\Big).$$

□

Proof: [Proof of Theorem 3] Let $T_{\ n,m}(x; y)$ be the Vallée Poussin trigonometric polynomials which realize the best approximation of the function $f(x, y)$. They are written as follows

$$T_{n,m}(x,y)=\iint\limits_{T^2} f(x+u;y+v)V_{n,m}(u;v)\,du\,dv,$$

$$\iint\limits_{T^2} |V_{n,m}(u;v)|\,du\,dv\le M,$$

$$f-f_{n,m}=\frac{1}{\pi^2}\iint\limits_{T^2}\big(f(x+u;y+v)-f(x;y)\big)D_n(u)D_n(v)\,du\,dv$$
$$=\frac{1}{\pi^2}\iint\limits_{T^2} g(u;v)D_n(u)D_n(v)\,du\,dv,$$

$$g=g(u;v)=g_{x;y}(u;v)$$
$$=f(x+u;y+v)-f(x;y)-T_{n,m}(x+u;y+v)-T_{n,m}(x;y),$$

$$\frac{1}{\pi^2}\iint\limits_{T^2}\big(T_{n,m}(x+u;y+v)-D_{n,m}(x;y)\big)D_n(u)D_n(v)\,du\,dv$$
$$=S_{n,m}(T_{n,m})-T_{n,m}=0,$$

$$\|f-T_{n,m}\|_{C(T^2)}\le E_{n,m}(f),\quad T=[0;2\pi],\quad T^2=[0;2\pi]^2,$$

$$\sup_{x;y}\sigma_p^k(i;T_{n,m})\le\sup_{x;y}\sigma_p^k(i;f),\quad k=1,2,$$

$$\sup_{x;y}\sigma_p^k(i;g)\le c\sup_{x;y}\sigma_p^k(i;f),$$

$$\sup_{x;y}\sigma_{1,2}(i;j;T_{n,m})\le c\sup_{x;y}|\sigma(i;j;f)|,\qquad |\sigma_{p,q}^{1,2}(i;j;f)|\le c(q)|\sigma_p^1(i;f)|,$$

$$\sup_{x;y}\sigma(i;j;g)\le c\sup_{x;y}\sigma(i;j;f),\qquad \sigma_{p,q}^{1,2}(i;j;f)\le c(p)i|\sigma_q^2(j;f)|.$$

In [6] we showed that

$$\Big|\int\limits_{-\pi}^{\pi} g(t)D_n(t)\,dt\Big|\le\sup_t\Big\{\sum_{i=1}^n\frac{|\sigma_p(i;\frac{\pi}{n};g)|}{i^2}+|g(t)|\Big\},$$

$$\Big|\iint\limits_{T^2} g(u;v)D_n(u)D_m(v)\,du\,dv\Big|=\Big|\int\limits_{-\pi}^{\pi}\Big(\int\limits_{-\pi}^{\pi} g(u;v)D_n(u)\,du\Big)D_n(v)\,dv\Big|$$

$$=\int\limits_{-\pi}^{\pi} p_n(v)D_n(v)\,dv\le C\Big(\sup_v\sum_{j=1}^m\frac{|\sigma_q(j;v_2;\frac{\pi}{m};p_n(v))|}{j^2}+\sup_v|p_n(v)|\Big),$$

$$p_n(v)=\int\limits_{-\pi}^{\pi} g(u;v)D_n(u)\,du,$$

$$\sup_v |p_n(v)| \le \sup_{u,v} \sum_{i=1}^{n} \frac{|\sigma_p^1(j;x;y;\frac{\pi}{n};g)|}{i^2} + \sup_{u;v} g(u;v),$$

$$\left|\sigma\Big(j;\frac{\pi}{m};v,p_n(v)\Big)\right| = \left|\sum_{i=1}^{j}(-1)^r \Delta_q\Big(v_r;\frac{\pi}{m};p_n(v)\Big)\right|$$

$$= \left|\int_{-\pi}^{\pi}\sum_{i=1}^{j}(-1)^r \Delta_q^2(u;v_q;g) D_n(u)\,du\right| = \left|\int_{-\pi}^{\pi}\sigma_1^2(j;g)D_n(u)\,du\right|$$

$$\le c\sup\sum_{i=1}^{n}\frac{\sigma^1(\sigma_q^2(g))}{i^2} + \sup\sigma^2(j;g),$$

$$\|f - S_{n,m}\|_C \le \sup\Bigg\{\sum_{i=1}^{n}\frac{|\sigma_p^1(i;\frac{\pi}{n};f)|}{i^2} + \sum_{i=1}^{m}\frac{|\sigma_q^2(i;\frac{\pi}{m};f)|}{i^2} + \sum_{i,j=1}^{n,m}\frac{|\sigma_p^1\sigma_q^2(i;j;f)|}{i^2j^2}\Bigg\},$$

$$\sum_{i,j=1}^{n,m}\frac{|\sigma_p^1\sigma_q^2(i;j;f)|}{i^2j^2} = \sum_{i,j=1}^{n,m}\frac{\sqrt{|\sigma_p^1\sigma_q^2(i;j)|}\sqrt{|\sigma_p^1\sigma_q^2(i;j)|}}{i^2j^2}$$

$$\le \sum_{i,j=1}^{n,m}\frac{\sqrt{|c(q)j\sigma_p^2(j;f)|}\sqrt{|c(p)i\sigma_q^1(j;f)|}}{i^2j^2}$$

$$= \sum_{i,j=1}^{n,m}\frac{\sqrt{|\sigma_p^1(i;f)|}\sqrt{|\sigma_q^2(j;f)|}}{i^{1+\frac12}j^{1+\frac12}} \le c(p)\sum_{i,j=1}^{n,m}\frac{\sqrt{|\sigma_p^1(i)}\sqrt{|\sigma_q^2(j)|}}{i^{1-\frac{\varepsilon}{2}}j^{\frac12+\frac{\varepsilon}{2}}j^{1-\frac{\varepsilon}{2}}i^{\frac12+\frac{\varepsilon}{2}}}$$

applicable $ab \le \frac{1}{2}\left(a^2+b^2\right)$

$$\le c(p)\Bigg(\sum_{i,j=1}^{n,m}\frac{|\sigma_p^1(i;f)|}{i^{2-\varepsilon}j^{1+\varepsilon}} + \sum_{i,j=1}^{n,m}\frac{|\sigma_q^2(j;f)|}{j^{2-\varepsilon}i^{1+\varepsilon}}\Bigg)$$

$$= c(p)\Bigg(\sum_{j=1}^{m}\frac{1}{j^{1+\varepsilon}}\sum_{i=1}^{n}\frac{|\sigma_p^1(i;f)|}{i^{2-\varepsilon}} + \sum_{i=1}^{n}\frac{1}{i^{1+\varepsilon}}\sum_{i,j=1}^{n,m}\frac{|\sigma_q^2(j;f)|}{j^{2-\varepsilon}}\Bigg)$$

$$\le c(p;\varepsilon)\Bigg(\sum_{i=1}^{n}\frac{|\sigma_p^1(i;f)|}{i^{2-\varepsilon}} + \sum_{j=1}^{m}\frac{|\sigma_q^2(i;f)|}{j^{1+\varepsilon}}\Bigg).$$

□

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