Bulletin of TICMI
Vol. 23, No. 2, 2019, 159–162

SHORT COMMUNICATIONS

A Nonlinear Equation for the Rectangular Dynamic Shell

Jemal Peradze*

Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University
2 University St., 0186, Tbilisi, Georgia
Department of Mathematics, Georgian Technical University
77 M. Kostava St., Tbilisi 0175, Georgia

(*Received March 22, 2018; Revised September 22, 2019; Accepted November 1, 2019*)

From an initial boundary value problem for the system of nonlinear differential equations, which describes the large deflection of a rectangular shell, we obtain a nonlinear integro-differential equation for the transverse displacement. This equation is analogous by its structure to the Kirchhoff equation for a string, the Woinowsky-Krieger equation for a beam and the Berger equation for a plate which are united under the common name Kirchhoff type equations.

Keywords: Shell vibration, Kirchhoff-Love hypothesis, Transverse displacement, Integro-differential equation, Kirchhoff type equations.

AMS Subject Classification: 74B20, 74K25.

Let us consider a problem of nonlinear vibration of a sloping shell. We will use the classical model based on the Kirchhoff-Love hypothesis [3, 6].

Equations of the problem can be taken in the form

$$
\begin{aligned}
&\frac{\partial N_1}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}+p_1=0,\\
&\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_2}{\partial x_2}+p_2=0,\\
&\rho h\frac{\partial^2 w}{\partial t^2}+D\Delta^2 w=\frac{\partial}{\partial x_1}\left(N_1\frac{\partial w}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial w}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(N_2\frac{\partial w}{\partial x_2}\right)\\
&\qquad\qquad+\frac{\partial}{\partial x_1}\left(N_{12}\frac{\partial w}{\partial x_2}\right)+N_1k_1+N_2k_2+q,\\
&\qquad (x_1,x_2)\in\Omega,\quad 0<t\le T,
\end{aligned}
\tag{1}
$$

* Email: j_peradze@yahoo.com

ISSN: 1512-0082 print

where for $N_1 = N_1(x_1, x_2, t)$, $N_2 = N_2(x_1, x_2, t)$, $N_{12} = N_{12}(x_1, x_2, t)$ we have

$$N_1 = \frac{Eh}{1-\nu^2}\left\{\frac{\partial u_1}{\partial x_1} - k_1 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2 + \nu\left[\frac{\partial u_2}{\partial x_2} - k_2 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2\right]\right\},$$
$$N_2 = \frac{Eh}{1-\nu^2}\left\{\frac{\partial u_2}{\partial x_2} - k_2 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2 + \nu\left[\frac{\partial u_1}{\partial x_1} - k_1 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2\right]\right\}, \quad (2)$$
$$N_{12} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right).$$

Here $u_l = u_l(x_1, x_2, t)$, $l = 1, 2$, and $w = w(x_1, x_2, t)$ are respectively longitudinal and transverse displacements of the point (x_1, x_2) of the shell midsurface at the moment of time t (Figure 1.), Ω is the domain occupied by the shell in plan, T is the upper boundary of the time interval, $k_l = k_l(x_1, x_2, t)$ are the shell curvatures, $l = 1, 2$, $p_l = p_l(x_1, x_2, t)$, $l = 1, 2$, $q = q(x_1, x_2, t)$ are the external force components, Δ is the Laplace operator, E and $0 < \nu < \frac{1}{2}$ are respectively Young's modulus and Poisson's ratio, D is the shell flexural rigidity, ρ is the mass density, h is the thickness.

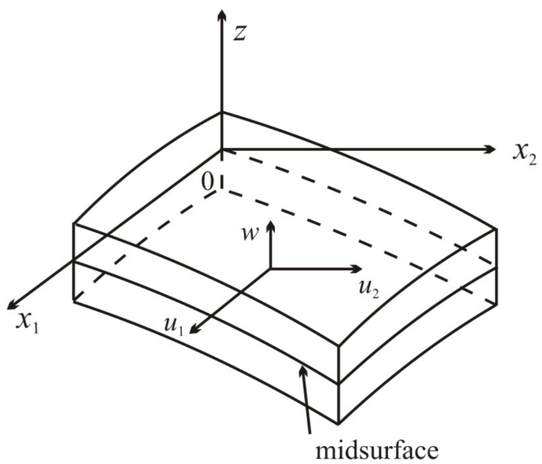

Figure 1.

System (1) does not contain the inertia terms $\frac{\partial^2 u_1}{\partial t^2}$ and $\frac{\partial^2 u_2}{\partial t^2}$. According to the author of the paper [7] where the solvability of system (1) is proved, this neglect can be justified by the fact that, as a rule, the frequencies of shell proper longitudinal vibrations are quite high and essentially higher than proper frequencies of transverse vibrations. Therefore such a simplification of the problem is practically admissible if principal frequencies of external forces are assumed much smaller than first proper frequencies of shell longitudinal vibrations.

The aim of this paper is to obtain from system (1) and formulas (2) the equation for the transverse displacement function w. To achieve this aim, from the first two equations of system (1) which form a linear subsystem with respect to u_1 and u_2, we have to express these functions through w,

$$u_l = \psi_l(w), \quad l = 1, 2, \quad (3)$$

and, after that, using (3) in the third equation of (1), to derive the sought equation for w. We have to choose a domain Ω and boundary conditions for the functions u_1 and u_2. We take respectively the rectangle and the homogeneous Dirichlet and Neumann conditions on the boundary $\partial\Omega$ of the domain Ω for the functions u_1 and u_2. So, let

$$\Omega = \{(x_1, x_2) \mid 0 < x_1 < \alpha, \;\; 0 < x_2 < \beta\}, \tag{4}$$

and

$$u_1|_{\partial\Omega} = 0, \quad \left.\frac{\partial u_2}{\partial n}\right|_{\partial\Omega} = 0, \tag{5}$$

where n is the external normal to the boundary $\partial\Omega$.

As to the initial and boundary conditions with the participation of the function w, they do not play any special role and can vary in form.

Using the Fourier series method [4], for the shell transverse displacement function $w(x_1, x_2, t)$ we have obtained [5] a nonlinear integro-differential equation of the form

$$\begin{aligned}
\rho h \frac{\partial^2 w}{\partial t^2} + D\Delta^2 w - \sum_{m=1}^{2}\sum_{n=1}^{2} \Bigg\{ \int_\Omega \Bigg[A_{mn}\left(\frac{1}{2}\left(\frac{\partial w}{\partial \xi_1}\right)^2 - k_1 w\right) \\
+ C_{mn}\frac{\partial w}{\partial \xi_1}\frac{\partial w}{\partial \xi_2} + B_{mn}\left(\frac{1}{2}\left(\frac{\partial w}{\partial \xi_2}\right)^2 - k_2 w\right) + d_{1mn}p_1 + d_{2mn}p_2\Bigg] d\xi_1\, d\xi_2 \\
+ \int_{\partial\Omega}\Bigg[a_{mn}\left(\frac{1}{2}\left(\frac{\partial w}{\partial \xi_1}\right)^2 - k_1 w\right) + c_{mn}\frac{\partial w}{\partial \xi_1}\frac{\partial w}{\partial \xi_2} \\
+ b_{mn}\left(\frac{1}{2}\left(\frac{\partial w}{\partial \xi_2}\right)^2 - k_2 w\right)\Bigg] ds \Bigg\} \left(\delta_{mn}k_m + \frac{\partial^2 w}{\partial x_m \partial x_n}\right) \\
+ p_1 \frac{\partial w}{\partial x_1} + p_2\frac{\partial w}{\partial x_2} = q, \quad (x_1, x_2) \in \Omega, \quad 0 < t \le T,
\end{aligned} \tag{6}$$

where the integrand coefficients A_{mn}, B_{mn}, C_{mn}, d_{1mn}, d_{2mn} and a_{mn}, b_{mn}, c_{mn}, $m, n = 1, 2$, depend on x_1, x_2 and ξ_1, ξ_2, ds is an element of the boundary $\partial\Omega$, δ_{mn} the Kronecker symbol, $m, n = 1, 2$.

Equation (6) is related to the string equation

$$\frac{\partial^2 w}{\partial t^2} - \left(c_0 + c_1\int_0^1 \left(\frac{\partial w}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 w}{\partial x^2} = 0, \tag{7}$$

introduced by Kirchhoff [2], the Woinowsky–Krieger beam equation [9]

$$\frac{\partial^2 w}{\partial t^2} + c_0\frac{\partial^4 w}{\partial x^4} - \left(c_1 + c_2\int_0^1 \left(\frac{\partial w}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 w}{\partial x^2} = 0, \tag{8}$$

and the Berger rectangular plate equation [1] which in Wah's interpretation [8] looks like

$$\frac{\partial^2 w}{\partial t^2} + c_0 \Delta^2 w - \left\{ \int_\Omega \left[\left(\frac{\partial w}{\partial \xi_1} \right)^2 + \left(\frac{\partial w}{\partial \xi_2} \right)^2 \right] d\xi_1 \, d\xi_2 \right\} \Delta w = 0. \tag{9}$$

Equations (7)–(9) together with numerous modifications and generalizations of the first two of them and also their analogues for static problems, form the class of integro-differential equations called the Kirchhoff class.

As to equation (6), the derivation of which the present paper is dedicated to, owing to its form and purpose it is a new equation belonging to this class. Note that there exists a vast bibliography on various aspects of Kirchhoff type equations. These are the works which deal with problems of the existence and uniqueness of a solution, stability and control, the construction and investigation of approximate algorithms, numerical computations. The interest in Kirchhoff type equations is permanent.

A few words about possible applications of equation (6). If the vibration of a shell, rectangular in the plan, is described by system (1) with certain initial and boundary conditions, which include (5) too, then such a problem reduces to the solution of only one equation (6). After finding w, the functions u_1 and u_2 are constructed using (3). As to the initial and boundary conditions for the function w which accompany equation (6), they are obtained from the corresponding relations of the initial problem provided that we use (3) in order to exclude u_1 and u_2 if they are contained in these relations.

In some cases, to describe the shell it suffices to know only the transverse displacement function. Then the problem reduces to the solution of only equation (6).

We have good reason to believe that the replacement of the domain Ω (4) and the boundary conditions (5) considered in this paper for u_1 and u_2 by some other domain and boundary conditions will not affect the type of equation (6) for w, but will change only the formulas only for the integrand coefficients A_{mn}, B_{mn}, C_{mn}, d_{1mn}, d_{2mn} and a_{mn}, b_{mn}, c_{mn}, which will not be necessarily written in terms of series. The insertion of inertia terms in the first two equations of system (1) will not evidently affect the type of the equation for transverse displacement either.

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