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Approximations by Multivariate Generalized Trigonometric Type Singular Integral Operators

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This research and survey work deals exclusively with the study of the approximation of generalized multivariate trigonometric type singular integrals to the identity-unit operator. Here we study quantitatively most of their approximation properties. These operators are not in general positive linear operators. In particular we study the rate of convergence of these integral operators to the unit operator, as well as the related simultaneous approximation. These are given via Jackson type inequalities and by the use of multivariate high order modulus of smoothness of the high order partial derivatives of the involved function. We also study the global smoothness preservation properties of these integral operators. These multivariate inequalities are nearly sharp and in one case the inequality is attained, that is sharp. Furthermore we give asymptotic expansions of Voronovskaya type for the error of approximation. The above properties are studied with respect to L_p norm, $1 \leq p \leq \infty$.

Keywords: Approximations by multivariate trigonometric type singular integral, Multivariate moduli of smoothness.

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1. Introduction

We start with our motivation for this work. The following comes from [5].

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Here it is $\xi \in (0, 1]$.

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}_+$, and $f^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$, $\beta \in \mathbb{N}$, we define for

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$x \in \mathbb{R}$, the trigonometric integral

$$T_{r,\xi}\left(f;x\right) := \frac{1}{W}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f\left(x+jt\right)\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta} dt, \tag{2}$$

where

$$W = \int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta} dt = 2\xi^{1-2\beta}\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^{2\beta} dt \overset{(6)}{=}$$

$$2\xi^{1-2\beta}\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^{k}\frac{k^{2\beta-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}. \tag{3}$$

$T_{r,\xi}$ operators are not positive operators, see [7].

We mention:

let p and m be integers with $1 \leq p \leq m$. We define the integral

$$I\left(m,p\right) := \int_{-\infty}^{\infty}\frac{\left(\sin x\right)^{2m}}{x^{2p}}dx = 2\int_{0}^{\infty}\frac{\left(\sin x\right)^{2m}}{x^{2p}}dx. \tag{4}$$

That is an (absolutely) convergent integral.

According to [9], page 210, item 1033, we obtain

$$I\left(m,p\right) = \pi\frac{\left(-1\right)^{p}\left(2m\right)!}{4^{m-p}\left(2p-1\right)!}\sum_{k=1}^{m}\left(-1\right)^{k}\frac{k^{2p-1}}{\left(m-k\right)!\left(m+k\right)!}. \tag{5}$$

In particular, for $p = m$ the above formula becomes

$$\int_{0}^{\infty}\frac{\left(\sin x\right)^{2m}}{x^{2m}}dx = \pi\left(-1\right)^{m}m\sum_{k=1}^{m}\left(-1\right)^{k}\frac{k^{2m-1}}{\left(m-k\right)!\left(m+k\right)!}. \tag{6}$$

We need the rth L_p-modulus of smoothness

$$\omega_r\left(f^{(n)},h\right)_p := \sup_{|t|\leq h}\left\|\Delta_t^r f^{(n)}\left(x\right)\right\|_{p,x}, \quad h>0, \tag{7}$$

where

$$\Delta_t^r f^{(n)}\left(x\right) := \sum_{j=0}^{r}\left(-1\right)^{r-j}\binom{r}{j}f^{(n)}\left(x+jt\right), \tag{8}$$

see [8], p. 44. Here we have $\omega_r\left(f^{(n)},h\right)_p < \infty,\ h>0$.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, ..., n \in \mathbb{N}. \tag{9}$$

Call

$$\tau (w, x) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)} (x + jw) - \delta_n f^{(n)} (x). \tag{10}$$

Notice also that

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

According to [2], p. 306, [1], we get

$$\tau (w, x) = \Delta_w^r f^{(n)} (x). \tag{11}$$

Thus

$$\|\tau (w, x)\|_{p,x} \leq \omega_r \left(f^{(n)}, |w| \right)_p, \quad w \in \mathbb{R}. \tag{12}$$

Using Taylor's formula one has

$$\sum_{j=0}^{r} \alpha_j [f (x + jt) - f (x)] = \sum_{k=1}^{n} \frac{f^{(k)} (x)}{k!} \delta_k t^k + R_n (0, t, x), \tag{13}$$

where

$$R_n (0, t, x) := \int_0^t \frac{(t - w)^{n-1}}{(n-1)!} \tau (w, x) \, dw, \ \ n \in \mathbb{N}. \tag{14}$$

Assume

$$c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi (t) \in \mathbb{R}, \quad k = 1, ..., n, \tag{15}$$

where

$$d\mu_\xi (t) := \frac{1}{W} \left(\frac{\sin \left(\frac{t}{\xi} \right)}{t} \right)^{2\beta} dt, \quad \forall \ t \in \mathbb{R}.$$

Using the above terminology we derive

$$\Delta (x) := T_{r,\xi} (f; x) - f (x) - \sum_{k=1}^{n} \frac{f^{(k)} (x)}{k!} \delta_k c_{k,\xi} = R_n^* (x), \tag{16}$$

where

$$R_n^*(x) := \int_{-\infty}^{\infty} R_n(0,t,x)\, d\mu_\xi(t), \quad n \in \mathbb{N}. \tag{17}$$

Let $\lceil \cdot \rceil$ denote the ceiling of a real number. We mention

Theorem 1.1: *([5]) Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n, \beta \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + np + 1}{2}$ and the rest as above. Then*

$$\|\Delta(x)\|_p \leq \frac{1}{((n-1)!)\,(q(n-1)+1)^{\frac{1}{q}}\,(rp+1)^{\frac{1}{p}}} \tag{18}$$

$$\left[\frac{1}{\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt} \sum_{j=1}^{\lceil rp \rceil + 1} \int_0^\infty \left[t^{np-1+j} \left(\frac{\sin t}{t}\right)^{2\beta}\right] dt\right]^{\frac{1}{p}} \xi^n \omega_r\left(f^{(n)}, \xi\right)_p.$$

Moreover, as $\xi \to 0$ we get that $\|\Delta(x)\|_p \to 0$.

The counterpart of Theorem 1.1 follows, case of $p = 1$.

Theorem 1.2: *([5]) Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\beta > \frac{r+1+n}{2}$. Then*

$$\|\Delta(x)\|_1 \leq \frac{1}{(r+1)(n-1)!\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]} \tag{19}$$

$$\sum_{j=1}^{r+1} \left(\int_0^\infty \left[t^{n-1+j} \left(\frac{\sin t}{t}\right)^{2\beta}\right] dt\right) \xi^n \omega_r\left(f^{(n)}, \xi\right)_1.$$

Hence as $\xi \to 0$ we obtain $\|\Delta(x)\|_1 \to 0$.

The case $n = 0$ is mentioned next.

Proposition 1.3: *([5]) Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\beta \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + 1}{2}$ and the rest as above. Then*

$$\|T_{r,\xi}(f) - f\|_p \leq \omega_r(f,\xi)_p \left(\frac{1}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]} \sum_{j=0}^{\lceil rp \rceil} \left[\int_0^\infty t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]\right)^{\frac{1}{p}}. \tag{20}$$

Also as $\xi \to 0$ we obtain $T_{r,\xi} \to$ unit operator I in the L_p norm, $p > 1$.

We also give

Proposition 1.4: *([5]) For $\beta \in \mathbb{N}$, $\beta > \frac{r+1}{2}$, we have*

$$\|T_{r,\xi}(f) - f\|_1 \leq \frac{\omega_r(f,\xi)_1}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]} \sum_{j=0}^{r} \left[\int_0^\infty t^j \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]. \tag{21}$$

Moreover as $\xi \to 0$ we get that $T_{r,\xi} \to I$ in the L_1 norm.

We also mention:
Case $\beta = 2$.

Corollary 1.5: *([5]) Let $f \in C^1(\mathbb{R})$ and $f' \in L_1(\mathbb{R})$. Then*

$$\left\| T_{1,\xi}(f;x) - f(x) \right\|_1 \leq \frac{3}{2\pi}\left(\ln 2 + \frac{\pi}{4}\right)\xi\omega_1\left(f',\xi\right)_1. \quad (22)$$

Corollary 1.6: *([5]) Let $f \in C^1(\mathbb{R})$ and $f' \in L_1(\mathbb{R})$. Then*

$$\left\| T_{2,\xi}(f;x) - f(x) \right\|_1 \leq \left(\frac{40}{33\pi}\ln\left(\frac{32^{\frac{27}{16}}}{4}\right) + \frac{5}{33} + \frac{5}{22\pi}\ln\frac{256}{27}\right)\xi\omega_2\left(f',\xi\right)_1. \quad (23)$$

Corollary 1.7: *([5]) It holds*

$$\left\| T_{1,\xi}(f) - f \right\|_4 \leq \omega_1(f,\xi)_4 \sqrt[4]{\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{15}{22\pi}\ln\frac{256}{27} + \frac{47}{22}}. \quad (24)$$

Also as $\xi \to 0$ we obtain $T_{1,\xi} \to$ unit operator I in the L_4 norm.

Corollary 1.8: *([5]) We have*

$$\left\| T_{6,\xi}(f) - f \right\|_1 \leq \omega_6(f,\xi)_1 \left(\frac{630}{151\pi}\ln\frac{2^{\frac{251}{60}}}{3^{\frac{9}{5}}} + \frac{5671}{2416}\right). \quad (25)$$

Moreover as $\xi \to 0$ we get that $T_{6,\xi} \to I$ in the L_1 norm.

We will use the following:

Remark 1: ([6]) Let $j, m \in \mathbb{Z}$, $m \geq 1$ such that $0 \leq j < 2m-1$. The integral

$$\int_{-\infty}^{\infty} x^j \left(\frac{\sin x}{x}\right)^{2m} dx = \begin{cases} 2\int_0^{\infty} x^j \left(\frac{\sin x}{x}\right)^{2m} dx, & \text{if } j \text{ is even} \\ 0, & \text{if } j \text{ is odd} \end{cases}, \quad (26)$$

is an (absolutely) convergent integral.

According to [9], page 210, item 1033, we obtain
case 1: j is even, $j < 2m-1$

$$\int_0^{\infty} x^j \left(\frac{\sin x}{x}\right)^{2m} dx = \frac{\pi(-1)^{\frac{2m-j}{2}}(2m)!}{2^{j+1}(2m-j-1)!}\sum_{k=1}^{m}(-1)^k \frac{k^{2m-j-1}}{(m-k)!\,(m+k)!}, \quad (27)$$

and
case 2: j is odd, $j < 2m-1$

$$\int_0^{\infty} x^j \left(\frac{\sin x}{x}\right)^{2m} dx = \frac{(-1)^{\frac{j-1}{2}}(2m)!}{2^j(2m-j-1)!}\sum_{k=1}^{m}(-1)^{m-k} \frac{k^{2m-j-1}\left[\ln(2k)\right]}{(m-k)!\,(m+k)!}. \quad (28)$$

In particular, for $j=0$ the formula (27) becomes

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^{2m}dx=\pi\left(-1\right)^m m\sum_{k=1}^{m}\left(-1\right)^k\frac{k^{2m-1}}{\left(m-k\right)!\left(m+k\right)!}. \tag{29}$$

In this work we study the approximation properties of general multivariate smooth trigonometric singular integral operators:

$$T_{r,n}^{[m]}\left(f;x_1,...,x_N\right)$$

$$:=\lambda_n^{-N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}f\left(x_1+s_1j,...,x_N+s_Nj\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N, \tag{30}$$

with $\beta\in N$, and

$$\lambda_n:=2\xi_n^{1-2\beta}\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}, \tag{31}$$

see [7], [9], p. 210, item 1033.

Notice that

$$\lambda_n^{-N}\int_{\mathbb{R}^N}\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N=1, \tag{32}$$

see also [7], [9], p. 210, item 1033, and [3], p. 16.

We call

$$\gamma:=2\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}, \tag{33}$$

that is

$$\lambda_n=\gamma\xi_n^{1-2\beta}. \tag{34}$$

Here $r\in\mathbb{N}$, $m\in Z_+$, and

$$\alpha_{j,r}^{[m]}:=\begin{cases}\left(-1\right)^{r-j}\binom{r}{j}j^{-m}, & \text{if } j=1,2,...,r,\\ 1-\sum\limits_{i=1}^{r}\left(-1\right)^{r-i}\binom{r}{i}i^{-m}, & \text{if } j=0,\end{cases} \tag{35}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{36}$$

See that $\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1$.

Here also $\xi_n \in (0,1]$, $n \in \mathbb{N}$, and $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function. The above operator $T_{r,n}^{[m]}$ is a special case of a more general operator $\theta_{r,n}^{[m]}$ studied in general in [3] by the author.

Next we mention about $\theta_{r,n}^{[m]}$.

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$.

We define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, ..., x_N) := \sum_{j=0}^{r} \alpha_{j.r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j)\, d\mu_{\xi_n}(s), \tag{37}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$.

The operators $\theta_{r,n}^{[m]}$ are not in general positive. For example, consider the function $\varphi(u_1, ..., u_N) = \sum_{i=1}^{N} u_i^2$ and also take $r = 2$, $m = 3$; $x_i = 0$, $i = 1, ..., N$. See that $\varphi \geq 0$, however

$$\theta_{2,n}^{[3]} (\varphi; 0, 0, ..., 0) = \left(\sum_{j=1}^{2} j^2 \alpha_{j,2}^{[3]} \right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2 \right) d\mu_{\xi_n}(s)$$

$$= \left(\alpha_{1,2}^{[3]} + 4\alpha_{2,2}^{[3]} \right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2 \right) d\mu_{\xi_n}(s) = \left(-2 + \frac{1}{2} \right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2 \right) d\mu_{\xi_n}(s) < 0. \tag{38}$$

assuming that $\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2 \right) d\mu_{\xi_n}(s) < \infty$.

Clearly in the case of $T_{r,n}^{[m]}$ we have

$$d\mu_{\xi_n}(s) = \lambda_n^{-N} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i =: d\varphi_{\xi_n}(s), \quad s \in \mathbb{R}^N. \tag{39}$$

Lemma 1.9: *The operator $\theta_{r,n}^{[m]}$ preserves the constant functions in N variables.*

We need the following definition.

Definition 1.10: Let $f \in C_B(\mathbb{R}^N)$, the space of all bounded and continuous functions or uniformly continuous on $\mathbb{R}^N$. Then, the rth multivariate modulus of smoothness of f is given by (see, e.g. [4])

$$\omega_r(f;h) := \sup_{\sqrt{u_1^2+...+u_N^2}\le h} \left\|\Delta^r_{u_1,u_2,...,u_N}(f)\right\|_\infty < \infty, \quad h>0, \tag{40}$$

where $\|\cdot\|_\infty$ is the sup-norm and

$$\Delta^r_u f(x) := \Delta^r_{u_1,u_2,...,u_N} f(x_1,...,x_N)$$

$$= \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1+ju_1, x_2+ju_2, ..., x_N+ju_N). \tag{41}$$

Let $m \in \mathbb{N}$ and let $f \in C^m(\mathbb{R}^N)$.

Suppose that all partial derivatives of f of order m are bounded, i.e.

$$\left\| \frac{\partial^m f(\cdot,\cdot,...,\cdot)}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}} \right\|_\infty < \infty, \tag{42}$$

for all $\alpha_j \in \mathbb{Z}^+$, $j=1,...,N$; $\sum_{j=1}^{N} \alpha_j = m$.

In this work we apply the general theory developed in [3] about $\theta^{[m]}_{r,n}$ to the operators $T^{[m]}_{r,n}$, so we can obtain computationaly specific results and show that the general theory has applications and it is a valid theory.

So for the very important in various branches of mathematics operators $T^{[m]}_{r,n}$ we prove the very essential properties of uniform approximation, L_p approximation, global smoothness preservation and simultaneously approximation, Voronovskaya asymptotic expansions and complex simultaneous approximation.

2. Auxilliary essential results

We will use

Lemma 2.1: *Let $N \in \mathbb{N}$, $r>0$, $z_i \in \mathbb{R}_+$, $i=1,...,N$. Then*

$$\left(1+\sum_{i=1}^{N} z_i\right)^r \le \prod_{i=1}^{N} (1+z_i)^r. \tag{43}$$

Proof: We have

$$\left(1+\sum_{i=1}^{N} z_i\right)^r \le \left(N+\sum_{i=1}^{N} z_i\right)^r = [(1+z_1)+(1+z_2)+...+(1+z_N)]^r$$

$$= \left(\sum_{i=1}^{N} (1 + z_i) \right)^r \leq \prod_{i=1}^{N} (1 + z_i)^r, \quad \text{by } 1 + z_i \geq 1, \ i = 1, ..., N.$$

□

We give

Theorem 2.2: *Let $r, N, \beta \in \mathbb{N}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m \in \mathbb{N}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Here we take $\beta > \frac{r+m+1}{2}$, and γ, λ_n are as in (33) and (34), respectively. Also we take $\lambda = 0, 1, ..., r$. When λ is even we define*

$$\psi_{1\lambda} := \frac{\pi (-1)^{\frac{2\beta-\lambda}{2}} (2\beta)!}{2^{\lambda+1} (2\beta - \lambda - 1)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-1}}{(\beta - k)! (\beta + k)!} \right), \tag{44}$$

and when λ is odd we define

$$\psi_{2\lambda} := \frac{(-1)^{\frac{\lambda-1}{2}} (2\beta)!}{2^{\lambda} (2\beta - \lambda - 1)!} \left(\sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-1} [\ln (2k)]}{(\beta - k)! (\beta + k)!} \right), \tag{45}$$

and we set

$$\psi_{\lambda} := \begin{cases} \psi_{1\lambda}, & \text{if } \lambda \text{ is even,} \\ \psi_{2\lambda}, & \text{if } \lambda \text{ is odd.} \end{cases} \tag{46}$$

Similarly, it is defined $\psi_{\lambda+m}$, just set in (44), (45), (46), $\lambda+m$ in place of λ. Then

$$A_{\xi_n} (\overline{\alpha}) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \prod_{i=1}^{N} \left(\frac{\sin \left(\frac{s_i}{\xi_n} \right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$\leq \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} [\psi_{\lambda} + \psi_{\lambda+m}] \right\}^N$$

$$\leq 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} [\psi_{\lambda} + \psi_{\lambda+m}] \right\}^N < +\infty, \tag{47}$$

uniformly bounded, and convergent to zero as $\xi_n \to 0$, when $n \to +\infty$.

Proof: We estimate

$$A_{\xi_n} (\overline{\alpha}) = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \prod_{i=1}^{N} \left(\frac{\sin \left(\frac{s_i}{\xi_n} \right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$= 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i}\right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^N ds_i \tag{48}$$

$$\leq 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i}\right) \left(1 + \sum_{i=1}^N \left(\frac{s_i}{\xi_n}\right)\right)^r \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^N ds_i$$

$$= \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i}\right) \left(1 + \sum_{i=1}^N z_i\right)^r \prod_{i=1}^N \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^N dz_i$$

$$\overset{(43)}{\leq} \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i}\right) \left(\prod_{i=1}^N (1 + z_i)^r\right) \prod_{i=1}^N \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^N dz_i$$

$$= \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \prod_{i=1}^N \left(\int_0^\infty z^{\alpha_i} (1+z)^r \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)$$

$$= \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \prod_{i=1}^N \left[\sum_{\lambda=0}^r \binom{r}{\lambda} \left(\int_0^\infty z^{\lambda+\alpha_i} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)\right]$$

$$= \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \prod_{i=1}^N \left\{\sum_{\lambda=0}^r \binom{r}{\lambda} \left[\int_0^1 z^{\lambda+\alpha_i} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right.\right.$$

$$\left.\left. + \int_1^\infty z^{\lambda+\alpha_i} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right]\right\} \tag{49}$$

$$\leq \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{\sum_{\lambda=0}^r \binom{r}{\lambda} \left[\int_0^1 z^{\lambda} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right.\right.$$

$$\left.\left. + \int_1^\infty z^{\lambda+m} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right]\right\}^N$$

$$\leq \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\int_0^\infty z^\lambda \left(\frac{\sin z}{z} \right)^{2\beta} dz \right.\right.$$

$$\left.\left. + \int_0^\infty z^{\lambda+m} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\}^N =: I. \tag{50}$$

Based on [9], p. 210, item 1033 and [6], see (27), (28), and by assuming $\mathbb{N} \ni \beta > \frac{r+m+1}{2}$, i.e. $\lambda < \lambda + m < 2\beta - 1$, for all $\lambda = 0, 1, ..., r$, we have the following calculations:

Let λ be even, then

$$\int_0^\infty z^\lambda \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{\pi \left(-1\right)^{\frac{2\beta-\lambda}{2}} (2\beta)!}{2^{\lambda+1} (2\beta - \lambda - 1)!} \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-1}}{(\beta-k)!\,(\beta+k)!} = \psi_{1\lambda}. \tag{51}$$

Let λ be odd, then

$$\int_0^\infty z^\lambda \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{\left(-1\right)^{\frac{\lambda-1}{2}} (2\beta)!}{2^{\lambda} (2\beta - \lambda - 1)!} \sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-1} \left[\ln (2k)\right]}{(\beta-k)!\,(\beta+k)!} = \psi_{2\lambda}. \tag{52}$$

Therefore

$$\int_0^\infty z^\lambda \left(\frac{\sin z}{z} \right)^{2\beta} dz = \psi_\lambda = \begin{cases} \psi_{1\lambda}, & \text{when } \lambda \text{ is even,} \\ \psi_{2\lambda}, & \text{when } \lambda \text{ is odd.} \end{cases} \tag{53}$$

Similarly, for $\lambda + m$ being even, we get

$$\int_0^\infty z^{\lambda+m} \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{\pi \left(-1\right)^{\frac{2\beta-\lambda-m}{2}} (2\beta)!}{2^{\lambda+m+1} (2\beta - \lambda - m - 1)!} \tag{54}$$

$$\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-m-1}}{(\beta-k)!\,(\beta+k)!} = \psi_{1(\lambda+m)}.$$

And when $\lambda + m$ is odd we get

$$\int_0^\infty z^{\lambda+m} \left(\frac{\sin z}{z} \right)^{2\beta} dz = \frac{\left(-1\right)^{\frac{\lambda+m-1}{2}} (2\beta)!}{2^{\lambda+m} (2\beta - \lambda - m - 1)!} \tag{55}$$

$$\sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-m-1} \left[\ln (2k)\right]}{(\beta-k)!\,(\beta+k)!} = \psi_{2(\lambda+m)}.$$

Therefore, it holds

$$\int_0^\infty z^{\lambda+m}\left(\frac{\sin z}{z}\right)^{2\beta} dz = \psi_{\lambda+m} = \begin{cases} \psi_{1(\lambda+m)}, & \text{when } \lambda+m \text{ is even,} \\ \psi_{2(\lambda+m)}, & \text{when } \lambda+m \text{ is odd.} \end{cases} \tag{56}$$

That is

$$I = \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m}\right] \right\}^N$$

$$\leq 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m}\right] \right\}^N < +\infty. \tag{57}$$

I.e. $A_{\xi_n}(\overline{\alpha})$ is uniformly bounded. The theorem is proved. □

We continue with

Theorem 2.3: *Let $r, n \in \mathbb{N}$, $\xi_n \in (0,1]$, $\beta \in \mathbb{N} : \beta > \frac{r+1}{2}$, $N \in \mathbb{N} - \{1\}$. Here γ, λ_n are as in (33) and (34), respectively, and ψ_λ is defined by (44), (45) and (46), $\lambda = 0, 1, ..., r$. Then*

$$B_{\xi_n} := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$\leq \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N$$

$$\leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N < +\infty, \tag{58}$$

uniformly bounded, and convergent to zero as $\xi_n \to 0$, when $n \to +\infty$.

Proof: We estimate

$$B_{\xi_n} = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$= 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \tag{59}$$

$$\leq 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1+\sum_{i=1}^{N}\left(\frac{s_i}{\xi_n}\right)\right)^r \prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$= \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(1+\sum_{i=1}^{N} z_i\right)^r \prod_{i=1}^{N}\left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i$$

$$\overset{(43)}{\leq} \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} (1+z_i)^r\right) \prod_{i=1}^{N}\left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i$$

$$= \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\int_0^{\infty} (1+z)^r \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)^N$$

$$= \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \left(\int_0^{\infty} z^{\lambda} \left(\frac{\sin z}{z}\right)^{2\beta} dz\right)\right]^N$$

$$\overset{(53)}{=} \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_{\lambda}\right]^N \tag{60}$$

$$\leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_{\lambda}\right]^N < +\infty,$$

under $\beta > \frac{r+1}{2}$. The theorem is proved. □

We also give

Theorem 2.4: *Let $p > 1$; $r, \beta, N \in \mathbb{N}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N$: $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m \in \mathbb{N}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Here we take $\beta > \frac{\lceil rp \rceil + m + 1}{2}$, and γ, λ_n are as in (33) and (34), respectively, and λ runs as $\lambda = 0, 1, ..., \lceil rp \rceil$. Furthermore ψ_λ is defined as in (44), (45) and (46). Similarly, it is defined $\psi_{\lambda+mp}$, just set in (44), (45), (46), $(\lambda + mp)$ instead of λ. Then*

$$C_{\xi_n}(\overline{\alpha}) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)^p \prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \tag{61}$$

$$\leq \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} [\psi_\lambda + \psi_{\lambda+mp}] \right\}^N$$

$$\leq 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} [\psi_\lambda + \psi_{\lambda+mp}] \right\}^N < +\infty,$$

uniformly bounded, and convergent to zero as $\xi_n \to 0$, when $n \to +\infty$.
Above $\lceil \cdot \rceil$ is the ceiling of the number.

Proof: We estimate

$$C_{\xi_n}(\overline{\alpha}) = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^N ds_i$$

$$= 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\left(\prod_{i=1}^N s_i^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^N ds_i$$

$$\leq 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{\alpha_i p} \right) \left(1 + \sum_{i=1}^N \left(\frac{s_i}{\xi_n} \right) \right)^{rp} \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^N ds_i \tag{62}$$

$$= 2^N \gamma^{-N} \xi_n^{2\beta(N-1)+mp} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(1 + \sum_{i=1}^N z_i \right)^{rp} \prod_{i=1}^N \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^N dz_i$$

$$\stackrel{(43)}{\leq} \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{\alpha_i p} \right) \left(\prod_{i=1}^N (1+z_i)^{rp} \right) \prod_{i=1}^N \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^N dz_i$$

$$= \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \prod_{i=1}^N \left(\int_0^\infty (1+z)^{rp} z^{\alpha_i p} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right)$$

$$\leq \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \prod_{i=1}^N \left(\int_0^\infty (1+z)^{\lceil rp \rceil} z^{\alpha_i p} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right) \tag{63}$$

$$= \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \prod_{i=1}^{N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\int_0^\infty z^{\lambda+\alpha_i p} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right) \right\}$$

$$= \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \prod_{i=1}^{N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left[\int_0^1 z^{\lambda+\alpha_i p} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right.\right.$$

$$\left.\left. + \int_1^\infty z^{\lambda+\alpha_i p} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\}$$

$$\leq \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left[\int_0^1 z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right.\right.$$

$$\left.\left. + \int_1^\infty z^{\lambda+mp} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\}^N$$

$$\leq \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left[\int_0^\infty z^{\lambda} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right.\right.$$

$$\left.\left. + \int_0^\infty z^{\lambda+mp} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right] \right\}^N$$

$$\overset{((53),\,(56))}{=} \xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left[\psi_\lambda + \psi_{\lambda+mp} \right] \right\}^N \tag{64}$$

$$\leq 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left[\psi_\lambda + \psi_{\lambda+mp} \right] \right\}^N < +\infty,$$

i.e. $C_{\xi_n}(\overline{\alpha})$ is uniformly bounded.

We assumed above that $\mathbb{N} \ni \beta > \frac{\lceil rp \rceil + m + 1}{2}$, i.e. $\lambda < \lambda + m < 2\beta - 1$, for all $\lambda = 0, 1, ..., \lceil rp \rceil$.

The theorem is proved. □

We also present

Theorem 2.5: *Let $p > 1$; $r, \beta \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Here we take $\beta > \frac{\lceil rp \rceil + 1}{2}$, and γ, λ_n are as in (33) and (34), respectively, and λ runs as $\lambda = 0, 1, ..., \lceil rp \rceil$. Furthermore ψ_λ is defined as in (44), (45) and (46). Then*

$$D_{\xi_n} := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$\leq \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda\right]^N \tag{65}$$

$$\leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda\right]^N < +\infty,$$

uniformly bounded, and convergent to zero as $\xi_n \to 0$, when $n \to +\infty$.

Proof: We estimate

$$D_{\xi_n} = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i \tag{66}$$

$$= 2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$\leq^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^{N} \left(\frac{s_i}{\xi_n}\right)\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$= 2^N \gamma^{-N} \xi_n^{2\beta(N-1)} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^{N} z_i\right)^{rp} \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i$$

$$\overset{(43)}{\leq} \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} (1 + z_i)^{rp}\right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i$$

$$= \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\int_0^\infty (1+z)^{rp} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right)^N$$

$$\leq \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left(\int_0^\infty (1+z)^{\lceil rp \rceil} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right)^N$$

$$= \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\int_0^\infty z^\lambda \left(\frac{\sin z}{z} \right)^{2\beta} dz \right) \right]^N \tag{67}$$

$$\overset{(53)}{=} \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda \right]^N$$

$$\leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda \right]^N < +\infty,$$

under $\beta > \frac{\lceil rp \rceil + 1}{2}$. The theorem is proved. □

We proceed to

Theorem 2.6: *Let $n, N \in \mathbb{N}$, $\xi_n \in (0,1]$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m \in \mathbb{N}$. Here $\beta \in \mathbb{N} : \beta > \frac{m+1}{2}$, and γ, λ_n are as in (33) and (34), respectively. Furthermore ψ_{α_i} is defined as in (44), (45) and (46), just replace λ by α_i, $i = 1, ..., N$. Then*

$$F_{\xi_n}(\overline{\alpha}) := \xi_n^{-m} \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i$$

$$\leq \xi_n^{2\beta(N-1)} 2^N \gamma^{-N} \max_{|\alpha|=m} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) \tag{68}$$

$$\leq 2^N \gamma^{-N} \max_{|\alpha|=m} \left(\prod_{i=1}^{N} \psi_{\alpha_i} \right) =: \varphi < +\infty.$$

Proof: We estimate

$$F_{\xi_n}\left(\overline{\alpha}\right)=\xi_n^{-m}\lambda_n^{-N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i$$

$$=\xi_n^{-m}\lambda_n^{-N}2^N\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i$$

$$=\xi_n^{2\beta(N-1)}2^N\gamma^{-N}\int_{\mathbb{R}_+^N}\left(\prod_{i=1}^{N}z_i^{\alpha_i}\right)\prod_{i=1}^{N}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}\prod_{i=1}^{N}dz_i \tag{69}$$

$$=\xi_n^{2\beta(N-1)}2^N\gamma^{-N}\prod_{i=1}^{N}\left(\int_0^{\infty}z^{\alpha_i}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right)$$

$$\overset{(53)}{=}\xi_n^{2\beta(N-1)}2^N\gamma^{-N}\left(\prod_{i=1}^{N}\psi_{\alpha_i}\right)\leq\xi_n^{2\beta(N-1)}2^N\gamma^{-N}\max_{|\alpha|=m}\left(\prod_{i=1}^{N}\psi_{\alpha_i}\right)$$

$$\leq 2^N\gamma^{-N}\max_{|\alpha|=m}\left(\prod_{i=1}^{N}\psi_{\alpha_i}\right)<+\infty,$$

under $\beta>\frac{m+1}{2}$, i.e. $\alpha_i<2\beta-1$, $i=1,...,N$. The theorem is proved. □

We make

Remark 1: As in Theorem 2.6, we denote

$$\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}:=\lambda_n^{-N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i, \tag{70}$$

where $\widetilde{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{j=1}^{N}\alpha_j=\widetilde{j}$.

By (68) we obtain

$$\left|\overline{c}_{\overline{\alpha},n}\right|=\left|\overline{c}_{\overline{\alpha},n,\widetilde{j}}\right|\leq\varphi\xi_n^m\leq\varphi. \tag{71}$$

3. Main results for $T_{r,n}^{[m]}$

3.1. *Uniform approximation*

We start with an application to $T_{r,n}^{[m]}$ of the following theorem.

Theorem 3.1 : *([3], p. 11) Let $m \in \mathbb{N}$, $f \in C^m\left(\mathbb{R}^N\right)$, $N \geq 1$, $x \in \mathbb{R}^N$. Assume $\left\| \frac{\partial^m f(\cdot,\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \right\|_\infty < \infty$, for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N : |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence.*

Suppose that for all $\overline{\alpha} := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$ we have

$$u_{\xi_n}(\overline{\alpha}) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{72}$$

For $\widetilde{j} = 1, \ldots, m$, and $\overline{\alpha} := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, call

$$c_{\overline{\alpha},n} := c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s_1, \ldots, s_N). \tag{73}$$

Then
i)

$$E_{r,n}^{[m]}(x) := \left| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right|$$

$$\leq \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0; \\ |\overline{\alpha}| = m}} \frac{\left(\omega_r\left(f_{\overline{\alpha}}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i \right)} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right). \tag{74}$$

$\forall\ x \in \mathbb{R}^N$.
ii)

$$\left\| E_{r,n}^{[m]} \right\|_\infty \leq R.H.S.(74). \tag{75}$$

Given that $\xi_n \to 0$, as $n \to \infty$, and u_{ξ_n} is uniformly bounded, then we derive that $\left\| E_{r,n}^{[m]} \right\| \to 0$ with rates.

iii) It holds also that

$$\left\|\theta_{r,n}^{[m]}(f)-f\right\|_{\infty} \leq \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}|=\widetilde{j}}} \frac{\left|c_{\overline{\alpha},n,\widetilde{j}}\right| \|f_{\overline{\alpha}}\|_{\infty}}{\prod_{i=1}^{N} \alpha_i!} \right) + R.H.S.(74), \tag{76}$$

given that $\|f_{\overline{\alpha}}\|_{\infty} < \infty$, *for all* $\overline{\alpha} : |\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$. *Furthermore, as* $\xi_n \to 0$ *when* $n \to \infty$, *assuming that* $c_{\overline{\alpha},n,\widetilde{j}} \to 0$, *while* u_{ξ_n} *is uniformly bounded, we conclude that*

$$\left\|\theta_{r,n}^{[m]}(f)-f\right\|_{\infty} \to 0 \tag{77}$$

with rates.

A uniform approximation result for $T_{r,n}^{[m]}$ follows:

Theorem 3.2: *Let* $r, N, \beta, m \in \mathbb{N}$, $f \in C^m\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$. *Assume* $\left\|\frac{\partial^m f(\cdot,\cdot,\dots,\cdot)}{\partial x_1^{\alpha_1}\dots\partial x_N^{\alpha_N}}\right\|_{\infty} < \infty$, *for all* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$. *Let* φ_{ξ_n} *be the Borel probability measure on* $\mathbb{R}^N$, *see (39), where* $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Here* $\beta > \frac{m+r+1}{2}$, *and* $A_{\xi_n}(\overline{\alpha})$ *as in (47), and* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70). Then*

i)

$$\overline{E}_{r,n}^{[m]}(x) := \left| T_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}|=\widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right|$$

$$\leq \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}|=m}} \frac{(\omega_r(f_{\overline{\alpha}}, \xi_n))}{\left(\prod_{i=1}^{N} \alpha_i!\right)} A_{\xi_n}(\overline{\alpha}), \tag{78}$$

$\forall\, x \in \mathbb{R}^N$.

ii)

$$\left\|\overline{E}_{r,n}^{[m]}\right\|_{\infty} \leq R.H.S.(78). \tag{79}$$

Given that $\xi_n \to 0$, *as* $n \to +\infty$, *we have that* $A_{\xi_n}(\overline{\alpha}) \to 0$ *and are uniformly bounded, and then we derive that* $\left\|\overline{E}_{r,n}^{[m]}\right\|_{\infty} \to 0$ *with rates.*

iii) It holds also that

$$\left\|T_{r,n}^{[m]}(f)-f\right\|_{\infty} \leq \sum_{\widetilde{j}=1}^{m} \left|\delta_{\widetilde{j},r}^{[m]}\right| \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}|=\widetilde{j}}} \frac{\left|\overline{c}_{\overline{\alpha},n,\widetilde{j}}\right| \|f_{\overline{\alpha}}\|_{\infty}}{\prod_{i=1}^{N} \alpha_i!} \right) + R.H.S.(78), \tag{80}$$

given that $\|f_{\overline{\alpha}}\|_\infty < +\infty$, *for all* $\overline{\alpha} : |\overline{\alpha}| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$. *Furthermore, as* $\xi_n \to 0$ *when* $n \to +\infty$, *we have that* $\overline{c}_{\overline{\alpha},n,\widetilde{j}} \to 0$ *and* $A_{\xi_n}(\overline{\alpha}) \to 0$, *and both are uniformly bounded, and we conclude that*

$$\left\| T_{r,n}^{[m]}(f) - f \right\|_\infty \to 0 \tag{81}$$

with rates.

Proof: Mainly by applying Theorem 3.1. By Theorem 2.2 we get that $A_{\xi_n}(\overline{\alpha}) \to 0$ and $A_{\xi_n}(\overline{\alpha})$ are uniformly bounded. By Theorem 2.6 and Remark **1** we get $\overline{c}_{\overline{\alpha},n} \to 0$ and $\overline{c}_{\overline{\alpha},n}$ are uniformly bounded. □

We mention

Theorem 3.3: *([3], p. 14) Let* $f \in C_B(\mathbb{R}^N)$, *uniformly continuous,* $N \geq 1$, $\xi_n \in (0,1]$. *Then*

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_r(f, \xi_n), \tag{82}$$

under the assumption

$$\Phi_{\xi_n} := \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{83}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *given that* Φ_{ξ_n} *are uniformly bounded, we derive*

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_\infty \to 0 \tag{84}$$

with rates.

We give

Theorem 3.4: *Let* $f \in C_B(\mathbb{R}^N)$, *uniformly continuous,* $\beta, r \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$, $\beta > \frac{r+1}{2}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. *Then*

$$\left\| T_{r,n}^{[0]} f - f \right\|_\infty \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \omega_r(f, \xi_n). \tag{85}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *we derive*

$$\left\| T_{r,n}^{[0]} f - f \right\|_\infty \to 0 \tag{86}$$

with rates.

Proof: By Theorems 2.3 and 3.3. □

3.2. L_p Approximation for $T_{r,n}^{[m]}$

We need

Definition 3.5: ([4], [8]) We call

$$\Delta_u^r f(x) := \Delta_{u_1,u_2,...,u_N}^r f(x_1,...,x_N) \tag{87}$$

$$:= \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, ..., x_N + ju_N).$$

Let $p \geq 1$, the modulus of smoothness of order r is given by

$$\omega_r(f;h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \tag{88}$$

$h > 0$.

We will apply

Theorem 3.6: *([3], p. 24) Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_{\overline{\alpha}} \in L_p(\mathbb{R}^N)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$. Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume for all $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{89}$$

For $\widetilde{j} = 1,...,m$, and $\overline{\alpha} := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, call

$$c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{90}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p = \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x} \tag{91}$$

$$\leq \left(\frac{m}{\left(q\left(m-1\right)+1\right)^{\frac{1}{q}}}\right)\left(\sum_{|\overline{\alpha}|=m}\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)$$

$$\left[\int_{\mathbb{R}^N}\left[\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right]^p d\mu_{\xi_n}(s)\right]^{\frac{1}{p}}\omega_r\left(f_{\overline{\alpha}},\xi_n\right)_p .$$

As $n\to\infty$ and $\xi_n\to 0$, by (91) we obtain $\left\|E_{r,n}^{[m]}\right\|\to 0$ with rates.

One also finds by (91) that

$$\left\|\theta_{r,n}^{[m]}(f;x)-f(x)\right\|_{p,x}\leq\sum_{\widetilde{j}=1}^{m}\left|\delta_{\widetilde{j},r}^{[m]}\right|\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\left|c_{\overline{\alpha},n,\widetilde{j}}\right|}{\prod_{i=1}^{N}\alpha_i!}\|f_{\overline{\alpha}}\|_p\right)+R.H.S.(91), \quad (92)$$

given that $\|f_{\overline{\alpha}}\|_p<\infty$, $|\overline{\alpha}|=\widetilde{j}$, $\widetilde{j}=1,...,m$.

Assuming that $c_{\overline{\alpha},n,\widetilde{j}}\to 0$, $\xi_n\to 0$, as $n\to\infty$, we get $\left\|\theta_{r,n}^{[m]}(f)-f\right\|_p\to 0$, that is $\theta_{r,n}^{[m]}\to I$ the unit operator, in L_p norm, with rates.

We present

Theorem 3.7: *Let $f\in C^m\left(\mathbb{R}^N\right)$, $r,\beta,N,m\in\mathbb{N}$, with $f_{\overline{\alpha}}\in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}|=m$, $x\in\mathbb{R}^N$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Here φ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ as in (39), for $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Let $\beta>\frac{\lceil rp\rceil+m+1}{2}$; $\alpha_j\in\mathbb{Z}^+$, $j=1,...,N:|\overline{\alpha}|:=\sum_{j=1}^{N}\alpha_j=m$. Here $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N:|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$. Then*

$$\left\|\overline{E}_{r,n}^{[m]}\right\|_p=\left\|T_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right)\right\|_{p,x}$$

$$\leq\left(\frac{m}{\left(q\left(m-1\right)+1\right)^{\frac{1}{q}}}\right)2^{\frac{N}{p}}\gamma^{-\frac{N}{p}}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left(\psi_\lambda+\psi_{\lambda+mp}\right)\right\}^{\frac{N}{p}}$$

$$\left(\sum_{|\overline{\alpha}|=m}\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\xi_n^{\left(\frac{2\beta(N-1)}{p}+m\right)}\omega_r\left(f_{\overline{\alpha}},\xi_n\right)_p. \tag{93}$$

As $n\to\infty$ and $\xi_n\to0$, by (93) we obtain $\left\|\overline{E}_{r,n}^{[m]}\right\|\to0$ with rates.

One also finds by (93) that

$$\left\|T_{r,n}^{[m]}(f;x)-f(x)\right\|_{p,x}\le\sum_{\widetilde{j}=1}^{m}\left|\delta_{\widetilde{j},r}^{[m]}\right|\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\left|\overline{c}_{\overline{\alpha},n,\widetilde{j}}\right|}{\prod_{i=1}^{N}\alpha_i!}\left\|f_{\overline{\alpha}}\right\|_p\right)+R.H.S.(93), \tag{94}$$

given that $\|f_{\overline{\alpha}}\|_p<\infty$, $|\overline{\alpha}|=\widetilde{j}$, $\widetilde{j}=1,...,m$.

Assuming that $\xi_n\to0$, as $n\to\infty$, we get $\left\|T_{r,n}^{[m]}(f)-f\right\|_p\to0$, that is $T_{r,n}^{[m]}\to I$ the unit operator, in L_p norm, with rates.

Proof: By Theorem 3.6. From Theorem 2.4 we get that $C_{\xi_n}(\overline{\alpha})$ is uniformly bounded, see (61) and $C_{\xi_n}(\overline{\alpha})\to0$, as $\xi_n\to0$, when $n\to\infty$. Also by Theorem 2.6 and Remark **1** we get that $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ are uniformly bounded and $\overline{c}_{\overline{\alpha},n}\to0$, as $\xi_n\to0$, when $n\to\infty$. □

We continue with an application of

Theorem 3.8: *([3], p. 26) Let $f\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$; $N\ge1$; $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$. Assume μ_{ξ_n} probability Borel measure on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}}>0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp}d\mu_{\xi_n}(s)<\infty. \tag{95}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_p \tag{96}$$

$$\le\left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp}d\mu_{\xi_n}(s)\right)^{\frac{1}{p}}\omega_r\left(f,\xi_n\right)_p.$$

As $\xi_n\to0$, when $n\to\infty$, we derive $\left\|\theta_{r,n}^{[0]}(f)-f\right\|_p\to0$, i.e. $\theta_{r,n}^{[0]}\to I$, the unit operator, in L_p norm.

We give

Theorem 3.9: *Let $f\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$; $N\in\mathbb{N}-\{1\}$, $\beta,r\in\mathbb{N}$; $p,q>1$:*

$\frac{1}{p}+\frac{1}{q}=1$; $\beta>\frac{\lceil rp\rceil+1}{2}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. *Then*

$$\left\|T_{r,n}^{[0]}(f)-f\right\|_p\leq 2^{\frac{N}{p}}\gamma^{-\frac{N}{p}}\left[\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\psi_\lambda\right]^{\frac{N}{p}}\xi_n^{\frac{2\beta(N-1)}{p}}\omega_r(f,\xi_n)_p. \tag{97}$$

As $\xi_n\to0$, *when* $n\to\infty$, *we derive* $\left\|T_{r,n}^{[0]}(f)-f\right\|_p\to0$, *i.e.* $T_{r,n}^{[0]}\to I$, *the unit operator, in* L_p *norm.*

Proof: By Theorems 3.8, 2.5. □

We mention

Theorem 3.10: *([3], p. 27) Let* $f\in\left(C\left(\mathbb{R}^N\right)\cap L_1\left(\mathbb{R}^N\right)\right)$; $N\geq1$. *Assume* μ_{ξ_n} *probability Borel measure on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}}>0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)<\infty. \tag{98}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_1\leq\left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\right)\omega_r(f,\xi_n)_1. \tag{99}$$

As $\xi_n\to0$, *we get* $\theta_{r,n}^{[0]}\to I$, *in* L_1 *norm.*

We give

Theorem 3.11: *Let* $f\in\left(C\left(\mathbb{R}^N\right)\cap L_1\left(\mathbb{R}^N\right)\right)$; $N\in\mathbb{N}-\{1\}$, $r,\beta\in\mathbb{N}$, $\beta>\frac{r+1}{2}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. *Then*

$$\left\|T_{r,n}^{[0]}(f)-f\right\|_1\leq 2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N\xi_n^{2\beta(N-1)}\omega_r(f,\xi_n)_1. \tag{100}$$

As $\xi_n\to0$, *we get* $T_{r,n}^{[0]}\to I$, *in* L_1 *norm.*

Proof: By Theorems 2.3, 3.10. □

We mention

Theorem 3.12: *([3], p. 29) Let* $f\in C^m\left(\mathbb{R}^N\right)$, $m,N\in\mathbb{N}$, *with* $f_{\overline{\alpha}}\in L_1\left(\mathbb{R}^N\right)$, $|\overline{\alpha}|=m$, $x\in\mathbb{R}^N$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n>0$, $(\xi_n)_{n\in\mathbb{N}}$ *is a bounded sequence. Suppose for all* $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^N\alpha_i=m$ *that we have*

$$\int_{\mathbb{R}^N}\left(\left(\prod_{i=1}^N|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)d\mu_{\xi_n}(s)<\infty. \tag{101}$$

For $\widetilde{j}=1,...,m,$ *and* $\overline{\alpha}:=(\alpha_1,...,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\ i=1,...,N,\ |\overline{\alpha}|:=\sum_{i=1}^N\alpha_i=\widetilde{j},$ *call*

$$c_{\overline{\alpha},n,\widetilde{j}}:=\int_{\mathbb{R}^N}\prod_{i=1}^N s_i^{\alpha_i}d\mu_{\xi_n}(s). \tag{102}$$

Then

$$\left\|E_{r,n}^{[m]}\right\|_1=\left\|\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^m\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{c_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}(x)}{\prod_{i=1}^N\alpha_i!}\right)\right\|_{1,x} \tag{103}$$

$$\le\sum_{|\overline{\alpha}|=m}\left(\frac{1}{\prod_{i=1}^N\alpha_i!}\right)\omega_r(f_{\overline{\alpha}},\xi_n)_1\int_{\mathbb{R}^N}\left(\prod_{i=1}^N|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s).$$

As $\xi_n\to0,$ *we get* $\left\|E_{r,n}^{[m]}\right\|_1\to0$ *with rates.*

From (103) we get

$$\left\|\theta_{r,n}^{[m]}f-f\right\|_1\le\sum_{\widetilde{j}=1}^m\left|\delta_{\widetilde{j},r}^{[m]}\right|\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\left|c_{\overline{\alpha},n,\widetilde{j}}\right|}{\prod_{i=1}^N\alpha_i!}\|f_{\overline{\alpha}}\|_1\right)+R.H.S.(103), \tag{104}$$

given that $\|f_{\overline{\alpha}}\|_1<\infty,\ |\overline{\alpha}|=\widetilde{j},\ \widetilde{j}=1,...,m.$

As $n\to\infty,$ *assuming* $\xi_n\to0$ *and* $c_{\overline{\alpha},n,\widetilde{j}}\to0,$ *we obtain* $\left\|\theta_{r,n}^{[m]}(f)-f\right\|_1\to0,$ *that is* $\theta_{r,n}^{[m]}\to I$ *in* L_1 *norm, with rates.*

We give

Theorem 3.13: *Let* $f\in C^m(\mathbb{R}^N),\ r,N,\beta,m\in\mathbb{N},$ *with* $f_{\overline{\alpha}}\in L_1(\mathbb{R}^N),$ *where* $\alpha_j\in\mathbb{Z}^+,\ j=1,...,N:|\overline{\alpha}|:=\sum_{j=1}^N\alpha_j=m,\ x\in\mathbb{R}^N,\ \xi_n\in(0,1],\ n\in\mathbb{N},$ *and* $\beta>\frac{m+r+1}{2}$. *Here* $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70), where* $\widetilde{j}=1,...,m,$ *and* $\overline{\alpha}:=(\alpha_1,...,\alpha_N),$ $\alpha_i\in\mathbb{Z}^+,\ i=1,...,N:|\overline{\alpha}|:=\sum_{i=1}^N\alpha_i=\widetilde{j}.$ *Besides, here* φ_{ξ_n} *is the Borel probability measure on* $\mathbb{R}^N$, *see (39). Then*

$$\left\|\overline{E}_{r,n}^{[m]}\right\|_1=\left\|T_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^m\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}(x)}{\prod_{i=1}^N\alpha_i!}\right)\right\|_{1,x} \tag{105}$$

$$\leq \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r \left(f_{\overline{\alpha}}, \xi_n \right)_1 \right) \xi_n^{2\beta(N-1)+m}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m} \right] \right\}^N .$$

As $\xi_n \to 0$, we get $\left\| \overline{E}_{r,n}^{[m]} \right\|_1 \to 0$ with rates.

From (105) we get

$$\left\| T_{r,n}^{[m]} f - f \right\|_1 \leq \sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\left| \overline{c}_{\overline{\alpha},n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \left\| f_{\overline{\alpha}} \right\|_1 \right) + R.H.S.(105), \tag{106}$$

given that $\left\| f_{\overline{\alpha}} \right\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

As $n \to \infty$, assuming $\xi_n \to 0$, we get $\overline{c}_{\overline{\alpha},n,\widetilde{j}} \to 0$ and $\left\| T_{r,n}^{[m]} (f) - f \right\|_1 \to 0$, that is $T_{r,n}^{[m]} \to I$ in L_1 norm, with rates.

Proof: By Theorem 3.12, also by Theorem 2.2, see (47) and by Theorem 2.6 and Remark **1**. □

3.3. *Global smoothness preservation and simultaneous approximation of $T_{r,n}^{[m]}$*

We need

Definition 3.14: ([3], p. 34) Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by

$$\omega_m (f; h)_p := \sup_{\|t\|_2 \leq h} \left\| \Delta_t^m (f) \right\|_{p,x} , \tag{107}$$

$h > 0$, where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jt) . \tag{108}$$

Denote

$$\omega_m (f; h)_\infty = \omega_m (f, h) . \tag{109}$$

Above, $x, t \in \mathbb{R}^N$.

We present the related global smoothness preservation result

Theorem 3.15: *We assume* $T_{r,n}^{[\widetilde{m}]}(f;x) \in \mathbb{R}$, $\widetilde{m} \in \mathbb{Z}_+$, $\forall\ x \in \mathbb{R}$. *Let* $h > 0$, $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$.

i) Assume $\omega_m(f,h) < \infty$. *Then*

$$\omega_m\left(T_{r,n}^{[\widetilde{m}]} f, h\right) \leq \left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m(f,h). \tag{110}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$. *Then*

$$\omega_m\left(T_{r,n}^{[\widetilde{m}]} f, h\right)_1 \leq \left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m(f,h)_1. \tag{111}$$

iii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p > 1$. *Then*

$$\omega_m\left(T_{r,n}^{[\widetilde{m}]} f, h\right)_p \leq \left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m(f,h)_p. \tag{112}$$

Proof: Direct application of ([3]) Theorem 3.2, p. 35. □

We make

Remark 1: Let $r = 1$, $\widetilde{m} \in \mathbb{Z}_+$, then $\alpha_{0,1}^{[\widetilde{m}]} = 0$, $\alpha_{1,1}^{[\widetilde{m}]} = 1$. Hence

$$T_{1,n}^{[\widetilde{m}]}(f;x) = \lambda_n^{-N}\int_{\mathbb{R}^N} f(x+s)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 \dots ds_N =: T_n(f;x). \tag{113}$$

By Theorem 3.15, we get

Theorem 3.16: *We suppose* $T_n(f;x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}$. *Let* $h > 0$, $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$.

i) Assume $\omega_m(f,h) < \infty$. *Then*

$$\omega_m(T_n f, h) \leq \omega_m(f,h). \tag{114}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$. *Then*

$$\omega_m(T_n f, h)_1 \leq \omega_m(f,h)_1. \tag{115}$$

iii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p > 1$. *Then*

$$\omega_m(T_n f, h)_p \leq \omega_m(f,h)_p. \tag{116}$$

Next, we get an optimality result

Proposition 3.17: *The above inequality (114):*

$$\omega_m (T_n f, h) \leq \omega_m (f, h)$$

is sharp, namely it is attained by any

$$f_j^* (x) = x_j^m, \quad j = 1, ..., N, \quad x = (x_1, ..., x_j, ..., x_N) \in \mathbb{R}^N. \tag{117}$$

Proof: Apply Proposition 3.5, p. 38, of [3]. □

We need

Theorem 3.18: *([3], p. 39) Let $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ a bounded sequence. Let $\overline{\beta} := (\beta_1, ..., \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, ..., N$; $|\overline{\beta}| := \sum_{i=1}^{N} \beta_i = l$. Here $f(x + sj)$, $x, s \in \mathbb{R}^N$, is μ_{ξ_n}-integrable wrt s, for $j = 1, ..., r$. There exist μ_{ξ_n}-integrable functions $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}, ..., h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j} \geq 0$ $(j = 1, ..., r)$ on $\mathbb{R}^N$ such that*

$$\left| \frac{\partial^{i_1} f(x + sj)}{\partial x_1^{i_1}} \right| \leq h_{i_1,j}(s), \quad i_1 = 1, ..., \beta_1, \tag{118}$$

$$\left| \frac{\partial^{\beta_1+i_2} f(x + sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,i_2,j}(s), \quad i_2 = 1, ..., \beta_2,$$

$$\vdots$$

$$\left| \frac{\partial^{\beta_1+\beta_2+...+\beta_{N-1}+i_N} f(x + sj)}{\partial x_N^{i_N} \partial x_{N-1}^{\beta_{N-1}} ... \partial x_2^{\beta_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j}(s), \quad i_N = 1, ..., \beta_N,$$

$\forall\ x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[\widetilde{m}]}(f; x)\right)_{\overline{\beta}} = \theta_{r,n}^{[\widetilde{m}]}\left(f_{\overline{\beta}}; x\right), \quad \widetilde{m} \in \mathbb{Z}_+. \tag{119}$$

In particular, it holds

$$\left(T_{r,n}^{[\widetilde{m}]}(f; x)\right)_{\overline{\beta}} = T_{r,n}^{[\widetilde{m}]}\left(f_{\overline{\beta}}; x\right), \tag{120}$$

when

$$d\mu_{\xi_n} = d\varphi_{\xi_n}(s), \quad s \in \mathbb{R}^N,$$

see (39).

Corollary 3.19: *(by Theorem 3.18, $r=1$) We have*

$$\left(T_n\left(f;x\right)\right)_{\overline{\beta}} = T_n\left(f_{\overline{\beta}};x\right). \tag{121}$$

We present simultaneous global smoothness results.

Theorem 3.20: *Let $h>0$ and the assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$. Here $\overline{\gamma} = 0, \overline{\beta}$ $(0 = (0,...,0))$, $\widetilde{m} \in \mathbb{Z}_+$.*
i) Assume $\omega_m\left(f_{\overline{\gamma}}, h\right) < \infty$. Then

$$\omega_m\left(\left(T_{r,n}^{[\widetilde{m}]}\left(f\right)\right)_{\overline{\gamma}}, h\right) \leq \left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m\left(f_{\overline{\gamma}}, h\right). \tag{122}$$

ii) Additionally suppose $f_{\overline{\gamma}} \in L_1\left(\mathbb{R}^N\right)$. Then

$$\omega_m\left(\left(T_{r,n}^{[\widetilde{m}]}\left(f\right)\right)_{\overline{\gamma}}, h\right)_1 \leq \left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m\left(f_{\overline{\gamma}}, h\right)_1. \tag{123}$$

iii) Additionally suppose $f_{\overline{\gamma}} \in L_p\left(\mathbb{R}^N\right)$, $p>1$. Then

$$\omega_m\left(\left(T_{r,n}^{[\widetilde{m}]}\left(f\right)\right)_{\overline{\gamma}}, h\right)_p \leq \left(\sum_{\widetilde{j}=0}^{r}\left|\alpha_{\widetilde{j},r}^{[\widetilde{m}]}\right|\right)\omega_m\left(f_{\overline{\gamma}}, h\right)_p. \tag{124}$$

We have

Corollary 3.21: *(to Theorem 3.20) Let $h>0$, $r=1$ and $\overline{\gamma} = 0, \overline{\beta}$.*
i) Assume $\omega_m\left(f_{\overline{\gamma}}, h\right) < \infty$. Then

$$\omega_m\left(\left(T_n\left(f\right)\right)_{\overline{\gamma}}, h\right) \leq \omega_m\left(f_{\overline{\gamma}}, h\right). \tag{125}$$

ii) Additionally suppose $f_{\overline{\gamma}} \in L_1\left(\mathbb{R}^N\right)$. Then

$$\omega_m\left(\left(T_n\left(f\right)\right)_{\overline{\gamma}}, h\right)_1 \leq \omega_m\left(f_{\overline{\gamma}}, h\right)_1. \tag{126}$$

iii) Additionally suppose $f_{\overline{\gamma}} \in L_p\left(\mathbb{R}^N\right)$, $p>1$. Then

$$\omega_m\left(\left(T_n\left(f\right)\right)_{\overline{\gamma}}, h\right)_p \leq \omega_m\left(f_{\overline{\gamma}}, h\right)_p. \tag{127}$$

Next comes multi-simultaneous approximation. We give

Theorem 3.22: *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m,l,N \in \mathbb{N}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Assume $\left\|f_{\overline{\gamma}+\overline{\alpha}}\right\|_\infty < \infty$, and let $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Here $\beta, r \in \mathbb{N}$, $\beta > \frac{m+r+1}{2}$, and $A_{\xi_n}\left(\overline{\alpha}\right)$ as in (47), and*

$\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70). Then*

$$\left\| \left(T_{r,n}^{[m]}(f;\cdot)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}(\cdot) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N\geq 0;\\ |\overline{\alpha}|=\widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(\cdot)}{\prod_{i=1}^{N}\alpha_i!} \right) \right\|_{\infty} \tag{128}$$

$$\leq \sum_{\substack{\alpha_1,\ldots,\alpha_N\geq 0;\\ |\overline{\alpha}|=m}} \frac{\left(\omega_r\left(f_{\overline{\gamma}+\overline{\alpha}},\xi_n\right)\right)}{\left(\prod_{i=1}^{N}\alpha_i!\right)} A_{\xi_n}(\overline{\alpha}).$$

Proof: Based on Theorems 3.2, 3.18. □

We continue with

Theorem 3.23: *Let $f \in C_B^l\left(\mathbb{R}^N\right)$, $r,l,\beta \in \mathbb{N}$ (functions l-times continuously differentiable and bounded), $N \in \mathbb{N}-\{1\}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$. Call $\overline{\gamma} = 0,\overline{\beta}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Let also $\beta > \frac{r+1}{2}$. Then*

$$\left\| \left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_{\infty} \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \omega_r\left(f_{\overline{\gamma}},\xi_n\right). \tag{129}$$

If $\xi_n \to 0$, as $n \to \infty$, then $\left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}} \to f_{\overline{\gamma}}$ uniformly.

Proof: By Theorems 3.4, 3.18. □

We present

Theorem 3.24: *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $r,\beta,N,m,l \in \mathbb{N}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call $\overline{\gamma} = 0,\overline{\beta}$. Let $f_{(\overline{\gamma}+\overline{\alpha})} \in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$, and $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. Let also $\beta > \frac{\lceil rp\rceil+m+1}{2}$; $\alpha_j \in \mathbb{Z}^+$, $j = 1,\ldots,N$, $|\overline{\alpha}| := \sum_{j=1}^{N}\alpha_j = m$. Here $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j} = 1,\ldots,m$, and $\overline{\alpha} := (\alpha_1,\ldots,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,\ldots,N$, $|\overline{\alpha}| := \sum_{i=1}^{N}\alpha_i = \widetilde{j}$. Then*

$$\left\| \left(T_{r,n}^{[m]}(f;x)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(x)}{\prod_{i=1}^{N}\alpha_i!} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{\left(q(m-1)+1\right)^{\frac{1}{q}}} \right) 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left\{ \sum_{\lambda=0}^{\lceil rp\rceil} \binom{\lceil rp\rceil}{\lambda} \left(\psi_\lambda + \psi_{\lambda+mp}\right) \right\}^{\frac{N}{p}}$$

$$\left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \xi_n^{\left(\frac{2\beta(N-1)}{p}+m\right)} \omega_r \left(f_{\overline{\gamma}+\overline{\alpha}}, \xi_n\right)_p . \tag{130}$$

Proof: Theorems 3.7 and 3.18. □

We continue with

Theorem 3.25: *Let $f \in C^l\left(\mathbb{R}^N\right)$, $\beta, r, l \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{\overline{\gamma}} \in L_p\left(\mathbb{R}^N\right)$ and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here $\beta > \frac{\lceil rp \rceil + 1}{2}$. Then*

$$\left\| \left(T_{r,n}^{[0]}(f)\right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_p \leq$$

$$2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda \right]^{\frac{N}{p}} \xi_n^{\frac{2\beta(N-1)}{p}} \omega_r \left(f_{\overline{\gamma}}, \xi_n\right)_p . \tag{131}$$

As $n \to +\infty$ and $\xi_n \to 0$, then $\left(T_{r,n}^{[0]}(f)\right)_{\overline{\gamma}} \stackrel{\|\cdot\|_p}{\to} f_{\overline{\gamma}}$.

Proof: By Theorems 3.9 and 3.18. □

We continue with

Theorem 3.26: *Let $f \in C^l\left(\mathbb{R}^N\right)$, $l \in \mathbb{N}$, $N \in \mathbb{N} - \{1\}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{\overline{\gamma}} \in L_1\left(\mathbb{R}^N\right)$ and $\beta, r \in \mathbb{N}$, $\beta > \frac{r+1}{2}$. Then*

$$\left\| \left(T_{r,n}^{[0]}(f)\right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_1 \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \omega_r \left(f_{\overline{\gamma}}, \xi_n\right)_1 . \tag{132}$$

As $n \to +\infty$ and $\xi_n \to 0$, then $\left(T_{r,n}^{[0]}(f)\right)_{\overline{\gamma}} \stackrel{\|\cdot\|_1}{\to} f_{\overline{\gamma}}$.

Proof: By Theorems 3.11, 3.18. □

We continue with

Theorem 3.27: *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $r, N, \beta, m, l \in \mathbb{N}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{(\overline{\gamma}+\overline{\alpha})} \in L_1\left(\mathbb{R}^N\right)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$, $\beta > \frac{m+r+1}{2}$. Here $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$.*

Then

$$\left\| \left(T_{r,n}^{[m]} (f;x) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} (x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}} (x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x}$$

$$\leq \left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r \left(f_{\overline{\gamma}+\overline{\alpha}}, \xi_n \right)_1 \xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} [\psi_\lambda + \psi_{\lambda+m}] \right\}^N . \tag{133}$$

Proof: By Theorems 3.13, 3.18. □

3.4. *Voronovskaya asymptotic expansions for* $T_{r,n}^{[m]}$

We will apply

Theorem 3.28: *([3], p. 53) Let $f \in C^m \left(\mathbb{R}^N\right)$, $m, N \in \mathbb{N}$, with all $\|f_{\overline{\alpha}}\|_\infty \leq M$, $M > 0$, all $\overline{\alpha} : |\overline{\alpha}| = m$. Let $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$.*

Call $c_{\overline{\alpha},n,\widetilde{j}} := \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) d\mu_{\xi_n}(s)$, all $|\overline{\alpha}| = \widetilde{j} = 1, ..., m-1$. Suppose $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n}(s) \leq \rho$, all $\overline{\alpha} : |\overline{\alpha}| = m$, $\rho > 0$, for any such $(\xi_n)_{n\in\mathbb{N}}$. Also $0 < \gamma^* \leq 1$, $x \in \mathbb{R}^N$. Then

$$\theta_{r,n}^{[m]} (f;x) - f(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{c_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}} (x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) + 0 \left(\xi_n^{m-\gamma^*} \right). \tag{134}$$

When $m = 1$, the sum collapses.

Above we assume $\theta_{r,n}^{[m]} (f;x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^N$.

We give

Theorem 3.29: *Let $r, m, \beta, N \in \mathbb{N}$, $\beta > \frac{m+1}{2}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Besides, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m$. Here $f \in C^m \left(\mathbb{R}^N\right)$, with all $\|f_{\overline{\alpha}}\|_\infty \leq M$, $M > 0$, for all $\overline{\alpha} : |\overline{\alpha}| = m$; and $d\mu_{\xi_n}(s) = d\varphi_{\xi_n}(s)$, as in (39), $\forall\ s \in \mathbb{R}^N$. Assume $T_{r,n}^{[m]} (f;x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^N$. Here $\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), all $|\overline{\alpha}| = \widetilde{j} = 1, ..., m-1$. Let*

$0<\gamma^*\leq 1$, $x\in\mathbb{R}^N$. Then

$$T_{r,n}^{[m]}\left(f;x\right)-f\left(x\right)=\sum_{\widetilde{j}=1}^{m-1}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}\left(x\right)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right)+0\left(\xi_n^{m-\gamma^*}\right). \tag{135}$$

When $m=1$, the sum collapses.

Proof: By Theorems 2.6, 3.28. Here $\rho=\varphi$, see (68). □

We give

Corollary 3.30: *(to Theorem 3.29) Let $f\in C^1\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}$, with all $\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\leq M$, $M>0$, $i=1,...,N$. Let $0<\gamma^*\leq 1$. Assume $T_{r,n}^{[1]}\left(f;x\right)\in\mathbb{R}$, $\forall\ x\in\mathbb{R}^N$. Here $r\in\mathbb{N}$ and $\beta\in\mathbb{N}-\{1\}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Then*

$$T_{r,n}^{[1]}\left(f;x\right)-f\left(x\right)=0\left(\xi_n^{1-\gamma^*}\right). \tag{136}$$

Proof: By Theorems 2.6, 3.29. Here it is $\rho=\varphi$, apply (68) for $m=1$. □

We continue with

Corollary 3.31: *(to Theorem 3.29) Let $f\in C^2\left(\mathbb{R}^2\right)$, with all $\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_\infty$, $\left\|\frac{\partial^2 f}{\partial x_2^2}\right\|_\infty$, $\left\|\frac{\partial^2 f}{\partial x_1\partial x_2}\right\|_\infty\leq M$, $M>0$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Call*

$$c_1=\int_{\mathbb{R}^2}s_1d\varphi_{\xi_n}^*\left(s\right),\quad c_2=\int_{\mathbb{R}^2}s_2d\varphi_{\xi_n}^*\left(s\right), \tag{137}$$

where

$$d\varphi_{\xi_n}^*=\lambda_n^{-2}\prod_{i=1}^{2}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1ds_2,\quad s=(s_1,s_2)\in\mathbb{R}^2.$$

Let $0<\gamma^*\leq 1$ and assume $T_{r,n}^{[2]}\left(f;x\right)\in\mathbb{R}$, $\forall\ x\in\mathbb{R}^2$. Here $r,\beta\in\mathbb{N}$ and $\beta>\frac{3}{2}$. Then

$$T_{r,n}^{[2]}\left(f;x\right)-f\left(x\right)=\left(\sum_{j=1}^{r}\alpha_{j,r}^{[2]}j\right)\left(c_1\frac{\partial f}{\partial x_1}\left(x\right)+c_2\frac{\partial f}{\partial x_2}\left(x\right)\right)+0\left(\xi_n^{2-\gamma^*}\right). \tag{138}$$

Proof: By Theorems 2.6, 3.29. □

We also give

Theorem 3.32: *Let $f\in C^{m+l}\left(\mathbb{R}^N\right)$, $m,l,N\in\mathbb{N}$. Assumptions of Theorem 3.18 are valid for $d\varphi_{\xi_n}\left(s\right)$, $s\in\mathbb{R}^N$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Call $\overline{\gamma}=0,\overline{\beta}$. Suppose $\left\|f_{\overline{\gamma}+\overline{\alpha}}\right\|_\infty\leq M$, $M>0$, for all $\overline{\alpha}:|\overline{\alpha}|=m$. Here $\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ is as in (70), all $|\overline{\alpha}|=\widetilde{j}=$*

$1,...,m-1$; $0<\gamma^{*}\leq 1$. Assume $T_{r,n}^{[m]}\left(f_{\overline{\gamma}};x\right)\in\mathbb{R}$, $\forall\ x\in\mathbb{R}^{N}$. Let also $r,\beta\in\mathbb{N}$ and $\beta>\frac{m+1}{2}$. Then

$$\left(T_{r,n}^{[m]}\left(f;x\right)\right)_{\overline{\gamma}}-f_{\overline{\gamma}}\left(x\right)=\sum_{\widetilde{j}=1}^{m-1}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\gamma}+\overline{\alpha}}\left(x\right)}{\left(\prod_{i=1}^{N}\alpha_{i}!\right)}\right)+0\left(\xi_{n}^{m-\gamma^{*}}\right). \quad (139)$$

When $m=1$, the sum collapses.

Proof: Use of Theorem 2.6 and Theorem 4.6, p. 54 of [3]. Here it is $\rho=\varphi$, see (68). □

3.5. *Simultaneous approximation by multivariate complex $T_{r,n}^{[m]}$*

We make

Remark 2: We consider here complex valued Borel measurable functions $f:\mathbb{R}^{N}\rightarrow\mathbb{C}$ such that $f=f_{1}+if_{2}$, $i=\sqrt{-1}$, where $f_{1},f_{2}:\mathbb{R}^{N}\rightarrow\mathbb{R}$ are implied to be real valued Borel measurable functions.

We define the multivariate complex Trigonometric singular operators

$$T_{r,n}^{[m]}\left(f;x\right):=T_{r,n}^{[m]}\left(f_{1};x\right)+iT_{r,n}^{[m]}\left(f_{2};x\right),\quad x\in\mathbb{R}^{N}. \quad (140)$$

We assume that $T_{r,n}^{[m]}\left(f_{j};x\right)\in\mathbb{R}$, $\forall\ x\in\mathbb{R}^{N}$, $j=1,2$.

One notices easily that

$$\left|T_{r,n}^{[m]}\left(f;x\right)-f\left(x\right)\right|\leq\left|T_{r,n}^{[m]}\left(f_{1};x\right)-f_{1}\left(x\right)\right|+\left|T_{r,n}^{[m]}\left(f_{2};x\right)-f_{2}\left(x\right)\right| \quad (141)$$

also

$$\left\|T_{r,n}^{[m]}\left(f;x\right)-f\left(x\right)\right\|_{\infty,x}\leq\left\|T_{r,n}^{[m]}\left(f_{1};x\right)-f_{1}\left(x\right)\right\|_{\infty,x}+\left\|T_{r,n}^{[m]}\left(f_{2};x\right)-f_{2}\left(x\right)\right\|_{\infty,x} \quad (142)$$

and

$$\left\|T_{r,n}^{[m]}\left(f\right)-f\right\|_{p}\leq\left\|T_{r,n}^{[m]}\left(f_{1}\right)-f_{1}\right\|_{p}+\left\|T_{r,n}^{[m]}\left(f_{2}\right)-f_{2}\right\|_{p},\quad p\geq 1. \quad (143)$$

Furthermore, it holds

$$f_{\overline{\alpha}}\left(x\right)=f_{1,\overline{\alpha}}\left(x\right)+if_{2,\overline{\alpha}}\left(x\right), \quad (144)$$

where $\overline{\alpha}$ denotes a partial derivative of any order and arrangement.

We give

Theorem 3.33: *Let $f:\mathbb{R}^{N}\rightarrow\mathbb{C}$, such that $f=f_{1}+if_{2}$, $j=1,2$. Here $r,N,\beta,m\in\mathbb{N}$, $f_{j}\in C^{m}\left(\mathbb{R}^{N}\right)$, $x\in\mathbb{R}^{N}$. Assume $\left\|\frac{\partial^{m}f_{j}(\cdot,\cdot,\ldots,\cdot)}{\partial x_{1}^{\alpha_{1}}\ldots\partial x_{N}^{\alpha_{N}}}\right\|_{\infty}<\infty$, for all*

$\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$. Let φ_{ξ_n} be the Borel probability measure on $\mathbb{R}^N$, see (39), where $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Here $\beta > \frac{m+r+1}{2}$, and $A_{\xi_n}(\overline{\alpha})$ as in (47), and $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70). Then

$$\left\| T_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{\infty,x}$$

$$\leq \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}| = m}} \frac{\left(\omega_r\left(f_{1,\overline{\alpha}}, \xi_n\right) + \omega_r\left(f_{2,\overline{\alpha}}, \xi_n\right)\right)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} A_{\xi_n}(\overline{\alpha}). \tag{145}$$

Proof: By Theorem 3.2. □

We proceed with

Theorem 3.34: *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $N \in \mathbb{N} - \{1\}$, $j = 1, 2$. Here $f_j \in C_B(\mathbb{R}^N)$ uniformly continuous, $\beta, r \in \mathbb{N}$, $\beta > \frac{r+1}{2}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Then*

$$\left\| T_{r,n}^{[0]} f - f \right\|_{\infty} \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N \xi_n^{2\beta(N-1)} \tag{146}$$

$$\left(\omega_r\left(f_1, \xi_n\right) + \omega_r\left(f_2, \xi_n\right)\right),$$

As $n \to \infty$ and $\xi_n \to 0$, we derive

$$\left\| T_{r,n}^{[0]} f - f \right\|_{\infty} \to 0 \tag{147}$$

with rates.

Proof: By Theorem 3.4. □

Next comes multi-simultaneous approximation.

Theorem 3.35: *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. Here $f_j \in C^{m+l}(\mathbb{R}^N)$, $N, m, l \in \mathbb{N}$. The assumptions of Theorem 3.18 are valid for f_j and $d\mu_{\xi_n} = d\varphi_{\xi_n}$. Call $\overline{\gamma} = 0, \overline{\beta}$. Assume $\|f_{j,\overline{\gamma}+\overline{\alpha}}\|_{\infty} < \infty$, and let $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Here $\beta, r \in \mathbb{N}$, $\beta > \frac{m+r+1}{2}$, and $A_{\xi_n}(\overline{\alpha})$ as in (47), and $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70). Then*

$$\left\| \left(T_{r,n}^{[m]}(f;\cdot)\right)_{\overline{\gamma}} - f_{\overline{\gamma}}(\cdot) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0; \\ |\overline{\alpha}| = \widetilde{j}}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(\cdot)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{\infty} \tag{148}$$

$$\leq \sum_{\substack{\alpha_1,...,\alpha_N \geq 0; \\ |\overline{\alpha}|=m}} \frac{\left(\omega_r\left(f_{1,\overline{\gamma}+\overline{\alpha}},\xi_n\right)+\omega_r\left(f_{2,\overline{\gamma}+\overline{\alpha}},\xi_n\right)\right)}{\left(\prod_{i=1}^{N}\alpha_i!\right)} A_{\xi_n}\left(\overline{\alpha}\right).$$

Proof: Based on Theorems 3.18, 3.22. □

We continue with

Theorem 3.36: *Let $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. Here $f_j\in C_B^l\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}-\{1\}$, $l\in\mathbb{N}$ (functions l-times continuously differentiable and bounded). The assumptions of Theorem 3.18 are valid for f_j and $d\mu_{\xi_n}=d\varphi_{\xi_n}$. Call $\overline{\gamma}=0,\overline{\beta}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Let also $\beta,r\in\mathbb{N}$, $\beta>\frac{r+1}{2}$. Then*

$$\left\|\left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}}-f_{\overline{\gamma}}\right\|_{\infty}\leq 2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N$$

$$\xi_n^{2\beta(N-1)}\left(\omega_r\left(f_{1,\overline{\gamma}},\xi_n\right)+\omega_r\left(f_{2,\overline{\gamma}},\xi_n\right)\right). \tag{149}$$

If $\xi_n\to 0$, as $n\to\infty$, then $\left(T_{r,n}^{[0]}f\right)_{\overline{\gamma}}\to f_{\overline{\gamma}}$ uniformly.

Proof: By Theorems 3.23 and 3.18. □

We proceed with L_p approximations

Theorem 3.37: *Let $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. Here $f_j\in C^m\left(\mathbb{R}^N\right)$, $r,\beta,N,m\in\mathbb{N}$, with $f_{j,\overline{\alpha}}\in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}|=m$, $x\in\mathbb{R}^N$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Here φ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ as in (39), for $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Let $\beta>\frac{\lceil rp\rceil+m+1}{2}$; $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N:|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=m$. Here $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$. Then*

$$\left\|T_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\widetilde{j}}\frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}}f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right)\right\|_{p,x}$$

$$\leq\left(\frac{m}{\left(q(m-1)+1\right)^{\frac{1}{q}}}\right)2^{\frac{N}{p}}\gamma^{-\frac{N}{p}}\left\{\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\left(\psi_\lambda+\psi_{\lambda+mp}\right)\right\}^{\frac{N}{p}}$$

$$\left(\sum_{|\overline{\alpha}|=m}\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\xi_n^{\left(\frac{2\beta(N-1)}{p}+m\right)}\left[\omega_r\left(f_{1,\overline{\alpha}},\xi_n\right)_p+\omega_r\left(f_{2,\overline{\alpha}},\xi_n\right)_p\right]. \tag{150}$$

Proof: By Theorem 3.7. □

We continue with

Theorem 3.38: *Let $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. Here $f_j\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$; $N\in\mathbb{N}-\{1\}$, $\beta,r\in\mathbb{N}$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $\beta>\frac{\lceil rp\rceil+1}{2}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Then*

$$\left\|T_{r,n}^{[0]}(f)-f\right\|_p\leq 2^{\frac{N}{p}}\gamma^{-\frac{N}{p}}\left[\sum_{\lambda=0}^{\lceil rp\rceil}\binom{\lceil rp\rceil}{\lambda}\psi_\lambda\right]^{\frac{N}{p}}$$

$$\xi_n^{\frac{2\beta(N-1)}{p}}\left[\omega_r\left(f_1,\xi_n\right)_p+\omega_r\left(f_2,\xi_n\right)_p\right]. \tag{151}$$

As $\xi_n\to 0$, when $n\to\infty$, we derive $\left\|T_{r,n}^{[0]}f-f\right\|_p\to 0$, i.e. $T_{r,n}^{[0]}\to I$, the unit operator, in L_p norm.

Proof: By Theorem 3.9. □

We also give

Theorem 3.39: *Let $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. Here $f_j\in\left(C\left(\mathbb{R}^N\right)\cap L_1\left(\mathbb{R}^N\right)\right)$; $N\in\mathbb{N}-\{1\}$, $r,\beta\in\mathbb{N}$, $\beta>\frac{r+1}{2}$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. Then*

$$\left\|T_{r,n}^{[0]}(f)-f\right\|_1\leq 2^N\gamma^{-N}\left[\sum_{\lambda=0}^{r}\binom{r}{\lambda}\psi_\lambda\right]^N$$

$$\xi_n^{2\beta(N-1)}\left(\omega_r\left(f_1,\xi_n\right)_1+\omega_r\left(f_2,\xi_n\right)_1\right). \tag{152}$$

As $\xi_n\to 0$, we get $T_{r,n}^{[0]}\to I$, in L_1 norm.

Proof: By Theorem 3.11. □

We further present

Theorem 3.40: *Let $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. Here $f_j\in C^m\left(\mathbb{R}^N\right)$, $N,\beta,m,r\in\mathbb{N}$, with $f_{j,\overline{\alpha}}\in L_1\left(\mathbb{R}^N\right)$, where $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=m$, $x\in\mathbb{R}^N$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$ and $\beta>\frac{m+r+1}{2}$. Here $\overline{c}_{\overline{\alpha},n}:=\overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$. Also* *here φ_{ξ_n} is the Borel probability measure on $\mathbb{R}^N$, see (39). Then*

$$\left\| T_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x} \tag{153}$$

$$\leq \left\{ \sum_{|\overline{\alpha}|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left[\omega_r(f_{1,\overline{\alpha}}, \xi_n)_1 + \omega_r(f_{2,\overline{\alpha}}, \xi_n)_1 \right] \right\} \xi_n^{2\beta(N-1)+m}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} [\psi_\lambda + \psi_{\lambda+m}] \right\}^N .$$

Proof: By Theorem 3.13. □

We continue with simultaneous L_p approximations.

Theorem 3.41: *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + i f_2$, $j = 1, 2$. Here $f_j \in C^{m+l}(\mathbb{R}^N)$, $r, \beta, N, m, l \in \mathbb{N}$. The assumptions of Theorem 3.18 are valid for $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$ and f_j. Call $\overline{\gamma} = 0, \overline{\beta}$. Let $f_{j,(\overline{\gamma}+\overline{\alpha})} \in L_p(\mathbb{R}^N)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\beta > \frac{\lceil rp \rceil + m + 1}{2}$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = m$. Here $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ as in (70), where $\widetilde{j} = 1, ..., m$, and $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. Then*

$$\left\| \left(T_{r,n}^{[m]}(f;x) \right)_{\overline{\gamma}} - f_{\overline{\gamma}}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} (\psi_\lambda + \psi_{\lambda+mp}) \right\}^{\frac{N}{p}} \tag{154}$$

$$\left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \xi_n^{\left(\frac{2\beta(N-1)}{p} + m \right)} \left[\omega_r(f_{1,\overline{\gamma}+\overline{\alpha}}, \xi_n)_p + \omega_r(f_{2,\overline{\gamma}+\overline{\alpha}}, \xi_n)_p \right].$$

Proof: By Theorems 3.18 and 3.24. □

We give also

Theorem 3.42: *Let* $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. *Here* $f_j \in C^l(\mathbb{R}^N)$, $N \in \mathbb{N} - \{1\}$, $l \in \mathbb{N}$. *The assumptions of Theorem 3.18 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$ *and* f_j. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Let* $f_{j,\overline{\gamma}} \in L_p(\mathbb{R}^N)$ *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here* $\beta, r \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + 1}{2}$. *Then*

$$\left\| \left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_p \leq 2^{\frac{N}{p}} \gamma^{-\frac{N}{p}} \left[\sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \psi_\lambda \right]^{\frac{N}{p}}$$

$$\xi_n^{\frac{2\beta(N-1)}{p}} \left[\omega_r (f_{1,\overline{\gamma}}, \xi_n)_p + \omega_r (f_{2,\overline{\gamma}}, \xi_n)_p \right]. \tag{155}$$

As $n \to +\infty$ *and* $\xi_n \to 0$, *then* $\left(T_{r,n}^{[0]} f \right)_{\overline{\gamma}} \stackrel{\|\cdot\|_p}{\to} f_{\overline{\gamma}}$.

Proof: By Theorems 3.18 and 3.25. □

We continue with

Theorem 3.43: *Let* $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. *Here* $f_j \in C^l(\mathbb{R}^N)$, $N \in \mathbb{N} - \{1\}$, $l \in \mathbb{N}$. *The assumptions of Theorem 3.18 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$ *and* f_j. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Let* $f_{j,\overline{\gamma}} \in L_1(\mathbb{R}^N)$ *and* $\beta, r \in \mathbb{N}$, $\beta > \frac{r+1}{2}$. *Then*

$$\left\| \left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} \right\|_1 \leq 2^N \gamma^{-N} \left[\sum_{\lambda=0}^{r} \binom{r}{\lambda} \psi_\lambda \right]^N$$

$$\xi_n^{2\beta(N-1)} \left[\omega_r (f_{1,\overline{\gamma}}, \xi_n)_1 + \omega_r (f_{2,\overline{\gamma}}, \xi_n)_1 \right]. \tag{156}$$

As $n \to +\infty$ *and* $\xi_n \to 0$, *then* $\left(T_{r,n}^{[0]}(f) \right)_{\overline{\gamma}} \stackrel{\|\cdot\|_1}{\to} f_{\overline{\gamma}}$.

Proof: By Theorems 3.18, 3.26. □

We finish with

Theorem 3.44: *Let* $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. *Here* $f_j \in C^{m+l}(\mathbb{R}^N)$, $N, \beta.r, m, l \in \mathbb{N}$. *The assumptions of Theorem 3.18 are valid for* $d\mu_{\xi_n} = d\varphi_{\xi_n}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$ *and* f_j. *Call* $\overline{\gamma} = 0, \overline{\beta}$. *Let* $f_{j,(\overline{\gamma}+\overline{\alpha})} \in L_1(\mathbb{R}^N)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$. *Here* $\beta > \frac{m+r+1}{2}$ *and* $\overline{c}_{\overline{\alpha},n} := \overline{c}_{\overline{\alpha},n,\widetilde{j}}$ *as in (70), where* $\widetilde{j} = 1, ..., m$, *and* $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$,

$\alpha_i \in \mathbb{Z}^+,\ i = 1, ..., N,\ |\overline{\alpha}| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$. *Then*

$$\left\| \left(T_{r,n}^{[m]} (f; x) \right)_{\overline{\gamma}} - f_{\overline{\gamma}} (x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\overline{\alpha}| = \widetilde{j}} \frac{\overline{c}_{\overline{\alpha},n,\widetilde{j}} f_{\overline{\gamma}+\overline{\alpha}} (x)}{\prod_{i=1}^{N} \alpha_i !} \right) \right\|_{1,x}$$

$$\leq \left(\sum_{|\overline{\alpha}| = m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i !} \right) \left[\omega_r \left(f_{1,\overline{\gamma}+\overline{\alpha}}, \xi_n \right)_1 + \omega_r \left(f_{2,\overline{\gamma}+\overline{\alpha}}, \xi_n \right)_1 \right] \right)$$

$$\xi_n^{2\beta(N-1)+m} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{r} \binom{r}{\lambda} \left[\psi_\lambda + \psi_{\lambda+m} \right] \right\}^N . \tag{157}$$

Proof: By Theorems 3.18, 3.27. □

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