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An Algorithm for Finding a Near-Optimal Rearrangement in the Steinitz Functional

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In this work we give the polynomial algorithm for finding the near-optimal rearrangement in the Steinitz functional for the vectors in the finite dimensional normed space. Maximum inequality and a Transference theorem, obtained by the authors, as well as a Monte-Carlo method are applied.

Keywords: Maximum inequality, transference theorem, rearrangement, signs, optimal, algorithm.

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1. Introduction

The main aim of this paper is to apply our maximum inequality and transference theorem [1, 2], presented in the next section, to the following problem which is an important subtask of many problems of machine learning, scheduling theory and discrepancy theory ([3] - [8]). The main problem is to find or estimate the minimum in π of the *Steinitz functional*

$$\Phi_x(\pi) = \max_{1 \le k \le n} \left\| \sum_{i=1}^{k} x_{\pi(i)} \right\|, \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is a fixed collection of elements of a finite dimensional normed space X and $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation. The problem was posed by E. Steinitz [9] who was solving the question on sum range of a conditionally convergent series in a finite dimensional space (the generalization of the famous Riemann problem).

The peculiarity of the related applied problems is that d, the dimension of X, can be very large, so that the *brute force* idea as a rule does not work. Though, based on the maximal inequalities for the rearrangements of vector summands, we

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construct a polynomial-time algorithm for finding a near-optimal permutation in (1).

2. The main maximal inequalities

In this section we give two maximal inequalities related to the problems of calculation or estimation of the Steinitz functional:

Theorem 2.1 : *([1, 2]) Let $x_1, \ldots, x_n \in X$ be a collection of elements of a normed space X with $\sum_{i=1}^{n} x_i = 0$. Then*
a) *For any collection of signs $\vartheta = (\vartheta_1, \ldots, \vartheta_n), \vartheta_i = \pm 1$, there is a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that*

$$\max_{1\le k\le n} \left\| \sum_{i=1}^{k} x_i \right\| + \max_{1\le k\le n} \left\| \sum_{i=1}^{k} \vartheta_i x_i \right\| \ge 2 \max_{1\le k\le n} \left\| \sum_{i=1}^{k} x_{\pi(i)} \right\| . \tag{2}$$

There is an explicit one-to-one correspondence between ϑ and $\pi = \pi(\vartheta)$.
b) *(Transference Theorem). There is a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that*

$$\max_{1\le k\le n} \left\| \sum_{i=1}^{k} x_{\sigma(i)} \right\| \le \max_{1\le k\le n} \left\| \sum_{i=1}^{k} \vartheta_i x_{\sigma(i)} \right\|$$

for any collection of signs $\vartheta = (\vartheta_1, \ldots, \vartheta_n), \vartheta_i = \pm 1$.

Remark 1 : The correspondence between ϑ and $\pi = \pi(\vartheta)$ in part a) is given as follows: if $\vartheta_{k_i} = +1, i = 1 \ldots, p$, and $\vartheta_{m_j} = -1, j = 1 \ldots, q, p + q = n$, then $\pi = (k_1, \ldots, k_p, m_q, \ldots, m_1)$.

2.1. *The greedy algorithm is not in general the best*

Given vectors $x_1, \ldots, x_n \in X$, *a greedy algorithm* chooses at each step a vector that minimizes the norm of the next partial sum. In other words, on step 1 it chooses an element x_{n_1} that has a minimum norm. On step 2 it selects an element $x_{n_2},\ x_{n_2} \ne x_{n_1}$ such that

$$\|x_{n_1} + x_{n_2}\| \le \|x_{n_1} + x_{n_k}\|$$

for any $n_k \ne n_1$, etc.
The following example constructed by Jakub Wojtaszchik (oral communication) shows that a greedy algorithm is not in general the best one even in a two-dimensional space.

Example 2.2 Consider n groups of vectors of l^2_∞, each consisting of the three following vectors: $(1, 1)$, $(2, -3)$ and $(-3, 2)$. Obviously, the greedy algorithm chooses at the first n steps the vectors $(1, 1), \ldots, (1, 1)$. Therefore, for the optimal permutation π_o and greedy permutation π_g we have respectively

$$\max_{1\le k\le n} \left\|x_{\pi_o(1)} + \cdots + x_{\pi_o(k)}\right\| = 3;$$

and

$$\max_{1\le k\le n} \left\|x_{\pi_g(1)} + \cdots + x_{\pi_g(k)}\right\| = n + 2.$$

In [10] we show that such sort of an example can be constructed in any 2-dimensional normed space.

2.2. *Corollaries to the Transference theorem (Theorem 2.1b)*

The Transference theorem allows us to get a permutation theorem given a sign theorem. Moreover, as we'll see in Section 3, if the sign algorithm is constructive, then a desired permutation can also be found constructively. As a first example we consider the classical Steinitz permutation theorem that we get from the following Barany-Grinberg-Sevostyanov sign theorem.

Theorem 2.3 : *([11, 12]) Let X be a normed space of dimension d, $x_1, \ldots, x_n \in X$, $\|x_i\| \le 1$, $i = 1, \ldots, n$. Then there exists a collection of signs $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ such that*

$$\max_{1\le k\le n} \|\vartheta_1 x_1 + \cdots + \vartheta_k x_k\| \le 2d.$$

The permutation version of Theorem 2.3 found by the Transference theorem can be stated as follows.

Corollary 2.4: *(The Steinitz inequality). Let X be a normed space of dimension d, $x_1, \ldots, x_n \in X$, $\|x_i\| \le 1$, $i = 1, \ldots, n$, and $x_1 + \cdots + x_n = 0$. Then there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that*

$$\max_{1\le k\le n} \left\|x_{\sigma(1)} + \cdots + x_{\sigma(k)}\right\| \le 2d.$$

Remark 2 : Steinitz [9] proved his inequality straightforwardly, however a proof through the sign version additionally allows to find the desired permutation constructively provided that the collection of signs in the sign version can be obtained constructively, by use of the Transference theorem (see Section 3)

Another sign-permutation duality example is provided by the case of the space l_∞^d.

Theorem 2.5 : *(Spencer [13]). Let $X = l_\infty^d$, $x_1, \ldots, x_n \in X$, $\|x_i\| \le 1$, $i = 1, \ldots, n$. Then there exists a collection of signs $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ such that*

$$\max_{1\le k\le n} \|\vartheta_1 x_1 + \cdots + \vartheta_k x_k\| \le \sqrt{2n\ ln 2d}.$$

Note that in the above theorem, Spencer also gives an effective way of finding thetas.

Due to the Transference theorem, the following dual permutational counterpart is also valid.

Corollary 2.6: *Let* $X = l_{\infty}^{d}$, $x_1, \ldots, x_n \in X$, $\|x_i\| \le 1$, $i = 1, \ldots, n$ *and* $x_1 + \cdots + x_n = 0$. *Then there exists a permutation* $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *such that*

$$\max_{1\le k\le n} \left\|x_{\sigma(1)} + \cdots + x_{\sigma(k)}\right\| \le \sqrt{2n \ ln2d}.$$

3. An algorithm for near-optimal permutation

In this section we show that the algorithm for near-optimal permutation for the Steinitz functional

$$\Phi_x(\pi) = \max_{1\le k\le n} \left\|\sum_{i=1}^{k} x_{\pi(i)}\right\|$$

can be reduced to the algorithm for the near-optimal sign algorithm. The reduction is based on the Transference theorem (Theorem 2.1b).

Let us first introduce the notations:

$$|x_\pi| \equiv \max_{1\le k\le n} \left\|\sum_{i=1}^{k} x_{\pi(i)}\right\|, \ |x_\pi \vartheta| \equiv \max_{1\le k\le n} \left\|\sum_{i=1}^{k} \vartheta_i x_{\pi(i)}\right\|.$$

Theorem 3.1: *Let* X *be a normed space,* $x_1, \ldots, x_n \in X$, $\|x_i\| \le 1$, $i = 1, \ldots, n$, *and* $x_1 + \cdots + x_n = 0$. *Assume that for any permutation* $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, *there is an algorithm with a polynomial complexity to define* $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ *such that*

$$|x_\pi \vartheta| \le D, \tag{3}$$

where D *does not depend on* π. *Then for any* $\varepsilon > 0$, *there is an algorithm with a polynomial complexity to define* $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *such that*

$$|x_\pi| \le D + \varepsilon.$$

The complexity of the algorithm is $\omega \cdot \log(\frac{n}{\varepsilon})$, *where* ω *is the complexity of the sign algorithm (for finding* ϑ*).*

Proof: According to (2), for any permutation π and any collection of signs ϑ, there is a permutation $\pi^* = \pi^*(\pi, \vartheta)$ such that

$$|x_\pi| + |x_\pi \vartheta| \ge 2\,|x_{\pi^*}|\,, \tag{4}$$

On the first step we take an arbitrary π_0 and due to Theorem 2.3, find ϑ_0 such that

$$|x_{\pi_0} \vartheta_0| \le D.$$

By (4), there is a permutation $\pi_1 = \pi^*(\pi_0, \vartheta_0)$ such that

$$|x_{\pi_1}| \leq \frac{1}{2}(|x_{\pi_0}| + D).$$

Choosing ϑ_1 so that

$$|x_{\pi_1}\vartheta_1| \leq D$$

and using again (4) for π_1, we find a permutation $\pi_2 = \pi^*(\pi_1, \vartheta_1)$ such that

$$|x_{\pi_2}| \leq \frac{1}{4}\,|x_{\pi_0}| + (1 - \frac{1}{4})D.$$

After the N-th iteration we get

$$|x_{\pi_N}| \leq \frac{1}{2^N}\,|x_{\pi_0}| + (1 - \frac{1}{2^N})D.$$

Therefore, taking $N > \log_2\big(\frac{|x_{\pi_0}|}{\epsilon}\big)$, we get the proof. $\square$

3.1. *Applying the Monte-Carlo*

Let $X = l_2^d,\quad x_1, \ldots, x_n \in X,\ \|x_i\| \leq 1,\ i = 1,\ \ldots,\ n$, and $\ x_1 + \cdots + x_n = 0$.

We choose at random k collections $\vartheta_1^{(1)},\ \ldots,\ \vartheta_k^{(1)}$, each of them being a collection of n signs and choose among them ϑ_1, that one for which $\left|\vartheta_1^{(1)}x\right|$ attains its minimum. Then we create the permutation $\pi_2 = \pi_2(\pi_1, \vartheta_1)$, generated by the initial permutation (denote it by π_1) and ϑ_1 according to Theorem 2.1 (a). Therefore, we'll have

$$|x_{\pi_2}| \leq \frac{1}{2}\,|x_{\pi_1}| + \frac{1}{2}\,|\vartheta_1 x_{\pi_1}|\,.$$

Then we choose at random (independently) $\vartheta_1^{(2)},\ \ldots,\ \vartheta_k^{(2)}$ and among them choose ϑ_2 minimizing $\left|\vartheta_i^{(2)}x_{\pi_2}\right|$. Carrying out these iterations $l-1, 1 < l < \infty$, times, we find a sequence of permutations $\pi_1, \ldots,\ \pi_l$ such that for the last permutation we get the following inequality

$$|x_{\pi_l}| \leq \frac{1}{2^l}\,|x_{\pi_1}| + (1 - \frac{1}{2^l})\max_{i\leq l}|\vartheta_i x_{\pi_i}|\,. \tag{5}$$

We now show that π_l for sufficiently large l is a near-optimal permutation. For these purposes let us make sure that the following probability is small enough after an appropriate choice of C, k and l:

$$P\left(\max_{i\leq l}\left|\vartheta_i x_{\pi(i)}\right| > C\sqrt{n}\ \right) \leq \sum_{i=1}^{l} P\left(\left|\vartheta_i x_{\pi(i)}\right| > C\sqrt{n}\ \right)$$

$$\leq \sum_{i=1}^{l} P\left\{\left(\left|\vartheta_1^{(i)} x_{\pi(i)}\right| > C\sqrt{n}\ \right) \cap \cdots \cap \left(\left|\vartheta_k^{(i)} x_{\pi(i)}\right| > C\sqrt{n}\ \right)\right\}$$

$$= \sum_{i=1}^{l} \left\{P\left(\left|\vartheta_1^{(i)} x_{\pi(i)}\right| > C\sqrt{n}\ \right)\right\}^k.$$

The next step is to use the estimation of the tail probability for the Rademacher random variables with values in a normed space (in our case it is the space l_2^d) (see the monograph by M. Ledoux and M.Talagrand [14], p.101). Then we get

$$P\left(\max_{i\leq l} \left|\vartheta_i x_{\pi(i)}\right| > C\sqrt{n}\ \right) \leq l \cdot 2^{k+1} exp\left\{-\frac{Cn}{32n}\right\}.$$

Up to now l and k were arbitrary. Letting $l = k$, as well as $C = 32 \cdot ln4$, we get

$$P\left(\max_{i\leq l} \left|\vartheta_i x_{\pi(i)}\right| > C\sqrt{n}\ \right) \leq \frac{k}{2^{k-1}}.$$

These computations along with (5) imply that with a large probability (which can be made arbitrarily close to one) the following inequality holds

$$|x_{\pi_l}| \leq \frac{1}{2^l}\,|x_{\pi_1}| + C\sqrt{n}.$$

According to Theorem 2.1b, the order of $\sqrt{n}$ is correct, and it is also known that it can not be improved.

Therefore, we proved the following

Theorem 3.2 : *The random algorithm described in this section leads to the nearly optimal permutation. The algorithm runs in a polynomial time.*

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