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On the Absolute Convergence of the Multiple Series of Fourier-Haar Coefficients

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As is well known, the Haar and Walsh systems are successfully applied in signal transmission processes. In this direction an important role is played by the study of the behavior of the signal, as a sum of the absolute values of the Fourier coefficients.

In this paper we study the problem of absolute convergence of the N-dimensional series of Fourier-Haar coefficients for the classes of functions with bounded partial p-variations.

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1. Introduction

The problems related to the behavior of single series of Fourier-Haar are well studied [10]. Namely, P. Ulianov [17] and B. Golubov [9] received the results related to the problems of absolute convergence of the series of Fourier-Haar coefficients. Some generalization of these results related were received by Z. Chanturia [4], T. Akhobadze [1], U. Goginava [8] and by the author [2]. In the term of modulus of smoothness the problem of absolute convergence of the series of Fourier-Haar coefficients was studied by V. Krotov [13]. Multidimensional analogies corresponding to the results of V. Krotov were formulated in the works of V. Tsagareishvili [16] and G. Tabatadze [15].

The estimates of Fourier coefficients of functions of bounded fluctuation with respect to Walsh system were studied in [14] and with respect to Vilenkin system were studied by G. Gát and R. Toledo [5] and by the author [3].

Let $I^N = [0,1]^N$ denote a cube in the N-dimensional Euclidean space R^N. The elements of R^N are denoted by $\overrightarrow{x} = (x_1, ..., x_N)$.

Let

$$\chi_{\overrightarrow{m}}\left(\overrightarrow{x}\right) = \prod_{i=1}^{N} \chi_{m_i}\left(x_i\right), \qquad x_i \in I = [0,1]\left(i = 1, 2, ..., N\right), \qquad N \geq 2$$

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is the multiple Haar system on $I^N = [0,1]^N$, where

$$\overrightarrow{x} = (x_1, ..., x_N), \quad \overrightarrow{m} = (m_1, ..., m_N) \qquad \left(m_i = 1, 2, ...; \quad i = \overline{1, N}\right)$$

As usual, $L_p(I^N)(p \geq 1)$ denotes the set of all measurable functions defined on I^N, for which

$$\|f\|_p = \left(\int\limits_{[0,1]^N} |f| d\overrightarrow{x} \right)^{\frac{1}{p}} < \infty$$

and $C(I^N)$ is the space of continuous functions on I^N equipped with the maximum norm

$$\|f\|_c = \max_{x_i \in I} |f(\overrightarrow{x})|.$$

Let us denote Fourier-Haar multiple coefficients of the function $f \in L([0,1])^N$ by $C_{\overrightarrow{m}}(f)$, i.e

$$C_{\overrightarrow{m}}(f) = \int\limits_{[0,1]^N} f \cdot \chi_{\overrightarrow{m}} d\overrightarrow{x}.$$

We say that $f \in Lip\alpha$ on $[0,1]^N$, if

$$\|f(\cdot + h) - f(\cdot)\|_C = O\left(\|h\|^\alpha\right),$$

where

$$\|h\| = \left(\sum_{i=1}^N h_i^2 \right)^{1/2}, \quad \alpha \in (0,1].$$

We have the following theorem.

Theorem 1.1 ([15]) *a) Let $f \in \mathrm{Lip}\,\alpha$ on $[0,1]^N, \alpha \in (0,1]$. If $\beta > 0$ and $\gamma + 1 < \beta(\frac{\alpha}{N} + \frac{1}{2})$, then*

$$\sum_{m_1=1}^\infty \cdots \sum_{m_N=1}^\infty \prod_{i=1}^N m_i^\gamma \left|C_{m_1,...,m_N}(f)\right|^\beta < \infty$$

b) Let $\gamma + 1 = \beta(\frac{\alpha}{N} + \frac{1}{2})$, for some $\alpha \in (0,1)$. Then there exists a function $f_\alpha \in \mathrm{Lip}\,\alpha$ for which

$$\sum_{m_1=1}^\infty \cdots \sum_{m_N=1}^\infty \prod_{i=1}^N m_i^\gamma \left|C_{m_1,...,m_N}(f_\alpha)\right|^\beta = \infty$$

The case for $\gamma = 0$ was considered earlier by V. Tsagareishvili [16].

Let f be a function defined on R^N, 1-periodic with respect to each variable. Denote by

$$\Delta_{h_j}(f,t) = f(t_1,\dots,t_j+h_j,\dots,t_N) - f(t_1,\dots,t_j,\dots,t_N)\,(j=1,\dots,N),$$
$$\Delta_{h_1,h_2,\dots,h_N}(f,t) \equiv \Delta_{h_N}\left(\Delta_{h_1,h_2,\dots,h_{N-1}},t\right).$$

Let $f \in L_p([0,1]^N)$. The partial integrated modulus of continuity are defined by

$$\omega_i(\delta,f)_p = \sup_{|h_i|\le\delta}\left(\int_{[0.1]^N}|\Delta_{h_i}(f,t)|^p\,dt\right)^{1/p}(i=1,\dots,N),\quad 0<\delta<1.$$

We also use the notion of mixed integrated modulus of continuity. It is defined as follows

$$\omega_{1,\dots,N}(\delta,f)_p = \sup\nolimits_{|h_i|\le\delta}\left(\int_{[0,1]^N}|\Delta_{h_1,h_2,\dots,h_N}(f,t)|^p\,dt\right)^{1/p},$$
$$(i=1,\dots,N), 0<\delta<1.$$

It is not difficult to show that

$$\omega_{1,\dots,N} \le 2^{N-1}\sqrt[N]{\omega_1}\cdots\sqrt[N]{\omega_N} \tag{1.1}$$

In 1881 C.Jordan [12] introduced a class of functions of bounded variation and, applying it to the theory of Fourier series proved that if a continuous function has bounded variation, then its Fourier series converges uniformly. In 1906 G. Hardy [11] generalized the Jordan criterion to double Fourier series and introduced the notion of bounded variation (HBV) for a function of two variables. He proved that if a continuous function of two variables has bounded variation (in the sense of Hardy), then its Fourier series converges uniformly in the sense of Pringsheim. In 1999 U. Goginava [6] introduced the notion of bounded partial p-variation (PBV_p) and proved that the class $PBV_p(p \ge 1)$ guarantees the uniform convergence of N-dimensional trigonometric Fourier series of the continuous function.

We study the problem of absolute convergence of the series of Fourier-Haar coefficients for the classes of functions with bounded partial p-variations, which were first considered by U.Goginava(see [6] for $p=1$ and [7] for $p>1$).

Definition 1.2: Let f be a function defined on $[0,1]^N$ and $1-$periodic with respect to each variable. f is said to be a function of bounded partial p-variation $\left(f \in PBV_p(I^N)\right)$, if for any $i=1,2,...,N$ and $n=1,2,...$

$$V_i(f) = \sup_{x_j,j\in\{1,\dots,N\}\setminus\{i\}}\sup_{\Pi}\sum_{k=0}^{n-1}\Big|f\left(x_1,...,x_{i-1},x_i^{(2k)},x_{i+1},...,x_N\right)$$
$$- f\left(x_1,...,x_{i-1},x_i^{(2k+1)},x_{i+1},...,x_N\right)\Big|^p < \infty,$$

where Π is an arbitrary system of disjoint intervals $\left(x_i^{(2k)},x_i^{(2k+1)}\right)$

$(k = 0, 1, ..., n-1)$ on $[0,1]$, i.e.

$$0 \leq x_i^{(0)} < x_i^{(1)} < x_i^{(2)} < ... < x_i^{(2n-2)} < x_i^{(2n-1)} \leq 1.$$

2. Main results

The main results of this paper are presented in the following propositions.

Theorem 2.1: *Let $f \in PBV_p(I^N), p \geq 1$ and $\beta > \frac{2pN}{2+pN}$. Then*

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} |C_{n_1,...,n_N}(f)|^{\beta} < \infty.$$

Theorem 2.2: *Let $f \in PBV_p(I^N), p \geq 1$ and $\alpha < \frac{1}{pN} - \frac{1}{2}$. Then*

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \prod_{i=1}^{N} (n_i + 1)^{\alpha} |C_{n_1,...,n_N}(f)| < \infty.$$

Theorem 2.3: *Let $f \in PBV_p(I^N), p \geq 1$ and $\beta > 0$, $\alpha + 1 < \beta \left(\frac{1}{pN} + \frac{1}{2}\right)$. Then*

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \prod_{i=1}^{N} (n_i + 1)^{\alpha} |C_{n_1,...,n_N}(f)|^{\beta} < \infty.$$

Since $\mathrm{Lip}\,\frac{1}{p} \subset PBV_p$ in case $p > 1$ the sharpness of Theorems 2.1-2.3 follows from the works [15]-[16].

3. Auxiliary results

Lemma 3.1: *Let $f \in PBV_p(I^N), p \geq 1$. Then*

$$\omega_i(\delta, f)_p \leq 3^{1/p} \cdot \delta^{1/p} \cdot \mathrm{V}_i(f)_p \quad (i = 1, \ldots, N), \quad 0 < \delta < 1,$$

where $\mathrm{V}_i(f)_p$ is a partial p-variation of function.

Using the method of [9], we can easily obtain the validity of Lemma 3.1 .

4. Proof of main results

Proof: (of Theorem 2.1.) We write

$$\sum_{q_1=0}^{\infty} \cdots \sum_{q_N=0}^{\infty} |C_{q_1,...,q_N}(f)|^{\beta}$$
$$= \sum_{q_1=0}^{\infty} |C_{q_1,0,...,0}(f)|^{\beta} + \sum_{q_2=1}^{\infty} |C_{0,q_2,0,...,0}(f)|^{\beta} + \cdots + \sum_{q_N=1}^{\infty} |C_{0,...,q_N}(f)|^{\beta}$$

$$+\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\left|C_{q_1,q_2,0,\ldots,0}(f)\right|^{\beta}+\cdots+\sum_{q_{N-1}=1}^{\infty}\sum_{q_N=1}^{\infty}\left|C_{0,\ldots,q_{N-1},q_N}(f)\right|^{\beta}$$
$$+\cdots+\sum_{q_1=1}^{\infty}\cdots\sum_{q_N=1}^{\infty}\left|C_{q_1,\ldots,q_N}(f)\right|^{\beta}. \tag{4.1}$$

where $q_k = 2^{n_k} + i_k,\ i_k = 1,\ldots,2^{n_k};\ n_k = 0,1,\ldots;\ k = 1,\ldots,N.$

Then using the Hölder inequality, from the Lemma 3.1 we get

$$\sum_{i_1=1}^{2^{n_1}}\left|C^{(i_1)}_{n_1,0,\ldots,0}(f)\right|^p$$
$$= 2^{\frac{pn_1}{2}}\sum_{i_1=1}^{2^{n_1}}\left|\int_{[0,1]^{N-1}}\left(\int_{\frac{2i_1-2}{2^{n_1+1}}}^{\frac{2i_1-1}{2^{n_1+1}}}[f(x_1,\ldots,x_N)\right.\right.$$
$$\left.\left.-\ f\left(x_1+\frac{1}{2^{n_1+1}},x_2,\ldots,x_N\right)\right]dx_1\right)dx_2\cdots dx_N\Bigg|^p$$
$$\le 2^{\frac{pn_1}{2}}\sum_{i_1=1}^{2^{n_1}}\left[\int_{[0,1]^{N-1}}\left(\int_{\frac{2i_1-2}{2^{n_1+1}}}^{\frac{2i_1-1}{2^{n_1+1}}}\left|\Delta_{\frac{1}{2^{n_1+1}}}(f,x)\right|dx_1\right)dx_2\cdots dx_N\right]^p$$
$$\le 2^{\frac{pn_1}{2}}\sum_{i_1=1}^{2^{n_1}}\left[\int_{[0,1]^{N-1}}\left(\int_{\frac{2i_1-2}{2^{n_1+1}}}^{\frac{2i_1-1}{2^{n_1+1}}}\left|\Delta_{\frac{1}{2^{n_1+1}}}(f,x)\right|^p dx_1\right)dx_2\cdots dx_N\right)^{1/p}$$
$$\times\left(\int_{[0,1]^{N-1}}\int_{\frac{2i_1-2}{2^{n_1+1}}}^{\frac{2i_1-1}{2^{n_1+1}}}1dx_1\cdots dx_N\right)^{1-1/p}\Bigg]^p$$
$$\le 2^{\frac{pn_1}{2}}\cdot\frac{1}{2^{n_1(p-1)}}\int_{[0,1]^{N-1}}\left|\Delta_{\frac{1}{2^{n_1+1}}}(f,x)\right|^p dx_1\cdots dx_N$$
$$\le 2^{n_1\left(1-\frac{p}{2}\right)}\omega_1^p\left(\frac{1}{2^{n_1+1}},f\right)_p \le 2^{n_1\left(1-\frac{p}{2}\right)}\cdot 3\cdot\frac{1}{2^{n_1}}V_1^p(f)_p$$
$$\le c\cdot 2^{-\frac{n_1p}{2}}\ \mathrm{V}_1^p(f)_p. \tag{4.2}$$

Let $\frac{2p}{1+p}<\beta<p$. Using the Hölder inequality, from (4.2) we get

$$\sum_{i_1=1}^{2^{n_1}}\left|C^{(i_1)}_{n_1,0,\ldots,0}(f)\right|^{\beta}\le\left(\sum_{i_1=1}^{2^{n_1}}\left|C^{(i_1)}_{n_1,0,\ldots,0}(f)\right|^p\right)^{\beta/p}\cdot 2^{n_1(1-\beta/p)}$$
$$\le 2^{n_1(1-\beta/p)}\cdot\left(c\cdot 2^{-\frac{n_1p}{2}}V_1^p(f)_p\right)^{\beta/p} \tag{4.3}$$
$$\le c\cdot 2^{n_1(1-\beta/p)}\cdot 2^{-n_1\beta/2}\le c\cdot 2^{n_1\left[1-\frac{\beta}{p}-\frac{\beta}{2}\right]}.$$

By (4.3) and from the condition of the Theorem 2.1 we obtain

$$\sum_{q_1=2}^{\infty} |C_{q_1,0,\ldots,0}(f)|^{\beta}$$

$$= \sum_{n_1=0}^{\infty} \sum_{i_1=1}^{2^{n_1}} \left| C_{n_1,0,\ldots,0}^{(i_1)}(f) \right|^{\beta} \le c \sum_{n_1=0}^{\infty} 2^{n_1\left[1-\frac{\beta}{p}-\frac{\beta}{2}\right]} < \infty.$$

Using the Hölder inequality, by (1.1) and from Lemma 3.1 we get

$$\sum_{i_1=0}^{2^{n_1}-1} \cdots \sum_{i_N=0}^{2^{n_N}-1} \left| \int_{\frac{2i_1}{2^{n_1}}}^{\frac{i_1+1}{2^{n_1}}} \cdots \int_{\frac{2i_N}{2^{n_N}}}^{\frac{i_N+1}{2^{n_N}}} f(x_1,\ldots,x_N) \right.$$

$$\left. \times \chi_{n_1}^{(i_1)}(x_1)\chi_{n_2}^{(i_2)}(x_2)\cdots\chi_{n_N}^{(i_N)}(x_N)dx_1\cdots dx_N \right|^p \le 2^{p\frac{n_1+n_2+\cdots+n_N}{2}}$$

$$\times \sum_{i_1=0}^{2^{n_1}-1} \cdots \sum_{i_N=0}^{2^{n_N}-1} \left[\int_{\frac{2i_1}{2^{n_1+1}}}^{\frac{2i_1+1}{2^{n_1+1}}} \cdots \int_{\frac{2i_N}{2^{n_N+1}}}^{\frac{2i_N+1}{2^{n_N+1}}} \left| \Delta_{\frac{1}{2^{n_1+1}},\ldots,\frac{1}{2^{n_N+1}}}(f,x) \right| dx_1\cdots dx_N \right]^p$$

$$\le 2^{p\frac{n_1+\cdots+n_N}{2}}$$

$$\times \sum_{i_1=0}^{2^{n_1}-1} \cdots \sum_{i_N=0}^{2^{n_N}-1} \left[\left(\int_{\frac{2i_1}{2^{n_1+1}}}^{\frac{2i_1+1}{2^{n_1+1}}} \cdots \int_{\frac{2i_N}{2^{n_N+1}}}^{\frac{2i_N+1}{2^{n_N+1}}} \left| \Delta_{\frac{1}{2^{n_1+1}},\ldots,\frac{1}{2^{n_N+1}}}(f,x) \right|^p dx_1\cdots dx_N \right)^{1/p} \right.$$

$$\left. \times \left(\int_{\frac{2i_1}{2^{n_1+1}}}^{\frac{2i_1+1}{2^{n_1+1}}} \cdots \int_{\frac{2i_N}{2^{n_N+1}}}^{\frac{2i_N+1}{2^{n_N+1}}} 1 dx_1\cdots dx_N \right)^{1-1/p} \right]^p$$

$$\le 2^{p\frac{n_1+\cdots+n_N}{2}} \frac{1}{2^{(n_1+\cdots+n_N)(p-1)}} \int_{[0,1]^N} \left| \Delta_{\frac{1}{2^{n_1+1}},\ldots,\frac{1}{2^{n_N+1}}}(f,x) \right|^p dx_1\cdots dx_N$$

$$\le 2^{(n_1+n_2+\cdots+n_N)\left(1-\frac{p}{2}\right)} \omega_{1,2,\ldots,N}^p \left(\frac{1}{2^{n_1+1}},\ldots,\frac{1}{2^{n_N+1}},f \right)_p$$

$$\le 2^{(n_1+\cdots+n_N)\left(1-\frac{p}{2}\right)} \cdot 2^{(N-1)p} \omega_1^{p/N} \left(\frac{1}{2^{n_1+1}},f \right)_p \cdots \omega_N^{p/N} \left(\frac{1}{2^{n_N+1}},f \right)_p$$

$$\le 2^{(n_1+\cdots+n_N)\left(1-\frac{p}{2}\right)} \cdot 2^{(N-1)p} \frac{1}{2^{\frac{n_1+\cdots+n_N}{N}}} \le c \cdot 2^{(n_1+\cdots+n_N)\left(1-\frac{p}{2}-\frac{1}{N}\right)}. \tag{4.4}$$

Let $\frac{2p}{1+p} < \beta < p$. Using the Hölder inequality, by (4.4) we write

$$\sum_{i_1=0}^{2^{n_1}-1} \cdots \sum_{i_N=0}^{2^{n_N-1}} \left| C_{n_1,\ldots,n_N}^{(i_1,\ldots,i_N)}(f) \right|^{\beta}$$

$$\le c \cdot 2^{(n_1+\cdots+n_N)\left(1-\frac{p}{2}-\frac{1}{N}\right)\frac{\beta}{p}} \cdot 2^{(n_1+\cdots+n_N)\left(1-\frac{\beta}{p}\right)} \tag{4.5}$$

$$= c \cdot 2^{(n_1+\cdots+n_N)\left[\frac{\beta}{p}-\frac{\beta}{2}-\frac{\beta}{pN}+1-\frac{\beta}{p}\right]} = c \cdot 2^{n_1\left[1-\frac{\beta}{2}-\frac{\beta}{pN}\right]} \cdots 2^{n_N\left[1-\frac{\beta}{2}-\frac{\beta}{pN}\right]}$$

By (4.5) and from the condition of the Theorem 2.1 we get

$$\sum_{q_1=1}^{\infty}\cdots\sum_{q_N=1}^{\infty}|C_{q_1,\ldots,q_N}(f)|^{\beta}=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\sum_{i_1=0}^{2^{n_1}-1}\cdots\sum_{i_N=0}^{2^{n_N}-1}\left|C_{n_1,\ldots,n_N}^{(i_1,\ldots i_N)}(f)\right|^{\beta}$$
$$\leq\sum_{n_1=0}^{\infty}2^{n_1\left[1-\frac{\beta}{2}-\frac{\beta}{pN}\right]}\sum_{n_2=0}^{\infty}2^{n_2\left[1-\frac{\beta}{2}-\frac{\beta}{pN}\right]}\ldots\sum_{n_N=0}^{\infty}2^{n_N\left[1-\frac{\beta}{2}-\frac{\beta}{pN}\right]}<\infty.$$

The convergence of the remaining terms of series (4.1) is proved similarly. The proof of Theorem 2.1 is complete. □

Proof: (of Theorem 2.2.) We write

$$\begin{aligned}&\sum_{q_1=0}^{\infty}\cdots\sum_{q_N=0}^{\infty}(q_1+1)^{\alpha}\cdots(q_n+1)^{\alpha}|C_{q_1,\ldots,q_N}(f)|\\
&=\sum_{q_1=0}^{\infty}(q_1+1)^{\alpha}|C_{q_1,0,\ldots,0}(f)|+\sum_{q_2=1}^{\infty}(q_2+1)^{\alpha}|C_{0,q_2,0,\ldots,0}(f)|\\
&+\cdots+\sum_{q_N=1}^{\infty}(q_N+1)^{\alpha}|C_{0,\ldots,q_N}(f)|\\
&+\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}(q_1+1)^{\alpha}(q_2+1)^{\alpha}|C_{q_1,q_2,0,\ldots,0}(f)|\\
&+\cdots+\sum_{q_{N-1}=1}^{\infty}\sum_{q_N=1}^{\infty}(q_{N-1}+1)^{\alpha}(q_N+1)^{\alpha}\left|C_{0,\ldots,q_{N-1},q_N}(f)\right|\\
&+\cdots+\sum_{q_1=1}^{\infty}\cdots\sum_{q_N=1}^{\infty}(q_1+1)^{\alpha}\cdots(q_N+1)^{\alpha}|C_{q_1,\ldots,q_N}(f)|.\end{aligned}\tag{4.6}$$

Let $\beta=1$, then from (4.3) we get

$$\begin{aligned}\sum_{i_1=1}^{2^{n_1}}(2^{n_1}+i_1+1)^{\alpha}\left|C_{n_1,0,\ldots,0}^{(i_1)}(f)\right|&\leq c\cdot2^{n_1\alpha}\sum_{i_1=1}^{2^{n_1}}\left|C_{n_1,0,\ldots,0}^{(i_1)}(f)\right|\\
&\leq c\cdot2^{n_1\alpha}\cdot2^{n_1\left[\frac{1}{2}-\frac{1}{p}\right]}=c\cdot2^{n_1\left[\alpha+\frac{1}{2}-\frac{1}{p}\right]}.\end{aligned}\tag{4.7}$$

By (4.7) and from the condition of Theorem 2.2 we obtain

$$\sum_{q_1=1}^{\infty}(q_1+1)^{\alpha}|C_{q_1,0,\ldots,0}(f)|=\sum_{n_1=0}^{\infty}\sum_{i_1=1}^{2^{n_1}}(2^{n_1}+i_1+1)^{\alpha}\left|C_{n_1,0,\ldots,0}^{(i_1)}(f)\right|$$

$$\leq c\sum_{n_1=0}^{\infty} 2^{n_1\alpha}\sum_{i_1=1}^{2^{n_1}}\left|C^{(i_1)}_{n_1,0,\dots,0}(f)\right| \leq c\sum_{n_1=0}^{\infty} 2^{n_1\left[\alpha+\frac{1}{2}-\frac{1}{p}\right]}<\infty.$$

Let $\beta=1$, then by (4.5) and from the condition of Theorem 2.2 we get

$$\sum_{q_1=1}^{\infty}\cdots\sum_{q_N=1}^{\infty}(q_1+1)^{\alpha}\cdots(q_N+1)^{\alpha}\left|C_{q_1,\dots,q_N}(f)\right|$$

$$\leq\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}2^{(n_1+\cdots+n_N)\alpha}\sum_{i_1=1}^{2^{n_1}}\cdots\sum_{i_N=1}^{2^{n_N}}\left|C^{(i_1,\dots,i_N)}_{n_1,\dots,n_N}(f)\right|$$

$$\leq c\sum_{n_1=0}^{\infty}2^{n_1\left[\frac{1}{2}+\alpha-\frac{1}{pN}\right]}\sum_{n_2=0}^{\infty}2^{n_2\left[\frac{1}{2}+\alpha-\frac{1}{pN}\right]}\dots\sum_{n_N=0}^{\infty}2^{n_N\left[\frac{1}{2}+\alpha-\frac{1}{pN}\right]}<\infty.$$

The convergence of the remaining terms of series (4.6) is proved similarly.
The proof of Theorem 2.2 is complete. □

Combining the methods of Theorems 2.1-2.2 we can prove validity of Theorem 2.3.

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