SOME MATHEMATICAL PROBLEMS OF POROELASTICITY: MODELING, ANALYSIS, DESIGN, AND ITS APPLICATIONS

T. Vashakmadze

I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University

Introduction

Our aim is to construct a spatial mathematical model of poroelastic anisotropic medium and consider different problems, when the bodies of the above mentioned type are weakened by couplages and cracks. In addition we do not consider or quote all the known works in this area. This aspect is successfully done in the web-site of A. Cheng see the part "Poronet", where the mentioned material is listed. Our objective is somehow different from the approach of such classic authors, as Jacob Frenkel and Mauris Biot. These scientists and their followers discovered and explained a great many physical events, which are characteristic for many processes and among them - for those generated by porosity . Biot also had constructed many interesting mathematical models of continuum medium. His models for poroelastic medium became the basis for many research, the number of which has been increasing with exceptional intensity during the last decade. The essential difference between Biot and our approches is following: the spatial part of the main opeartor corresponding to the mathematical model constructed below is the strongly elliptic while for the model of Biot (see e.g., [1], system of equation (2.6)) the same one is an indefinite opeartor as it contains the factor - operator *graddiv* type having a double degeneration.

1. Construction of the mathematical model of the poroelastic medium

It is obvious, that each medium can consist of solid, liquid and gaseous material. We consider the case when we have the mixture of solid and liquid media, at that time we introduce the concept of measure of the mixture: the whole volume of the considered solid body is described by pores, the size of which is considerably greater than molecular sizes, but much less than the size of the body. At the same time the distance between the closest pores is of order of their diameter. Due to penetration of a liquid into the pores the solid body is being saturatd by a liquid. The evident examples of this are the icebergs of large volume saturated by salty water, diseased bones, soil of earth and so on. If after the mixing the medium keeps as a solid body, then we say, that we have the saturation of the first type and mean that we have the medium which represents the binary mixture of solid with solid. If the medium obtained due to mixing has the properties of a liquid, we say, that we have the saturation of the second type. We consider the first case and in each cell, or, in a macropoint (the size of macropoints is considerably greater than the size of pores, but much less than the size of body) we consider two tensors of tension and deformation corresponding to components of mixture and two displacement vectors. In addition, writing their connection rule, we base on experimental works of Biot, Frenkel. For the mixture of solid body and a liquid Biot introduced the tensors of tension and deformation $\overline{\sigma}_{ij}$ and $\overline{\varepsilon}_{ij}$ ([2], F. (61)-(71)), which characterizes the mixture, as a whole medium. If $\overline{\sigma}$ is an obtainable tensor for the mixture, then it is clear, that

$$\overline{\sigma} = \sigma + (\overline{\sigma} - \sigma), \tag{1.1}$$

where σ is a tension tensor of the solid body. If in the result of this process the mixture of a solid body and a liquid remains as a solid body, then for this description it is necessary to introduce the quantities, which generally are characteristics of a solid body. Thus, as mentioned above, we consider the case, when the interaction of a solid body and a liquid generates the solid-solid mixture which, if it is only a mechanical mixture, is less rigid than the initial body. When in the mixture besides the mechanical one there exist chemical and other processes, it is available to obtain the mixtures of higher rigidity. The analogous picture must take place when the mixture of a solid body and a liquid transforms into the liquid, that is the solid body melts. In this case the mixture is called a mechanical mixture, if its viscosity is more than the viscosity of the given liquid. Thus, while constructing the mathematical model of binary mixture it is necessary to consider its prehistory.

2. The basic system of three-dimensional equations with respect to spatial variables

We denote the domain in the three dimensional Euclidean space R^3 by Ω. In the Cartesian coordinates the point is denoted by $x = (x_1, x_2, x_3)$ or (x, y, z). Time changes in the interval $t \in (0.T)$, $\quad Q_T = \Omega \times (0, T)$.

Thus, in each point of the mixture (macropoint) we consider the following average quantities

$$\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{32}, \sigma_{31}, \sigma_{21})^T, \quad \varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{32}, \varepsilon_{31}, \varepsilon_{21})^T,$$

$$u = (u_1, u_2, u_3)^T, \quad p = (p_{11}, p_{22}, p_{33}, p_{32}, p_{31}, p_{21})^T,$$

$$\zeta = (\zeta_{11}, \zeta_{22}, \zeta_{33}, \zeta_{32}, \zeta_{31}, \zeta_{21})^T, \quad w = (w_1, w_2, w_3)^T,$$

where for the components of the deformation $\varepsilon_{ij}, \zeta_{ij}$ we have the formulas

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i.j} + u_{j,i} + u_{k,i}u_{k,j}\right), \quad \zeta_{ij} = \frac{1}{2}\left(w_{i.j} + w_{j,i} + w_{k,i}w_{k,j}\right). \tag{2.1}$$

The equilibrium equations for the mixture have the following form

$$\begin{cases} \partial_j\left(\sigma_{ij} + \sigma_{kj}u_{i,k}\right) = \partial_{tt}\left(\rho_1 u_i + \rho_2 w_i\right) + f_i, \\ \partial_j\left(p_{ij} + p_{kj}w_{i,k}\right) = \partial_{tt}\left(\rho_2 u_i + \rho_3 w_i\right) + \frac{\eta}{\mu}\partial_t w_i + \varphi_i, \quad (x, T) \in Q_T, \end{cases} \tag{2.2}$$

where $f=(f_1,f_2,f_3)^T$, $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T$ are the vectors of volume forces, ρ_i are densities, and the quantity $\frac{\eta}{\mu}$ is defined analogously to [1].

Instead of law (5.1) of [1] we define the Hooke's type law (basing also on the experimental data of A. Cheng [3]) as follows:

$$\sigma = B\varepsilon + C\zeta, \tag{2.3}$$

$$p = C\varepsilon + M\zeta, \quad (x,t)\in Q_T, \tag{2.4}$$

where $B,\ A=B^{-1}$- are rigidity and pliability 6×6 symmetric matrices, respectively. $C=\{c_{ij}\}_{6\times 6}$, $M=\{m_{ij}\}_{6\times 6}$ - are also symmetric matrices.

As we mentioned above, we will mean, that in each point of the body passes at least one plane of flexible symmetry, which is parallel to $O_{x_1x_2}$ plane. That is to say, that in matrices C and M there are at most 13 constants not equal to zero and

$$\begin{aligned} b_{i4}=b_{i5}=b_{46}=b_{56}=c_{i4}=c_{i5}=c_{46}=c_{56} &= m_{i4}=m_{i5} \\ &= m_{46}=m_{56}=0. \end{aligned} \tag{2.5}$$

From (2.3) and (2.4) we have

$$\sigma_{ii}=B_i\varepsilon+C_i\zeta, \qquad p_{ii}=C_i\varepsilon+M_i\zeta, \tag{2.6}$$

$$\sigma_{ij}=B_{9-(i+j)}\varepsilon+C_{9-(i+j)}\zeta, \qquad p_{ij}=C_{9-(i+j)}\varepsilon+M_{9-(i+j)}\zeta, \quad i\neq j, \tag{2.7}$$

where B_i, C_i, M_i - are i-th lines of corresponding matrices.

Let us introduce the following notations

$$\tau_{ij}=(\sigma_{ij},p_{ij})^T, \quad \in_{ij}=(\varepsilon_{ij},\zeta_{ij})^T, \quad U_i=(u_i,w_i)^T, \tag{2.8}$$

On the basis of the introduced notations and assumptions (2.5) the equilibrium equations (2.2) and relations (2.3), (2.4) are written in the following form

$$\partial_j\left(\tau_{ij}+\tau_{kj}\odot U_{i,k}\right)=\rho\partial_{tt}U_i+\rho_0\partial_t U_i+F_i, \tag{2.9}$$

$$\tau_{ii}=A_{i1}\in_{11}+A_{i2}\in_{22}+A_{i3}\in_{33}+A_{i6}\in_{12}, \quad i=1,2,3, \tag{2.10}$$

$$\tau_{\alpha 3}=A_{6-\alpha\ 4}\in_{32}+A_{6-\alpha}\in_{31}, \tag{2.11}$$

$$\tau_{12}=A_{16}\in_{11}+A_{26}\in_{22}+A_{36}\in_{33}+A_{66}\in_{12}, \tag{2.12}$$

where

$$A_{mn}=\begin{pmatrix} b_{mn} & c_{mn} \\ c_{mn} & m_{mn}\end{pmatrix}, \quad \rho=\begin{pmatrix}\rho_1 & \rho_3\\ \rho_3 & \rho_2\end{pmatrix}, \quad \rho_0=\begin{pmatrix}0 & 0\\ 0 & \frac{\eta}{\mu}\end{pmatrix}, \ F_i=(f_i,\varphi_i)^T.$$

Symbol $\odot$ denotes the following operation

$(a_1,a_2)^T\odot(b_1,b_2)^T=(a_1b_1,a_2b_2)^T$.

Furter, in the main part of this lecture we reported also the two-dimensional new models of von Karman-Reissner type systems of differential equations with variable thickness constructed by methods of Ch.1, [4].

REFERENCES

[1] Biot M.A., Generalized Theory of Acoustic Propagation of Porous Dissipative Media. J. Acoust. Soc. Am. v.35, No.5, part 1, 1962, pp. 1254-1264

[2] Biot M.A. Nonlinear and Semilinear Rheology of Porous Media. J. Geophys. Res. 78, 1973, pp. 4924-4937

[3] Cheng A.H.-D. Material Coefficients of Anisotropic Poroelasticity. Inf. J.Rock Mech. Min. Sci., v. 34, No.2, 1997, pp. 199-205

[4] Vashakmadze T., The theory of Anisotropic Elastic Plates. Kluwer Acad. Publ. Dortrecht-Boston-London, 1999

Recieved October 15, 2002; revised November 25, 2002; accepted December 8, 2002.