DESCRIPTION OF THE NUMERICAL ALGORITHM OF MULTIDIMENSIONAL DOMAINS IMAGING

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Abstract. Domain imaging means in the first place, creation of method allowing mapping of the given domain to another one. For a creation of such an algorithm certain common requirements can be necessary [1], in particular:

1. Simplicity;
2. Speed;
3. Regularity (conformity) of created mapping;
4. Transparency of the algorithm.

The method described in the present paper allows creating of a mapping of domains based on a mathematical model simulating stationary distribution of the temperature in a rod construction. The siggested algorithm models a pseudo-regular mapping and allows avoiding the solution of Laplace equation (needed for creation of regular mapping).

Key words: mesh generation, domain imaging, numerical algorithm, triangulation.

MSC 2000: 65L50, 65M50, 65K05, 90C99.

1. Introduction

A domain considered in the paper is surface and an algorithm is developed on the example of mapping of the surface to rectangle.

Further we will call the above algorithm as a parameterization of the surface. The presentation of piecewise smooth surfaces in parametrical form (parameterization) allows to carry out with great efficiency:

1. repartition of the surface;
2. interpolation of the surface;
3. morphing (continuous transformation of one surface into another one);

This and other operations on the parameterized surface find an application in the creation of the finite element mesh generation, interpolation of domain, 3D graphical visualization, animation, and etc.

The algorithm could easily be generalized for the case of multidimensional areas.

2. Task Formulation

Let be given a finite piecewise smooth surface, defined in three-dimensional Euclidean space by means of the finite number of triangles under the following requirements:

a) the surface should be simply connected;
b) each triangle defining the surface should have not more than one adjacent triangle from each side[1)];
c) the surface should have not less than four border segments each border segment must have only one adjacent border segment to each of its vertices[2)];
d) triangles having common vertex should be adjacent in pairs[3)];
e) no triangle [1] (when considering triangles as an open set of points) should have common points with other triangle[4)].

We need to create homomorphous (one-to-one and continuous) mapping of a given piecewise smooth surface to plane rectangle (further referred to as parametrical rectangle).

3. Describtion of the Algorithm

The whole border of the surface is divided into four boundary parts as shown in Fig. 1.
1. numbers of even and odd borders are opposite to each other;
2. each border starts and ends by the vertex of triangle (i.e., it is impossible that the border include only the part of the side of a triangle).
Items (a), (c) ensure the fulfillment of the above listed requirements.
Let us consider each side of the triangle as heat conducting[5)] and heat isolated pivot.
Let us formulate a boundary-contact problem of the stationary heat distribution in pivotal construction assuming that temperature in each pivot is distributed according to the linear law: $T_i(l) = \frac{a_i}{L_i} l + b_i$, L_i is the length of the pivot, a_i, b_i are the unknown coefficients, l is the variable such that $0 \leq l \leq L_i$.
Let us introduce the following notation:
P is a point of the contact of pivots (vertex of triangle);
S_P is an array with pivots numbers having point with number P as a vertex;

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1) Let us note that if triangle doesn't have adjacent triangle on one of its sides, then this side is part of the border of the surface under consideration and further will be called border segment.

2) i.e., the surface should have one border without branching points.

3) *Adjacent in pairs* are triangles with common vertex and

$$A_1|A_2|...|A_n|A_1$$

($|$ means adjacent to a side).

4) i.e., no triangle can cross or coincide with other triangle.

5) Conductivity coefficients for all pivots are the same.

N_P is a size of the array S_p (total amount of pivots having point P as a beginning or an end)

$$\delta_i^p = \begin{cases} 0 & \text{if the point } P \text{ is beginning of the pivot with number } i, \\ L_i & \text{if the point } P \text{ (see Fig. 2) is the end of the pivot.} \end{cases}$$

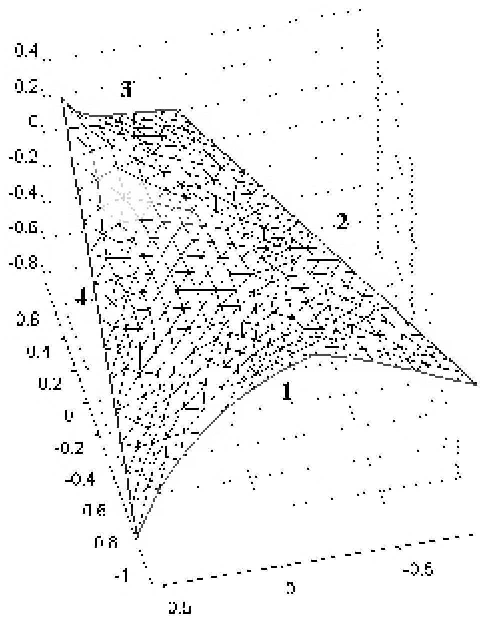

Fig. 1.

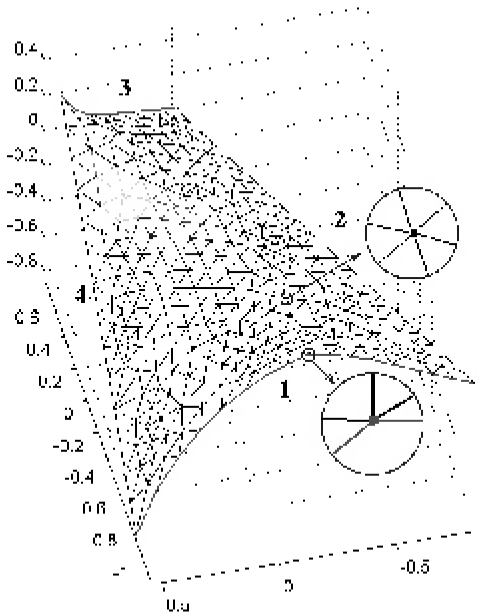

Fig. 2.

For each point P belonging to the bound 1 we give the value of the temperature u_{max} for each pivot adjoining to it

$$T_j(\delta_j^P) = u_{max}, \tag{1}$$

where $j = S_P(1)$, $S_P(2)$, ..., $S_P(N_P - 1)$, $S_P(N_p)$.

For each point P belonging to the bound 3 we give the value of the temperature u_{min}, for each pivot adjoining to it:

$$T_j(\delta_j^P) = u_{min}, \tag{2}$$

where $j = S_P(1)$, $S_P(2)$, ..., $S_P(N_P - 1)$, $S_P(N_p)$.

In all other points (including points belonging to the borders 2 and 4) two conditions hold.

The first one requires that the sum of heat streams from each pivot must be equal to zero at the point P:

$$\frac{1}{\max\limits_j(L_j)} \sum_j \left(\frac{dT_j}{dl}\right) = 0, \tag{3}$$

where $j = S_P(1)$, $S_P(2)$, ..., $S_P(N_P - 1)$, $S_P(N_p)$.

The second one requires equality of temperatures at the point of contact P for each pivot:

$$T_j(\delta_j^P) = T_k(\delta_k^P) \tag{4}$$

where $j = S_P(1)$, $S_P(2)$, ..., $S_P(N_P - 1)$, $S_P(N_p)$,
$k = S_P(2)$, $S_P(3)$, ..., $S_P(N_P)$, $S_P(1)$.

Equations (1), (2), (3), (4) represent a linear system of equations with a_i, b_i as unknown coefficients. Solving system we determine value of temperature in each point. Let us denote temperature at each point by u_P.

In the same manner we formulate second problem of finding v_P, giving the temperature v_{max} on the border 2 and v_{min} on the border 4.

After solving above mentioned problems in each vertex of the triangle two values (u, v) will be given, which we will interpret as coordinates of this point on the parametrical rectangle (parametrical coordinates). Thus, each vertex of triangle has also parametrical coordinates apart from Cartesian coordinates.

4. Mappings

Mapping from (x, y, z) to (u, v).

We can create (see Fig. 4) for each triangle a mapping of points in (x,y,z) coordinates to the (u,v) coordinates (see Fig. 4)

$$u = F_u(x, y, z), \quad v = F_v(x, y, z).$$

Over each vertex of triangle (x_i, y_i, z_i) $(i = 1, 2, 3)$ (see Fig. 3) we create new point with coordinates $(x_i + u_i n_x, y_i + u_i n_y, z_i + u_i n_z)$, where $\vec{n}(n_x, n_y, n_z)$ $(\|\vec{n}\| = 1)$ the normal vector perpendicular to the surface of the triangle.

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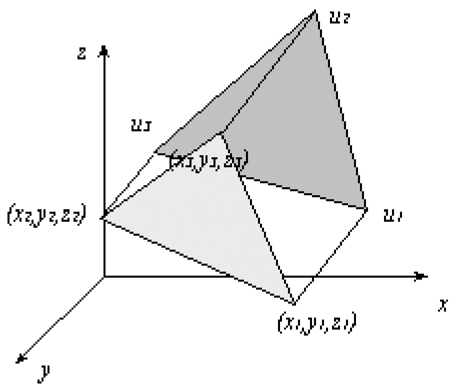

Fig. 3

The plane with the coefficients A_u, B_u, C_u, D_u is created on the new points

$$A_u x + B_u y + C_u z + D_u = 0.$$

For each point (x, y, z) belonging to the triangle we find corresponding meaning of u by solving $A_u(x + un_x) + B_u(y + un_y) + C_u(z + un_z) + D_u = 0$:

$$u = F_u(x, y, z) = -\frac{A_u x + B_u y + C_u z + D_u}{A_u n_x + B_u n_y + C_u n_z}.$$

In the same manner we find:

$$v = F_v(x, y, z) = -\frac{A_v x + B_v y + C_v z + D_v}{A_v n_x + B_v n_y + C_v n_z}.$$

Mapping from (u,v) to (x,y,z).

We are looking for the reverse mapping of the point with coordinates (u, v) to the point with the coordinates (x, y, z) in the following form:

$$x = F_x(u, v) = a_x u + b_x v + c_x,$$
$$y = F_y(u, v) = a_y u + b_y v + c_y,$$
$$z = F_z(u, v) = a_z u + b_z v + c_z.$$

Coefficients a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z could be found by means of solving the system of equations $(i = 1, 2, 3)$

$$x_i = F_x(u_i, v_i) = a_x u_i + b_x v_i + c_x,$$
$$y_i = F_y(u_i, v_i) = a_y u_i + b_y v_i + c_y,$$
$$z_i = F_z(u_i, v_i) = a_z u_i + b_z v_i + c_z.$$

Remark 1 Let us note that constructed mapping for each triangle is affine.

Remark 2 Mapping of the whole surface on parametrical rectangle (which, in general is not affine mapping) is constructed by means of affine mapping of each triangle.

Remark 3 Without special efforts the algorithm can be generalized for the case of multi-dimensional areas. In the case of three-dimensional areas, there will be the following differences:

1. tetrahedrons should be considered instead of triangles;
2. the border of the area should be surface and should be divided into eight parts;
3. the area should be mapped on the rectangular parallelepiped.

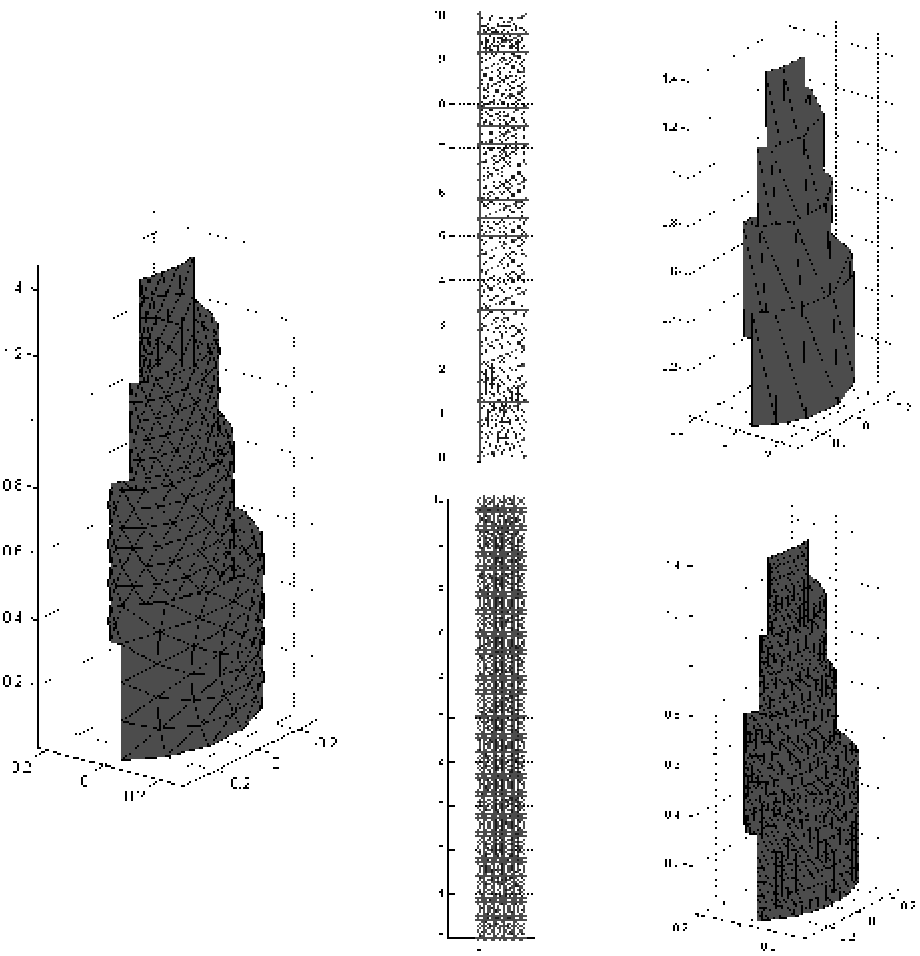

Fig. 4

5. Application of the Algorithm

Parameterization of piecewise smooth surfaces could find its application in all those cases when the work with surface itself is complicated.

Presentation of the Surface by Means of Different Polygons

The surface defined by means of triangle mesh could be:

1. presented by means of another triangle mesh (coarsened or re-meshed);
2. presented by means of different polygons (quadrangles, hexagons, and etc.).

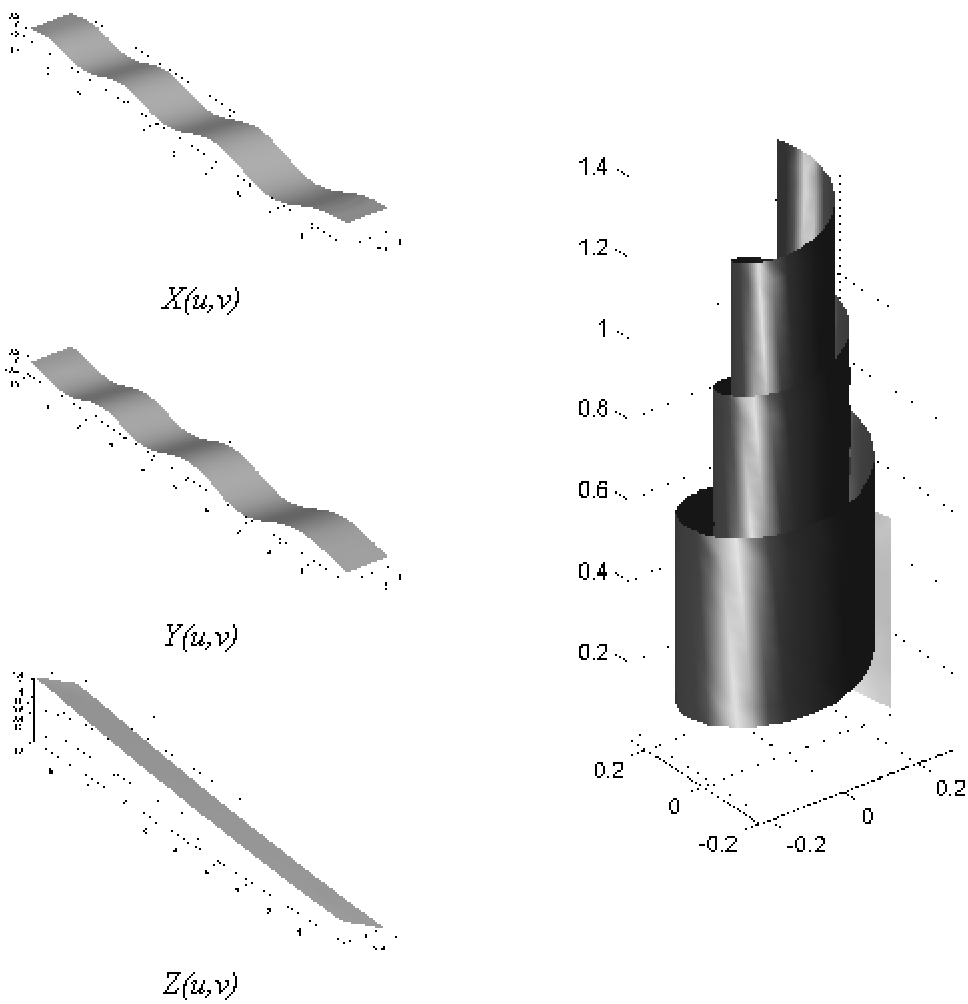

Fig. 5

In both, above mentioned cases at first step initial surface is parameterized (see Fig. 4), the next step is to cover parametrical rectangle with desired polygons and finally mapp points (u, v) to the (x, y, z) coordinates.

7. Presentation of a Surface in an Analytical Form

The surface should be interpolated by means of special functions (B-spline [2], orthogonal polynomials, trigonometric functions and etc), in the first turn the set of points (x_i, y_i, z_i), $i = 1..N$, should be selected on the surface and relevant (u_i, v_i) coordinates should be found. Then by means of special functions we define the function of two variables $X(u, v)$ (see Fig. 5) in such a way that functional

$$S_x = \sum_{i=1}^{N} \|X(u_i, v_i) - x_i\|$$

would be of minimal value. In the same manner we define functions $Y(u, v)$ and $Z(u, v)$.

R e f e r e n c e s

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