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ON LEMPERT FUNCTIONS IN C^2

by Witold Jarnicki

Abstract. We give a characterization of all cartesian products $D_1 \times D_2 \subset \mathbb{C}^2$ for which the Lempert function and the injective Lempert function coincide. In particular, we show that there exist domains in $\mathbb{C}^2$ for which they are different.

1. Introduction. The main result of this paper is very similar to the one presented in [**2**], which concerns equality between the Kobayashi-Royden and Hahn pseudometrics for product domains in $\mathbb{C}^2$. The ideas and techniques used here are mostly the same; therefore, only essentially different parts are presented.

For a domain $D \subset \mathbb{C}^n$, the *Lempert function* L and the *injective Lempert function* H are defined by the formulae:

$$L_D(z_1, z_2) := \inf\{p(\lambda_1, \lambda_2) \colon \exists_{f \in \mathcal{O}(E,D)}\ f(\lambda_1) = z_1,\ f(\lambda_2) = z_2\},\ z_1, z_2 \in D,$$
$$H_D(z_1, z_2) := \inf\{p(\lambda_1, \lambda_2) \colon \exists_{f \in \mathcal{O}(E,D)}\ f(\lambda_1) = z_1,\ f(\lambda_2) = z_2,\ f \text{ is injective}\}, \quad z_1, z_2 \in D,\ z_1 \neq z_2,$$

where E denotes the unit disc and p denotes the Poincaré distance (cf. [**1**])[1]. Put $H_D(z, z) := 0$. Obviously, $L \leq H$. It is known that both functions are invariant under biholomorphic mappings, i.e., if $f \colon D \longrightarrow \widetilde{D}$ is biholomorphic, then

$$H_D(z_1, z_2) = H_{\widetilde{D}}(f(z_1), f(z_2)), \quad L_D(z_1, z_2) = L_{\widetilde{D}}(f(z_1), f(z_2)), \quad z_1, z_2 \in D.$$

It is also known that $H_{\mathbb{C}} \equiv L_{\mathbb{C}} \equiv 0$ and that for a hyperbolic (in the sense of the uniformization theorem) domain $D \subset \mathbb{C}$ and for any $z_1, z_2 \in D$, $z_1 \neq z_2$ we have $H_D(z_1, z_2) \equiv L_D(z_1, z_2)$ iff D is simply connected. Using methods similar to [**3**], one can prove that $H_D \equiv L_D$ for any domain $D \subset \mathbb{C}^n$, $n \geq 3$.

[1] Observe that for any $z_1, z_2 \in D$, $z_1 \neq z_2$, there exists an injective holomorphic disc $f \colon E \longrightarrow D$ such that $z_1, z_2 \in f(E)$. Indeed, first we take an injective $\mathcal{C}^1$-curve $\alpha : [0, 1] \longrightarrow D$ with $\alpha(0) = z_1$, $\alpha(1) = z_2$, and $\alpha'(t) \neq 0$ for all $t \in [0, 1]$. Next, we take a $\mathcal{C}^1$–approximation of α by a polynomial mapping P with $P(0) = z_1$ and $P(1) = z_2$; P has to be injective when close enough to α. Finally, we proceed as in Remark 3.1.1 in [**1**].

Let $D_1, D_2 \subset \mathbb{C}$. The aim of this paper is to show that $H_{D_1\times D_2} \equiv L_{D_1\times D_2}$ iff at least one of D_1, D_2 is simply connected or biholomorphic to $\mathbb{C}_*$. In particular, there are domains $D \subset \mathbb{C}^2$ for which $H_D \not\equiv L_D$.

2. The main result.

Theorem 1. *Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then:*

1. *If at least one of D_1, D_2 is simply connected, then $H_{D_1\times D_2} \equiv L_{D_1\times D_2}$.*
2. *If at least one of D_1, D_2 is biholomorphic to $\mathbb{C}_*$, then $H_{D_1\times D_2} \equiv L_{D_1\times D_2}$.*
3. *Otherwise, $H_{D_1\times D_2} \not\equiv L_{D_1\times D_2}$.*

Let $p_j\colon D_j^* \longrightarrow D_j$ be a holomorphic universal covering of D_j ($D_j^* \in \{\mathbb{C}, E\}$), $j = 1, 2$. Recall that if D_j is simply connected, then $H_{D_j} \equiv L_{D_j}$. If D_j is not simply connected and D_j is not biholomorphic to $\mathbb{C}_*$, then, by the uniformization theorem, $D_j^* = E$ and p_j is not injective.

Hence, Theorem 1 is an immediate consequence of the following three propositions (we keep the above notation).

Proposition 2. *If $H_{D_1} \equiv L_{D_1}$, then $H_{D_1\times D_2} \equiv L_{D_1\times D_2}$ for any domain $D_2 \subset \mathbb{C}$.*

Proposition 3. *If D_1 is biholomorphic to $\mathbb{C}_*$, then $H_{D_1\times D_2} \equiv L_{D_1\times D_2}$ for any domain $D_2 \subset \mathbb{C}$.*

Proposition 4. *If $D_j^* = E$ and p_j is not injective, $j = 1, 2$, then $H_{D_1\times D_2} \not\equiv L_{D_1\times D_2}$.*

Observe that for any domain $D \subset \mathbb{C}^n$ we have:
$H_D \equiv L_D$ iff for any $f \in \mathcal{O}(E, D)$, $0 < \alpha < \vartheta < 1$ with $f(0) \neq f(\alpha)$, there exists an injective $g \in \mathcal{O}(E, D)$ such that $g(0) = f(0)$ and $g(\vartheta) = f(\alpha)$. $(*)$

Proof of Proposition 2. Let $f = (f_1, f_2) \in \mathcal{O}(E, D_1 \times D_2)$, $0 < \alpha < \vartheta < 1$, and $f(0) \neq f(\alpha)$.

First, consider the case where $f_1(0) \neq f_1(\alpha)$.

By $(*)$, there exists an injective function $g_1 \in \mathcal{O}(E, D_1)$ such that $g_1(0) = f_1(0)$ and $g_1(\vartheta) = f_1(\alpha)$. Put $g(z) := (g_1(z), f_2(\frac{\alpha}{\vartheta}z))$.

Obviously, $g \in \mathcal{O}(E, D_1 \times D_2)$ and g is injective. Moreover, $g(0) = f(0)$ and $g(\vartheta) = (g_1(\vartheta), f_2(\alpha)) = (f_1(\alpha), f_2(\alpha)) = f(\alpha)$.

Suppose now that $f_1(0) = f_1(\alpha)$. Take $0 < d < \operatorname{dist}(f_1(0), \partial D_1)$ [2] and put

$$h(z) := \frac{f_2(\frac{\alpha}{\vartheta}z) - f_2(0)}{f_2(\alpha) - f_2(0)}, \quad M := \max\{|h(z)|\colon z \in \overline{E}\},$$

$$g_1(z) := f_1(0) + \frac{d}{M + \frac{1}{\vartheta}}\left(h(z) - \frac{z}{\vartheta}\right), \quad g(z) := \left(g_1(z), f_2\left(\frac{\alpha}{\vartheta}z\right)\right), \quad z \in E.$$

[2]$\operatorname{dist}(z_0, A) := \inf\{\|z - z_0\|\colon z \in A\}$, where $\|\cdot\|$ is the Euclidean norm; $\operatorname{dist}(z_0, \emptyset) := +\infty$.

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $|g_1(z) - f_1(0)| < d$, we get $g_1(z) \in B(f_1(0), d) \subset D_1$, [3] $z \in E$. Hence $g \in \mathcal{O}(E, D_1 \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally $g(0) = (g_1(0), f_2(0)) = (f_1(0) + \frac{d}{M+\frac{1}{\vartheta}} h(0), f_2(0)) = f(0)$ and $g(\vartheta) = (g_1(\vartheta), f_2(\alpha)) = (f_1(0) + \frac{d}{M+\frac{1}{\vartheta}}(h(\vartheta) - 1), f_2(\alpha)) = (f_1(0), f_2(\alpha)) = f(\alpha)$. □

Proof of Proposition 3. We may assume that $D_1 = \mathbb{C}_*$ and $D_2 \neq \mathbb{C}$. Using $(*)$, let $f = (f_1, f_2) \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$, $0 < \alpha < \vartheta < 1$, and $f(0) \neq f(\alpha)$. Applying an appropriate automorphism of $\mathbb{C}_*$, we may assume that $f_1(0) = 1$.

For the case where $f_2(0) = f_2(\alpha)$, we apply the above construction to the domains $\widetilde{D}_1 = f_2(0) + \operatorname{dist}(f_2(0), \partial D_2)E$, $\widetilde{D}_2 = \mathbb{C}_*$ and mappings $\widetilde{f}_1 \equiv f_2(0)$, $\widetilde{f}_2 = f_1$.

Now, consider the case where $f_2(0) \neq f_2(\alpha)$ and $f_1(\alpha) = 1 + \vartheta$. We put

$$g_1(z) := 1 + z, \quad g(z) := \left(g_1(z), f_2\left(\frac{\alpha}{\vartheta} z\right)\right), \quad z \in E.$$

Obviously, $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$ and g is injective. We have $g(0) = (1, f_2(0)) = f(0)$ and $g(\vartheta) = (1 + \vartheta, f_2(\alpha)) = f(\alpha)$.

In all other cases, define a sequence (d_k) such that we have

$$d_k^k = \frac{f_1(\alpha)}{1 + \vartheta}, \quad k \in \mathcal{N},$$
$$\operatorname{Arg}(d_k) \longrightarrow 0.$$

Observe that $d_k \longrightarrow 1$. Let $M := \max\{|f_2(z)| \colon |z| \le \frac{\alpha}{\vartheta}\}$. Take a $k \in \mathcal{N}$ such that $|c_k| > M$, where

$$c_k := \frac{f_2(\alpha) - d_k f_2(0)}{1 - d_k}.$$

Put

$$h(z) := \frac{f_2(\frac{\alpha}{\vartheta} z) - c_k}{f_2(0) - c_k},$$
$$g_1(z) := (1 + z) h^k(z), \quad g_2(z) := f_2\left(\frac{\alpha}{\vartheta} z\right), \quad g(z) := (g_1(z), g_2(z)), \quad z \in E.$$

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $h(z) \neq 0$, we have $g_1(z) \neq 0$, $z \in E$. Hence $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally, $g(0) = (h^k(0), f_2(0)) = f(0)$ and

[3] $B(z_0, r) := \{z \in \mathbb{C}^n \colon \|z - z_0\| < r\}$.

$$
\begin{aligned}
g(\vartheta) = (g_1(\vartheta), f_2(\alpha)) &= \Big((1+\vartheta)\Big(\frac{f_2(\alpha)-c_k}{f_2(0)-c_k}\Big)^k, f_2(\alpha)\Big) \\
&= \Big((1+\vartheta)\Big(\frac{f_2(\alpha)(1-d_k)-f_2(\alpha)+d_k f_2(0)}{f_2(0)(1-d_k)-f_2(\alpha)+d_k f_2(0)}\Big)^k, f_2(\alpha)\Big) \\
&= \Big((1+\vartheta)\Big(\frac{d_k(f_2(0)-f_2(\alpha))}{f_2(0)-f_2(\alpha)}\Big)^k, f_2(\alpha)\Big) \\
&= ((1+\vartheta)d_k^k, f_2(\alpha)) = f(\alpha).
\end{aligned}
$$

□

Proof of Proposition 4. One can show (see [**2**]) that there exist $\varphi_1, \varphi_2 \in \mathrm{Aut}(E)$ and a point $q = (q_1, q_2) \in E^2$, $q_1 \neq q_2$, such that $p_j(\varphi_j(q_1)) = p_j(\varphi_j(q_2))$, $j = 1, 2$, and $\det[(p_j \circ \varphi_j)'(q_k)]_{j,k=1,2} \neq 0$. Put $\widetilde{p}_j := p_j \circ \varphi_j$, $j = 1, 2$, and suppose that $H_{D_1 \times D_2} \equiv L_{D_1 \times D_2}$. Put $z = (z_1, z_2) := (\widetilde{p}_1(0), \widetilde{p}_2(0))$ and $w = (w_1, w_2) := (\widetilde{p}_1(r), \widetilde{p}_2(r))$, where $r \in (0, 1)$ is such that $\widetilde{p}_j \colon \overline{B(0, r)} \longrightarrow D_j$ is injective.

Let $(1, 1/\sqrt{r}) \ni \alpha_n \searrow 1$. Fix an $n \in \mathcal{N}$. Since $L_{D_1 \times D_2}(z, w) = p(0, r)$, there exists $f_n \in \mathcal{O}(E, D_1 \times D_2)$ such that $f_n(0) = z$ and $f_n(\alpha_n r) = w$. By $(*)$, there exists an injective holomorphic mapping $g_n = (g_{n,1}, g_{n,2}) \colon E \longrightarrow D_1 \times D_2$ such that $g_n(0) = z$ and $g_n(\alpha_n^2 r) = w$. Let $\widetilde{g}_{n,j}$ be the lifting with respect to $\widetilde{p}_j$ of $g_{n,j}$ with $\widetilde{g}_{n,j}(0) = 0$, $j = 1, 2$. Observe that $\widetilde{g}_{n,j}(\alpha_n^2 r) = r$ for n large enough, $j = 1, 2$.

By the Montel theorem, we may assume that the sequence $(\widetilde{g}_{n,j})_{n=1}^{\infty}$ is locally uniformly convergent, $\widetilde{g}_{0,j} := \lim_{n \longrightarrow \infty} \widetilde{g}_{n,j}$. We have $\widetilde{g}_{0,j}(0) = 0$, $\widetilde{g}_{0,j}(r) = r$ and $\widetilde{g}_{0,j} \colon E \longrightarrow E$. By the Schwarz lemma we have $\widetilde{g}_{0,j} = \mathrm{id}_E$, $j = 1, 2$. From now on, we proceed as in [**2**]. □

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Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków
Poland
e-mail: wmj@im.uj.edu.pl