UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIX
2001

GEOMETRIC DEGREE OF GENERICALLY FINITE EXTENSIONS

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Abstract. Let V, W be irreducible algebraic subsets of $\mathbb{C}^n$. Every dominant, generically finite mapping $f : V \to W$ can be extented to a dominant, generically finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$. We show that if f is a projection then there exists a dominant, generically finite extension F of f with the same geometric degree. The same is shown for an arbitrary f with the assumption that V, W are smooth sets and $4 \cdot \dim V + 2 \leq n$.

1. Introduction. Let $f : V \to W$ be any polynomial mapping of irreducible algebraic subsets of $\mathbb{C}^n$. It is known (see [**2**] Lemma 5.4) that the mapping f can be extended to a dominant, generically finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$. If f is, also, a dominant generically finite mapping, then a natural question arises concerning the relation between the number $\operatorname{gdeg} F$ (defined as the number of points in the generic fiber of F and called geometric degree of the mapping F) and the number $\operatorname{gdeg} f$.

If f is a finite mapping, then f can be extended to a finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ (see [**7**]). For relations between $\operatorname{gdeg} F$ and $\operatorname{gdeg} f$ in this situation see [**3**], [**4**], [**5**] and [**6**].

In this paper we will prove that any generically finite projection $\pi : V \to \pi(V)$ can be extended to a dominant, generically finite mapping with the same geometric degree. Applying this fact we will show that any dominant, generically finite mapping between smooth algebraic sets (with a large codimension) can be extended to a dominant, generically finite mapping with the same geometric degree.

2. Notation and basic facts. Let $f : V \to W$ be any polynomial mapping of irreducible algebraic sets. The mapping f is called generically finite if

Paper partially supported by FNP (Polish Science Foundation).

there exists an open and dense subset U of W such that $\#f^{-1}(y)$ is finite for all $y \in U$. The mapping f is called dominant if $\overline{f(V)} = W$.

It is known that if f is a dominant and generically finite mapping then $\mathbb{C}(V)$ is a finite field extension of the field $f^*(\mathbb{C}(W)) \simeq \mathbb{C}(W)$, where f^* denotes the homomorphism of coordinate rings $f^* : \mathbb{C}[V] \ni \varphi \mapsto \varphi \circ f \in \mathbb{C}[W]$, and its extension to the homomorphism of the fields $f^* : \mathbb{C}(V) \to \mathbb{C}(W)$. In this situation $\operatorname{gdeg} f = [\mathbb{C}(V) : \mathbb{C}(W)] = \dim_{\mathbb{C}(W)} \mathbb{C}(V)$ (see [**8**]). If f is dominant and generically finite, then $\dim V = \dim W$. Conversely, if $\dim V = \dim W$, then $f : V \to W$ is dominant if and only if f is generically finite.

3. Projections. Now we will prove the following

THEOREM 3.1. *Let $V \subset \mathbb{C}^k \times \mathbb{C}^n$ be an irreducible algebraic set, $\pi : V \to 0 \times \mathbb{C}^n$ the natural projection. If $\pi : V \to \overline{\pi(V)}$ is generically finite, then there exists a generically finite mapping $\Pi : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ such that $\Pi|_V = \pi$ and*

$$\operatorname{gdeg} \Pi = \operatorname{gdeg} \pi.$$

PROOF. Let $x = (x_1, \ldots, x_k)$ be coordinates in $\mathbb{C}^k$ and let $y = (y_1, \ldots, y_n)$ be coordinates in $\mathbb{C}^n$.

To begin, let us assume that $k = 1$. We know that $\mathbb{C}(V)$ is a finite field extension of the field $\mathbb{C}(\overline{\pi(V)})$. Thus $x_1|_V$ is algebraic over $\mathbb{C}(\overline{\pi(V)})$. Let $\tilde{P} = T^l + \tilde{a}_1 T^{l-1} + \ldots + \tilde{a}_l \in \mathbb{C}(\overline{\pi(V)})[T]$ be the minimal polynomial for $x_1|_V$ over $\mathbb{C}(\overline{\pi(V)})$. Multiplying $\tilde{P}$ by a common multiple of the denominators of $\tilde{a}_1, \ldots, \tilde{a}_l$ we obtain a polynomial $P = a_0 T^l + a_1 T^{l-1} + \ldots + a_l \in \mathbb{C}[\overline{\pi(V)}][T]$ such that $P(x_1|_V) = 0$ and $a_0 \neq 0$. By an extension of coefficients $a_0, \ldots, a_l$ from the set $\overline{\pi(V)}$ to the whole space $0 \times \mathbb{C}^n$ we obtain a polynomial $\bar{P}$ such that $\bar{P}|_V = 0$ and $\deg \bar{P} = \deg \tilde{P}$. Now let $\Pi : \mathbb{C} \times \mathbb{C}^n \ni (x, y) \mapsto (\bar{P}(x, y), y) \in \mathbb{C} \times \mathbb{C}^n$. It is easy to see that Π is a generically finite mapping such that $\Pi|_V = \pi$, and $\operatorname{gdeg} \Pi = \deg \bar{P} = \deg \tilde{P} = \operatorname{gdeg} \pi$.

For $k > 1$ we proceed by induction. Choose a system of coordinates in $\mathbb{C}^k$ in such a way that $\pi_1 : V \to \overline{\pi_1(V)}$ is a generically finite mapping, where $\pi_1 : V \ni ((x_1, \ldots, x_k), y) \mapsto ((x_2, \ldots, x_k), y) \in \mathbb{C}^{k-1} \times \mathbb{C}^n$. It follows that $\pi_2 : \overline{\pi_1(V)} \to \overline{\pi(V)}$ is a generically finite mapping, where $\pi_2 : \overline{\pi_1(V)} \ni ((x_2, \ldots, x_k), y) \mapsto y \in 0 \times \mathbb{C}^n$. Of course, there is $\pi = \pi_2 \circ \pi_1$ and $\operatorname{gdeg} \pi = \operatorname{gdeg} \pi_2 \cdot \operatorname{gdeg} \pi_1$. By induction, there exists a generically finite mapping $\tilde{\Pi}_2 : \mathbb{C}^{k-1} \times \mathbb{C}^n \to \mathbb{C}^{k-1} \times \mathbb{C}^n$ such that $\tilde{\Pi}_2|_{\pi_1(V)} = \pi_2$ and $\operatorname{gdeg} \tilde{\Pi}_2 = \operatorname{gdeg} \pi_2$. By the first part of the proof, there exists a generically finite mapping $\Pi_1 : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ such that $\Pi_1|_V = \pi_1$ and $\operatorname{gdeg} \Pi_1 = \operatorname{gdeg} \pi_1$. Set $\Pi_2 : \mathbb{C}^k \times \mathbb{C}^n \ni ((x_1, \ldots, x_k), y) \mapsto (x_1, \tilde{\Pi}_2((x_2, \ldots, x_k), y)) \in \mathbb{C}^k \times \mathbb{C}^n$, and

$$\Pi = \Pi_2 \circ \Pi_1.$$

It is easy to check that $\Pi : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}^k \times \mathbb{C}^n$ is a generically finite mapping such that $\Pi|_V = \pi$ and $\operatorname{gdeg} \Pi = \operatorname{gdeg} \pi$.

$\square$

Let us notice that there can exist a dominant, generically finite extension with a smaller geometric degree than the geometric degree of the extended mapping. We have the following

EXAMPLE 1. Let $F : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1x_2, x_2, x_3, \ldots, x_n) \in \mathbb{C}^n$, then F is generically finite and $\operatorname{gdeg} F = 1$. Let $W \in \mathbb{C}[x_3, \ldots, x_n][x_1]$ be any irreducible polynomial of degree $k \in \mathbb{N}$ (with respect to x_1) and set $V = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : x_2 = 0, W(x_1, x_3, \ldots, x_n) = 0 \}$. Then $F|_V : V \to F(V)$ is a generically finite mapping and $\operatorname{gdeg}(F|_V) = k$.

4. Mappings of smooth variety. Let us recall some facts about embeddings. A polynomial mapping $f : V \to \mathbb{C}^n$ is called an embedding if $f(V) = \overline{f(V)}$ and f is an isomorphism on the image. We have the following well-known lemma (see e.g.[**1**])

LEMMA 4.1. *If $X \subset \mathbb{C}^n$ is a closed algebraic smooth set,* $\dim X = k$ *and* $n > 2k + 1$, *then we can change coordinates in such a way that the projection*

$$\phi : X \ni (x, y) \mapsto (0, y) \in 0 \times \mathbb{C}^{2k+1},$$

is an embedding.

We also have the following

THEOREM 4.2. [**1**, Thm 1.2] *Let $X \subset \mathbb{C}^n$ be a closed algebraic set which is smooth and not necessarily irreducible of dimension (not necessarily pure) k. Let $\phi : X \to \mathbb{C}^n$ be an embedding. If $n \geq 4k + 2$ then there exists an isomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\Phi|_X = \phi.$$

Now we are in a position to prove the following

THEOREM 4.3. *Let $V, W \subset \mathbb{C}^n$ be smooth, irreducible algebraic sets and let $f : V \to W$ be a dominant, generically finite mapping. If V, W are smooth and $4 \cdot \dim V + 2 \leq n$, then there exists a generically finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F|_V = f$ and*

$$\operatorname{gdeg} F = \operatorname{gdeg} f.$$

PROOF. By Lemma 4.1 we can assume that the projections:

$$\phi_1 : V \to 0 \times \mathbb{C}^{2k+1} \text{ and } \phi_2 : W \to 0 \times \mathbb{C}^{n-2k-1}$$

are embeddings. Put

$$\tilde{V} = \phi_1(V), \qquad \tilde{W} = \phi_2(W)$$

and

$$\tilde{f} = \phi_2 \circ f \circ \phi_1^{-1} : \tilde{V} \to \tilde{W}.$$

The mapping $\tilde{f}$ is generically finite with $\operatorname{gdeg} \tilde{f} = \operatorname{gdeg} f$.

Since $\tilde{V} \subset 0 \times \mathbb{C}^{2k+1}$, $\tilde{W} \subset 0 \times \mathbb{C}^{n-2k-1}$, we can consider the sets $\tilde{V}$ and $\tilde{W}$ as subsets of $\mathbb{C}^{2k+1}$ and $\mathbb{C}^{n-2k-1}$, respectively. The following mapping

$$\psi : \tilde{V} \ni x \mapsto (x, \tilde{f}(x)) \in \hat{V} \subset \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1},$$

where $\hat{V} = \psi(\tilde{V})$, is an isomorphism. Thus for the projection:

$$\pi : \hat{V} \ni (x, y) \mapsto (0, y) \in 0 \times \mathbb{C}^{n-2k-1}$$

we have $\tilde{f} = \pi \circ \psi$, and since ψ is an isomorphism, it follows that $\pi : \hat{V} \to \overline{\tilde{W} = \pi(\hat{V})}$ is a generically finite mapping and $\operatorname{gdeg} \pi = \operatorname{gdeg} \tilde{f} = \operatorname{gdeg} f$. By Theorem 3.1 there exists a generically finite mapping $\Pi : \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1} \to \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1}$ such that $\Pi|_{\hat{V}} = \pi$ and

$$\operatorname{gdeg} \Pi = \operatorname{gdeg} \pi.$$

Applying Theorem 4.2 to $\psi : \tilde{V} \to \hat{V}$, $\phi_1 : V \to \tilde{V}$ and $\phi_2 : W \to \tilde{W}$, we conclude that there exist isomorphisms Ψ, Φ_1, $\Phi_2 : \mathbb{C}^n \to \mathbb{C}^n$ such that:

$$\Psi|_{\tilde{V}} = \psi,\ \Phi_1|_V = \phi_1,\ \Phi_2|_W = \phi_2.$$

Now

$$F = \Phi_2^{-1} \circ \Pi \circ \Psi \circ \Phi_1$$

is a generically finite extension of f such that:

$$\operatorname{gdeg} F = \operatorname{gdeg} f.$$

□

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Received October 31, 2000

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