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IMPROPER INTERSECTION OF ANALYTIC CURVES

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Dedicated to Professor Tadeusz Winiarski on the occasion of his 60th birthday

Abstract. We give an effective formula for the improper intersection cycle of analytic curves in terms of local parametrizations of the curves.

1. Introduction. A new geometric improper intersection theory in the complex analytic geometry was initiated by Achilles, Tworzewski and Winiarski in [**2**] (for isolated improper intersections). In general case the theory was introduced by Tworzewski in [**18**]. For arbitrary analytic sets X, Y (or more generally for analytic cycles X, Y) in a complex manifold M we obtain an analytic cycle $X \bullet Y$ in M which reflects the geometric structure of intersection of X and Y in M. It has also been generalized to arbitrary analytic spaces M by Rams [**15**]. The theory has found applications in the separation of analytic sets [**4**], [**5**], [**6**], [**10**], [**17**]. The main idea of construction of $X \bullet Y$ in M is as follows (see [**18**]): for any $x \in M$ we define the index of intersection $i(X, Y; x) \in \mathbb{Z}$ of X and Y at x. Since $i(X, Y; \cdot) : M \to \mathbb{Z}$ is an analytically constructible function, it generates an analytic cycle in M, just $X \bullet Y$. If X and Y intersect properly in M then $X \bullet Y$ is the ordinary cycle of intersection of X and Y in M in the sense of Draper [**7**]. Much more complicated case is when X and Y intersect improperly. We will consider this case when X and Y are analytic curves in M i.e. analytic sets of pure dimension one in M. If X and Y are irreducible analytic curves in M then two cases may occur:

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1. $X \cap Y$ is an isolated set in M. Then

$$X \bullet Y = \sum_{P \in X \cap Y} i(X \bullet Y; P)P$$

where $i(X \bullet Y; P) \in \mathbb{N}$ (in this case $i(X, Y; P) = i(X \bullet Y; P)$). An effective formula for $i(X \bullet Y; P)$ was given in [**3**],

2. $X = Y$. In this case

$$X \bullet X = X + \sum_{P \in \mathrm{Sing}(X)} i(X \bullet X; P)P$$

where $\mathrm{Sing}(X)$ is the set of singular points of X. The main result of the paper is an effective formula for the coefficients $i(X \bullet X; P)$ in terms of local parametrizations of X near P (Th. 4).

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2. Intersection algorithm. Since in the proof of the main theorem we will use notions from the Tworzewski intersection algorithm, we first recall it (see [**18**]).

Let M be a complex manifold of dimension n. An *analytic cycle on* M is a formal sum

$$A = \sum_{j \in J} \alpha_j C_j,$$

where $\mathbb{Z} \ni \alpha_j \neq 0$ for $j \in J$ and $\{C_j\}_{j \in J}$ is a locally finite family of distinct irreducible analytic subsets of M. By $\mathrm{Supp}\, A$ we mean $\bigcup_{j \in J} C_j$. If $U \subset M$ is an open set then by $A|U$ we mean the restriction of A to U (defined in an obvious way). The *degree* $\mu(A; x)$ *of* A *at* $x \in M$ is defined to be the sum

$$\sum_{j \in J} \alpha_j \mu(C_j; x),$$

where $\mu(C_j; x)$ stands for the degree of the component C_j at x. Then the function

$$M \ni x \mapsto \mu(A; x) \in \mathbb{Z}$$

is analytically constructible, and inversely, for each analytically constructible function $f : M \to \mathbb{Z}$ there exists a unique analytic cycle A on M such that its degree equals the value of f at every point of M i.e.

$$f(x) = \mu(A; x), \quad x \in M.$$

Each analytic cycle A has the unique decomposition into the sum of analytic cycles T_d of pure dimension d

$$A = \sum_{d=0}^{n} T_d.$$

The *extended degree of A at x* is defined by

$$\mu^{ext}(A; x) := (\mu(T_n; x), ..., \mu(T_0; x)) \in (\mathbb{N}_0)^{n+1},$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $\Delta \subset M$ is an analytic submanifold then the *part of A supported by Δ* is defined by

$$A^{\Delta} := \sum_{j \in J,\, C_j \subset \Delta} \alpha_j C_j.$$

Now we may recall the Tworzewski algorithm. Since the intersection cycle is a biholomorphic invariant we will lead considerations in open sets of $\mathbb{C}^n$.

Let X and Y be pure dimensional analytic sets in an open set $\Omega \subset \mathbb{C}^n$. Let $r := \dim X$ and $s := \dim Y$. Denote by $\Delta \subset \mathbb{C}^n \times \mathbb{C}^n$ the diagonal, i.e.

$$\Delta := \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : x_1 = y_1, \dots, x_n = y_n\}.$$

For any open set $U \subset \mathbb{C}^n \times \mathbb{C}^n$ such that $U \cap \Delta \neq \emptyset$, we denote by $\mathcal{H}(U, X \times Y)$ the family of all systems $\mathcal{H} = (H_1, ..., H_n)$ of analytic hypersurfaces in U (i.e. analytic sets of codimension 1 in U) such that:

(a) H_j is a nonsingular hypersurface and contains Δ,

(b) $\bigcap_{j=1}^{n} T_{(x,x)} H_j = T_{(x,x)} \Delta$ for $x \in U \cap \Delta$,

(c) $(U \backslash \Delta) \cap (X \times Y) \cap H_1 \cap ... \cap H_j$ is an analytic subset of $(U \backslash \Delta)$ of pure dimension $r + s - j$ (or empty) for $j = 1, ..., n$.

For any $\mathcal{H} = (H_1, ..., H_n) \in \mathcal{H}(U, X \times Y)$ we define an analytic cycle $(X \times Y) \cdot \mathcal{H}$ in U by the following procedure:

Step 0. Let $Z_0 := (X \times Y) \cap U$, treated as an analytic cycle. Then $Z_0 = Z_0^{\Delta} + (Z_0 - Z_0^{\Delta})$, where Z_0^{Δ} is the part of Z_0 supported by $U \cap \Delta$ (usually $Z_0^{\Delta} = 0$ unless $X = Y = \{\text{one point}\}$).

Step 1. Let $Z_1 := (Z_0 - Z_0^{\Delta}) \cdot H_1$ – it is the intersection cycle in the sense of Draper of $(Z_0 - Z_0^{\Delta})$ and H_1 (note that the intersection of these sets is proper). Then $Z_1 = Z_1^{\Delta} + (Z_1 - Z_1^{\Delta})$, where Z_1^{Δ} is the part of Z_1 supported by $U \cap \Delta$.

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Step n. Let $Z_n := (Z_{n-1} - Z_{n-1}^{\Delta}) \cdot H_n$. Then $Z_n = Z_n^{\Delta} + (Z_n - Z_n^{\Delta})$, where Z_n^{Δ} is the part of Z_n supported by $U \cap \Delta$. In this last case $\mathrm{Supp}(Z_n - Z_n^{\Delta}) \cap \Delta = \emptyset$.

Then we define

$$(X \times Y) \cdot \mathcal{H} := Z_0^{\Delta} + ... + Z_n^{\Delta}.$$

Now we may define the basic notions of the intersection theory. For any $x \in \Omega$ we define *the extended index of intersection of X and Y at x* by

$$(1) \qquad i^{ext}(X,Y;x) := \min_{lex}\{\mu^{ext}((X \times Y) \cdot \mathcal{H};(x,x)) \in (\mathbb{N}_0)^{n+1} : \mathcal{H} \in \mathcal{H}(U, X \times Y),\ U \ni (x,x)\}$$

where minimum in $(\mathbb{N}_0)^{n+1}$ is taken with respect to the lexicographic order. Next, we define *the index of intersection of X and Y at x* by

$$i(X,Y;x) := \sum i^{ext}(X,Y;x),$$

where $\sum v$ is the sum of coordinates of $v \in \mathbb{Z}^{n+1}$. The function

$$\Omega \ni x \mapsto i(X,Y;x) \in \mathbb{Z}$$

in analytically constructible in Ω. So, it generates an analytic cycle in Ω. We denote it by $X \bullet Y$ and call *the intersection cycle of X and Y in Ω*. If

$$X \bullet Y = \sum_{j \in J} \alpha_j C_j, \quad \alpha_j \in \mathbb{Z}$$

then α_j is called *the intersection multiplicity of X and Y along C_j* and is denoted by $i(X \bullet Y; C_j)$.

We extend this definition to the case of arbitrary analytic cycles in the usual way, i.e. by $\mathbb{Z}$–linearity.

In the sequel we will need results concerning the above algorithm and the intersection cycle. First we quote two formal properties of the intersection cycle.

THEOREM 1. *1. Let X_1, X_2 and Y be pure dimensional analytic sets in an open set $\Omega \subset \mathbb{C}^n$. If X_1, X_2 are irreducible in Ω, $X_1 \neq X_2$, and $\dim X_1 = \dim X_2$ then*

$$(X_1 \cup X_2) \bullet Y = X_1 \bullet Y + X_2 \bullet Y.$$

2. Let X and Y be pure dimensional analytic sets in an open set $\Omega \subset \mathbb{C}^n$. If $\tilde{\Omega} \subset \Omega$ is an open set and $\tilde{X} := X \cap \tilde{\Omega}$, $\tilde{Y} := Y \cap \tilde{\Omega}$ then

$$\tilde{X} \bullet \tilde{Y} = (X \bullet Y)\,|\tilde{\Omega}.$$

PROOF. Ad 1. See Corollary 5 in [**1**] or Prop. 3, Ch. III in [**13**].

Ad 2. It follows from the fact that the definition of the extended index of intersection and, a fortiori, the index of intersection is local, i.e.

$$i(X,Y;x) = i(\tilde{X},\tilde{Y};x), \quad x \in \tilde{\Omega}.$$

This gives $\tilde{X} \bullet \tilde{Y} = (X \bullet Y)\,|\tilde{\Omega}$. □

The next results concern the algorithm. The first is that in the above algorithm it suffices to take for H_j linear hyperplanes, and the second one that there are many such hyperplanes. To formulate precisely these results we have to fix some notions. Since an arbitrary hyperplane H in $\mathbb{C}^n \times \mathbb{C}^n$ containing Δ has the equation

$$A_1(x_1 - y_1) + ... + A_n(x_n - y_n) = 0, \quad A_1, ..., A_n \in \mathbb{C} \text{ and not all vanish,}$$

the set of such hyperplanes will be identified with $\mathbb{P}^{n-1}$. For $A \in \mathbb{P}^{n-1}$ we denote by H^A the hyperplane generated by A and for $\mathbf{A} \in (\mathbb{P}^{n-1})^n$ by $\mathcal{H}^{\mathbf{A}}$ an appropriate system of hyperplanes. If $x \in X \cap Y$ then we define

$$\begin{aligned}\mathcal{J}(x) &= \{\mathbf{A} \in (\mathbb{P}^{n-1})^n : \mathcal{H}^{\mathbf{A}} \text{ realizes minimum in the intersection algorithm}\} \\ &= \{\mathbf{A} \in (\mathbb{P}^{n-1})^n : i^{ext}(X, Y; x) = \mu^{ext}((X \times Y) \cdot \mathcal{H}^{\mathbf{A}}; (x, x))\}.\end{aligned}$$

THEOREM 2. *For $x \in X \cap Y$*

$$\mathcal{J}(x) \neq \emptyset.$$

PROOF. See Nowak [**12**], Cor. 6, or Achilles and Rams [**1**], Cor. 3. □

THEOREM 3. *Fix $x \in X \cap Y$, $i \in \{1, ..., n\}$ and $\mathbf{A} = (A^1, ..., A^n) \in \mathcal{J}(x)$. Then there exists an neighbourhood $W \subset \mathbb{P}^{n-1}$ of A^i such that for any $A \in W$ we have*

$$(A^1, ..., A^{i-1}, A, A^{i+1}, ..., A^n) \in \mathcal{J}(x).$$

PROOF. See Rams [**15**], Th. 3.3, Achilles and Rams [**1**], Cor. 3, Nowak [**13**], Prop. 5, Ch. III, or Spodzieja [**16**], Th. 3. □

3. Improper intersections of analytic curves. Let X, Y be irreducible analytic curves in an open set $\Omega \subset \mathbb{C}^n$. Then either $X \cap Y$ is an isolated set in Ω or $X = Y$. In the first case from the intersection algorithm we easily obtain that

$$\operatorname{Supp}(X \bullet Y) = X \cap Y$$

(see [**18**], Th. 6.6). So,

$$X \bullet Y = \sum_{P \in X \cap Y} i(X \bullet Y; P) P.$$

The effective formulas for $i(X \bullet Y; P)$ were given in [**3**], Th. 1, in terms of local parametrizations of X and Y near P. Namely, without loss of generality (by the biholomorphic invariance of intersection cycle and Theorem 1) we may assume that $P = 0 \in X \cap Y$ is an isolated point of intersection of X and Y, the germs of X and Y at 0 are irreducible and that

$$\begin{aligned}&\mathbb{C} \supset K_1 \ni t \mapsto (t^p, \phi(t)) \in X, \quad \operatorname{ord} \phi \geqslant p, \\ &\mathbb{C} \supset K_2 \ni \tau \mapsto (\tau^q, \psi(\tau)) \in Y, \quad \operatorname{ord} \psi \geqslant q\end{aligned}$$

are local parametrizations of X and Y in a neighbourhood of 0 (K_1, K_2 are neighbourhoods of 0 in $\mathbb{C}$). Then

$$
\begin{aligned}
i(X \bullet Y; P) &= (1/q) \sum_{i=1}^{q} \operatorname{ord}(\phi(t^q) - \psi(\eta^i t^p)) \\
&= (1/p) \sum_{i=1}^{p} \operatorname{ord}(\psi(t^p) - \phi(\varepsilon^i t^q))
\end{aligned}
\tag{2}
$$

where η, ε are primitive roots of unity of degree q and p, respectively (in the above formulas by $\operatorname{ord} \lambda$ of a holomorphic mapping $\lambda = (\lambda_1, ..., \lambda_n)$ defined in a neighbourhood of $0 \in \mathbb{C}^k$ we mean $\min_{i=1}^{n}(\operatorname{ord} \lambda_i)$).

Consider now the other case, $X = Y$.

THEOREM 4. *Let X be an irreducible analytic curve in an open set $\Omega \subset \mathbb{C}^n$. Then the intersection cycle $X \bullet X$ is equal to*

$$
X \bullet X = X + \sum_{P \in \operatorname{Sing}(X)} i(X \bullet X; P) P \tag{3}
$$

where $i(X \bullet X; P)$ is given by the following formulas (for simplicity we assume that $P = 0$):

1. If the germ of X at 0 is irreducible and

$$
\mathbb{C} \supset K \ni t \mapsto (t^p, \phi(t)) \in X, \quad \operatorname{ord} \phi > p > 1,
$$

is a local parametrization of X in a neighbourhood of 0 (K is a neighbourhood of 0 in $\mathbb{C}$), then

$$
i(X \bullet X; P) = \sum_{i=1}^{p-1} \operatorname{ord}(\phi(t) - \phi(\varepsilon^i t)) \tag{4}
$$

where ε is a primitive root of unity of degree p,

2. If the germ of X at 0 is reducible and

$$
(X)_0 = (X_1)_0 \cup ... \cup (X_k)_0
$$

is the decomposition of the germ $(X)_0$ of X at 0 into irreducible components, then

$$
i(X \bullet X; P) = \sum_{\substack{i,j=1 \\ i \neq j}}^{k} i(X_i \bullet X_j; P) + \sum_{i=1}^{k} i(X_i \bullet X_i; P) \tag{5}
$$

and $i(X_i \bullet X_j; P)$ can be calculated by formula (2) and $i(X_i \bullet X_i; P)$ by formula (4).

Proof. Take an arbitrary point $P \in X$. We may assume that $P = 0$.

1. Assume that the germ of X at 0 is irreducible. We will calculate the index $i(X, X; P)$. Let

$$\mathbb{C} \supset K \ni t \mapsto \Phi(t) := (t^p, \phi(t)) = (t^p, \phi_2(t), ..., \phi_n(t)) \in X, \quad \operatorname{ord} \phi > p \geqslant 1,$$

be a local parametrization of X in a neighbourhood of 0. We may also assume (shrinking Ω) that

$$X = \{\Phi(t) : t \in K\}.$$

Then we have

$$X \times X = \{(t^p, \phi(t), \tau^p, \phi(\tau)) : t, \tau \in K\}.$$

Now we apply the intersection algorithm. Take an open set $U = \tilde{U} \times \tilde{U} \subset \Omega \times \Omega$, $(0, 0) \in U$ and a system of hyperplanes $\mathcal{H} = (H_1, ..., H_n) \in \mathcal{H}(U, X \times Y)$, $H_i = \{(x, y) : A_1^i(x_1 - y_1) + ... + A_n^i(x_n - y_n) = 0, \quad A_1^i, ..., A_n^i \in \mathbb{C}$ and not all vanish$\}$. By Theorem 3 we may assume that $A_1^1 \neq 0$. Consider two cases:

(i) P is a nonsingular point of X. Then $p = 1$. Consider the step 0 of the algorithm. Since $X \times X$ is an irreducible analytic set of pure dimension 2 and $X \times X \not\subseteq \Delta$ then $Z_0^\Delta = 0$ and $Z_0 - Z_0^\Delta = (X \times X) \cap U$. Let us pass to the step 1 of the algorithm. We have to find $Z_1 = (Z_0 - Z_0^\Delta) \cdot H_1 = ((X \times X) \cap U) \cdot H_1$. Notice first that

$$\begin{aligned}
&(X \times X) \cap U \cap H_1 \\
&= \{(t, \phi(t), \tau, \phi(\tau)) : t, \tau \in \Phi^{-1}(\tilde{U}),\ A_1^1(t - \tau) + ... + A_n^1(\phi_n(t) - \phi_n(\tau)) = 0\} \\
&= \{(t, \phi(t), t, \phi(t)) : t \in \Phi^{-1}(\tilde{U})\} \\
&\cup \{(t, \phi(t), \tau, \phi(\tau)) : t, \tau \in \Phi^{-1}(\tilde{U}),\ A_1^1 + A_2^1 \frac{(\phi_2(t) - \phi_2(\tau))}{t - \tau} + ... \\
&+ A_n^1 \frac{(\phi_n(t) - \phi_n(\tau))}{t - \tau} = 0\}.
\end{aligned}$$

The first set is equal to $X^\Delta \cap U \subset \Delta$, where $X^\Delta := \{(x, x) : x \in X\}$. So, it is biholomorphic to X near 0. Since the vector $[1, 0, ..., 0]$ belongs to the tangent space $T_{(0,0)}(X \times X)$ and not to H_1 (because $A_1^1 \neq 0$), the proper intersection of $(X \times X) \cap U$ with H_1 is transversal along $X^\Delta \cap U$. Hence

$$Z_1^\Delta = X^\Delta \cap U$$

and

$$\mu(Z_1^\Delta; (0, 0)) = 1.$$

Since $A_1^1 \neq 0$ and $\operatorname{ord} \phi > 1$, the second set is empty for sufficiently small U. Then for such U we have $Z_1 - Z_1^\Delta = \emptyset$. Hence

$$i^{ext}(X, Y; 0) = (0, ..., 0, 1, 0)$$

and consequently

$$i(X, Y; 0) = 1.$$

(ii) P is a singular point of X. Then $p > 1$. The step 0 of the algorithm is the same as in the first case. We have $Z_0^\Delta = 0$ and $Z_0 - Z_0^\Delta = (X \times X) \cap U$. Let us pass to the step 1 of the algorithm. We have to find

$$Z_1 = (Z_0 - Z_0^\Delta) \cdot H_1 = ((X \times X) \cap U) \cdot H_1.$$

Notice first that

$$\begin{aligned}
&(X \times X) \cap U \cap H_1 \\
&= \{(t^p, \phi(t), \tau^p, \phi(\tau)) : t, \tau \in \Phi^{-1}(\tilde{U}),\ A_1^1(t^p - \tau^p) + ... + A_n^1(\phi_n(t) - \phi_n(\tau)) = 0\} \\
&= \{(t^p, \phi(t), t^p, \phi(t)) : t \in \Phi^{-1}(\tilde{U})\} \\
&\cup \{(t^p, \phi(t), \tau^p, \phi(\tau)) : t, \tau \in \Phi^{-1}(\tilde{U}),\ A_1^1(t^{p-1} + ... + \tau^{p-1}) \\
&+ A_2^1 \frac{(\phi_2(t) - \phi_2(\tau))}{t - \tau} + ... + A_n^1 \frac{(\phi_n(t) - \phi_n(\tau))}{t - \tau} = 0\}
\end{aligned}$$

The first set is equal to $X^\Delta \cap U \subset \Delta$, where $X^\Delta := \{(x, x) : x \in X\}$. So, it is biholomorphic to X near 0. Moreover, the proper intersection of $(X \times X) \cap U$ with H_1 is transversal along $X^\Delta \cap U$. In fact, for sufficiently small $t \in K$, $t \neq 0$, the vector $[pt^{p-1}, \phi'(t), 0, ..., 0]$ belongs to the tangent space

$$T_{(pt^{p-1}, \phi'(t), pt^{p-1}, \phi'(t))}(X \times X)$$

to $X \times X$ at a nonsingular point $(pt^{p-1}, \phi'(t), pt^{p-1}, \phi'(t))$ and does not belong to H_1 (because $A_1^1 \neq 0$). Hence

$$Z_1^\Delta = X^\Delta \cap U$$

and

$$\mu(Z_1^\Delta; (0, 0)) = \mu(X^\Delta \cap U; (0, 0)) = \mu(X; 0) = p.$$

Now, we will analyse the second set. Since $A_1^1 \neq 0$, we may for simplicity put $A_1^1 = 1$. First we consider the analytic set

$$\Psi(t, \tau) = 0$$

in a neighbourhood of $0 \in \mathbb{C}^2_{(t,\tau)}$ where we put

$$\Psi(t, \tau) := (t^{p-1} + ... + \tau^{p-1}) + A_2^1 \frac{(\phi_2(t) - \phi_2(\tau))}{t - \tau} + ... + A_n^1 \frac{(\phi_n(t) - \phi_n(\tau))}{t - \tau}.$$

Since $\operatorname{ord} \phi > p$, this analytic set generates $(p-1)$ irreducible nonsingular germs at $0 \in \mathbb{C}^2_{(t,\tau)}$. So, shrinking U, we may assume that this analytic set is a sum of $(p-1)$ irreducible analytic sets V_i, $i = 1, ..., p-1$ and each of them has a parametrization

$$\Phi_i : K_i \to V_i, \quad 0 \in K_i \subset \mathbb{C},$$
$$\Phi_i(s) = (s, \psi_i(s)) = (s, \varepsilon^i s + ...)$$

where ε is a primitive root of unity of degree p. Since $\Psi(\Phi_i(s)) \equiv 0$, easy calculations give a more precise form of the $\Phi_i(s)$

$$\Phi_i(s) = (s, \varepsilon^i s + (a_2^i A_2^1 + ... + a_n^i A_n^1) s^{k_i} + ...),$$

for some $k_i > 1$ and $a_2^i, ..., a_n^i \in \mathbb{C}$ which do not all vanish.

The above considerations show that

$$(X \times X) \cap U \cap H_1 = (X^\Delta \cap U) \cup \tilde{V}_1 \cup ... \cup \tilde{V}_{p-1},$$

where

$$\tilde{V}_i := \{(t^p, \phi(t), \tau^p, \phi(\tau)) : (t, \tau) \in V_i\}, \quad i = 1, ..., p-1.$$

Similarly as above we prove that the proper intersection of $(X \times X) \cap U$ with H_1 is transversal along each $\tilde{V}_i$, $i = 1, ..., p-1$. Summing up, we obtain

$$Z_1 = (X^\Delta \cap U) + \tilde{V}_1 + ... + \tilde{V}_{p-1},$$
$$Z_1^\Delta = X^\Delta \cap U,$$
$$Z_1 - Z_1^\Delta = \tilde{V}_1 + ... + \tilde{V}_{p-1}.$$

Let us pass to the step 2 of the algorithm. We have to find

$$Z_2 = (Z_1 - Z_1^\Delta) \cdot H_2 = (\tilde{V}_1 + ... + \tilde{V}_{p-1}) \cdot H_2.$$

Since $\dim(Z_1 - Z_1^\Delta) = 1$ and $\dim(Z_1 - Z_1^\Delta) \cap H_2 = 0$, then shrinking U we have

$$(Z_1 - Z_1^\Delta) \cap H_2 = \{(0,0)\}.$$

Hence

$$Z_2 = \alpha\{(0,0)\},$$

where α is the sum of multiplicities of the proper isolated intersection of the hyperplane H_2 with the analytic curves $\tilde{V}_i$ at $(0,0)$ for $i = 1, ..., p-1$. Since each $\tilde{V}_i$ has a parametrization

$$\tilde{\Phi}_i(s) = (s^p, \phi(s), \psi(s)^p, \phi(\psi(s))),$$

there is

$$\alpha = \sum_{i=1}^{p-1} \operatorname{ord}(A_1^2(s^p - \psi_i(s)^p) + A_2^2(\phi_2(s) - \phi_2(\psi_i(s)) + ...).$$

Since

$$(X \times X) \cdot \mathcal{H} = (X^\Delta \cap U) + \alpha\{(0,0)\},$$

there is

$$i^{ext}(X,X;0) := \min_{\substack{(1,A_2^1,...,A_n^1)\\(A_1^2,A_2^2,...,A_n^2)}} \{(0,...,0,\mu(X;0),\alpha)\}.$$

So, we have to calculate

$$\min_{\substack{(1,A_2^1,...,A_n^1)\\(A_1^2,A_2^2,...,A_n^2)}} \sum_{i=1}^{p-1} \operatorname{ord}(A_1^2(s^p - \psi_i(s)^p) + A_2^2(\phi_2(s) - \phi_2(\psi_i(s)) + ...).$$

Notice that it is equal to

$$\min_{(1,A_2^1,...,A_n^1)} \sum_{i=1}^{p-1} \operatorname{ord}(s^p - \psi_i(s)^p, \phi_2(s) - \phi_2(\psi_i(s)), ...).$$

So, to conclude the proof it suffices to prove that for each $i \in \{1,...,p-1\}$ the equality

$$(6)\qquad \begin{aligned} &\min_{(1,A_2^1,...,A_n^1)} \operatorname{ord}(s^p - \psi_i(s)^p, \phi_2(s) - \phi_2(\psi_i(s)), ..., \phi_n(s) - \phi_n(\psi_i(s)))\\ &= \operatorname{ord}(\phi_2(s) - \phi_2(\varepsilon^i s), ..., \phi_n(s) - \phi_n(\varepsilon^i s)) \end{aligned}$$

holds. Then fix $i \in \{1,...,p-1\}$. Put

$$u := \operatorname{ord}(\phi_2(s) - \phi_2(\varepsilon^i s), ..., \phi_n(s) - \phi_n(\varepsilon^i s)).$$

It means that

$$\begin{aligned} u &:= \min_{j=2}^{n} u_j\\ u_j &:= \min\{r \in \mathbb{N} : r \in \operatorname{Supp}\phi_j,\ \varepsilon^{ir} \neq 1\}, \end{aligned}$$

where $\operatorname{Supp}\phi$ for a series $0 \neq \phi(s) = c_{n_1}s^{n_1} + c_{n_2}s^{n_2} + ...$, $c_{n_i} \neq 0$, denotes the set $\{n_1, n_2, ...\}$. We have

$$\begin{aligned} &\min_{(1,A_2^1,...,A_n^1)} \operatorname{ord}(s^p - \psi_i(s)^p, \phi_2(s) - \phi_2(\psi_i(s)), ..., \phi_n(s) - \phi_n(\psi_i(s)))\\ &\leqslant \min_{(1,A_2^1,...,A_n^1)} \operatorname{ord}(\phi_2(s) - \phi_2(\psi_i(s)), ..., \phi_n(s) - \phi_n(\psi_i(s)))\\ &= \min_{\substack{(1,A_2^1,...,A_n^1)\\2\leqslant j\leqslant n}} \operatorname{ord}(\phi_j(s) - \phi_j(\varepsilon^i s + (a_2^i A_2^1 + ... + a_n^i A_n^1)s^{k_i} + ...)). \end{aligned}$$

Notice that for a fixed $j \in \{2,...,n\}$:

(a) if $\varepsilon^{i \operatorname{ord}\phi_j} \neq 1$ then

$$\operatorname{ord}(\phi_j(s) - \phi_j(\varepsilon^i s + (a_2^i A_2^1 + ... + a_n^i A_n^1)s^{k_i} + ...)) = u_j,$$

(b) if $\varepsilon^{i\,\mathrm{ord}\,\phi_j} = 1$ then

$$\min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}(\phi_j(s) - \phi_j(\varepsilon^i s + (a_2^i A_2^1 + \dots + a_n^i A_n^1)s^{k_i} + \dots))$$
$$= \min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}(\gamma_j(a_2^i A_2^1 + \dots + a_n^i A_n^1)s^{\mathrm{ord}\,\phi_j - 1 + k_i} + \dots + \delta_j(1-\varepsilon^{iu_j})s^{u_j} + \dots)$$

for some constants $\gamma_j, \delta_j \neq 0$, which do not depend on $A_2^1, \dots, A_n^1$. Since $a_2^i A_2^1 + \dots + a_n^i A_n^1 \neq 0$ in $\mathbb{C}[A_2^1, \dots, A_n^1]$, this last expression is equal to

$$\min(\mathrm{ord}\,\phi_j - 1 + k_i, u_j) \leqslant u_j.$$

So, from these cases we obtain the inequality " $\leqslant$ " in formula (6). To prove the opposite inequality we assume to the contrary that there is a strict inequality "$<$" in formula (6). Then two cases may happen:

A.

$$\min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}(\phi_2(s) - \phi_2(\psi_i(s)), \dots, \phi_n(s) - \phi_n(\psi_i(s))) < u$$

Then from the last considerations there exists $j \in \{2, \dots, n\}$ such that

$$\min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}(\phi_2(s) - \phi_2(\psi_i(s)), \dots, \phi_n(s) - \phi_n(\psi_i(s)))$$
$$= \mathrm{ord}\,\phi_j - 1 + k_i \geqslant \mathrm{ord}\,\phi - 1 + k_i.$$

But we have

$$\Psi(s, \psi_i(s)) \equiv 0$$

Since $\psi_i(s)$ is not equal to s, the last identity is equivalent to the following one

$$\underbrace{(s^p - \psi_i(s)^p)}_{I(s)} + \underbrace{(A_2^1(\phi_2(s) - \phi_2(\psi_i(s)) + \dots)}_{II(s)} \equiv 0. \tag{7}$$

We have

$$\mathrm{ord}\,I(s) = p - 1 + k_i < \mathrm{ord}\,\phi - 1 + k_i,$$
$$\min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}\,II(s) \geqslant \mathrm{ord}\,\phi - 1 + k_i,$$

which gives a contradiction in this case.

B.

$$\min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}(s^p - \psi_i(s)^p)$$
$$< \min_{(1,A_2^1,\dots,A_n^1)} \mathrm{ord}(\phi_2(s) - \phi_2(\psi_i(s)), \dots, \phi_n(s) - \phi_n(\psi_i(s))).$$

But this is impossible by (7).

Summing up, we obtain

$$i^{ext}(X, X; 0) = \{(0, ..., 0, \mu(X; 0), \sum_{i=1}^{p-1} \operatorname{ord}(\phi(s) - \phi(\varepsilon^i s))\}.$$

Hence

$$i(X, X; 0) = \mu(X; 0) + \sum_{i=1}^{p-1} \operatorname{ord}(\phi(s) - \phi(\varepsilon^i s)).$$

2. Assume that the germ of X at 0 is reducible. Then formula (5) follows from Theorem 1.

This concludes the proof. □

4. Plane curves. Let X be an analytic curve in a neighbourhood $U \subset \mathbb{C}^2$ of the origin $P = (0, 0) \in X$, and let $f = 0$ be its (reduced) equation, where $f \in \mathbb{C}\{x, y\}$. We denote by $\mu_P(X)$ the Milnor number of X at P. Then we have

$$\mu_P(X) = \mu_P(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}),$$

where $\mu_P(g, h)$ stands for the multiplicity of a holomorphic mapping (g, h) at P. Now, we recall two well known formulas (see e.g. [**14**]):

1. the Teissier formula: if $f(0, y) \neq 0$ then

$$\mu_P(f, \frac{\partial f}{\partial y}) = \mu_P(X) + \operatorname{ord} f(0, y) - 1,$$

2. the relation between the Milnor number and the virtual number of double points $\delta_P(X)$ of X at P: if $P \in X$ and $r_P(X)$ is the number of irreducible germs of X at P, then

$$2\delta_P(X) = \mu_P(X) + r_P(X) - 1.$$

PROPOSITION 5. *If $P = (0, 0)$ is a singular point of X and the axis $x = 0$ is not tangent to the curve X at P then*

$$i(X \bullet X; P) = \mu_P(f, \frac{\partial f}{\partial y}). \tag{8}$$

PROOF. First assume that the germ of X at P is irreducible and that the tangent line to the curve X at P is the axis $y = 0$. Then in a neighbourhood $U' \subset U$ of P the curve X has a parametrization

$$\mathbb{C} \supset K \ni t \mapsto (t^p, \phi(t)) \in X, \quad \operatorname{ord} \phi > p.$$

(K is a neighbourhood of 0 in $\mathbb{C}$). Then, from the main theorem and the Puiseux Theorem, there follows

$$i(X \bullet X; P) = \sum_{i=1}^{p-1} \operatorname{ord}(\phi(t) - \phi(\varepsilon^i t)) = \mu_P(f, \frac{\partial f}{\partial y})$$

where ε is a primitive root of unity of degree p. If the germ of X at P is still irreducible and the tangent line to the curve X at P is not the axis $y = 0$ (and also not the axis $x = 0$ by assumption) then by a linear change of variables L in $\mathbb{C}^2$ we obtain from the above case that for $\tilde{f}(\tilde{x}, \tilde{y}) := f \circ L(\tilde{x}, \tilde{y})$ we have

$$i(X \bullet X; P) = \mu_P(\tilde{f}, \frac{\partial \tilde{f}}{\partial \tilde{y}}).$$

But from the Teissier formula we obtain that for such a linear change of variables we have

$$\mu_P(\tilde{f}, \frac{\partial \tilde{f}}{\partial \tilde{y}}) = \mu_P(f, \frac{\partial f}{\partial y}).$$

Assume now that the germ of X at P is reducible. Let $(X)_P = (X_1)_P \cup \ldots \cup (X_k)_P$ be the decomposition of the germ $(X)_P$ into irreducible germs. Then we also have $f = f_1 \ldots f_k$ in a neighbourhood $U' \subset U$ of P, where each f_i is holomorphic and describes X_i in U'. Then from the additivity of the intersection cycle (Theorem 1), cases considered above and properties of the multiplicity of mappings we have

$$\begin{aligned} & i(X \bullet X; P) \\ &= \sum_{i=1}^{k} i(X_i \bullet X_i; P) + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} i(X_i \bullet X_j; P) \\ &= \sum_{i=1}^{k} \mu_P(f_i, \frac{\partial f_i}{\partial y}) + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \mu_P(f_i, f_j) \\ &= \sum_{i=1}^{k} \mu_P(f_i, \frac{\partial f}{\partial y}) = \mu_P(f, \frac{\partial f}{\partial y}). \end{aligned}$$

□

Corollary 6. *Under the assumptions of Proposition 5 we have*

$$\begin{aligned} i(X \bullet X; P) &= \mu_P(X) + \mu(X; P) - 1 \\ &= 2\delta_P(X) - r_P(X) + \mu(X; P). \end{aligned} \tag{9}$$

Proof. It follows from the fact that from the assumption on the tangent line we have

$$\mu(X; P) = \operatorname{ord} f(0, y).$$

□

Remark 7. Notice that in the case that X is an algebraic plane curve in the projective plane $\mathbb{P}^2$ over $\mathbb{C}$ the coefficients $i(X \bullet X; P)$ in the intersection cycle $X \bullet X$ are the same as coefficients of singular points in the Stückrad–Vogel intersection cycle $v(X,X)$. Namely, formula (8) is given in **[9]**, Ch. 3, S. 2(2) and formulas (9) in **[8]**, Example 2.5.16).

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