UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIX
2001

INVARIANTS OF SINGULARITIES OF POLYNOMIALS IN TWO COMPLEX VARIABLES AND THE NEWTON DIAGRAMS

by Mateusz Masternak

Abstract. For any polynomial mapping $f : \mathbf{C}^2 \to \mathbf{C}$ with a finite number of critical points we consider the Milnor number $\mu(f)$, the jump of the Milnor numbers at infinity $\lambda(f)$, the number of branches at infinity $r_\infty(f)$ and the genus $\gamma(f)$ of the generic fiber $f^{-1}(t_{gen})$. The aim of this note is to estimate these invariants of f in terms of the Newton diagram $\Delta_\infty(f)$.

1. Introduction. Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with a finite number of critical points. We define the *global Milnor number* $\mu(f)$ by putting

$$\mu(f) := \sum_{P \in \mathbf{C}^2} \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y} \right)_P$$

where the symbol $(\cdot, \cdot)_P$ denotes the multiplicity of intersection at the point $P \in \mathbf{C}^2$. Note that $\mu(f) < +\infty$.

Let $C^t \subset \mathbf{P}^2(\mathbf{C})$ be the projective closure of the fiber $f^{-1}(t)$ where $t \in \mathbf{C}$. If $d = \deg f$ and $F(X, Y, Z)$ is the homogeneous form corresponding to $f = f(X, Y) = \sum_{\alpha+\beta \le d} c_{\alpha\beta} X^\alpha Y^\beta$, then C^t is given by the equation $F(X, Y, Z) - tZ^d = 0$. Let $L_\infty \subset \mathbf{P}^2(\mathbf{C})$ be the line at infinity given by $Z = 0$ and let $(C^t)_\infty = C^t \cap L_\infty$. Obviously, $(C^t)_\infty = (C^0)_\infty$. In the sequel we write $C = C^0$ and $C_\infty = (C^0)_\infty$. If $f^+(X, Y) = \sum_{\alpha+\beta=d} c_{\alpha\beta} X^\alpha Y^\beta$ is the *leading part* of the polynomial f, then

$$C_\infty = \{(x : y : z) \in \mathbf{P}^2(\mathbf{C}) : \ z = 0 \text{ and } f^+(x, y) = 0\} .$$

For every $P \in C^t$ we denote by $\mu_P^t = \mu_P^t(C^t)$ the *Milnor number* of the curve C^t at the point P. There exist numbers $\mu_P^{gen} \ge 0$ ($P \in C_\infty$) such that

$\mu_P^t \geq \mu_P^{gen}$ for all $t \in \mathbf{C}$. Moreover, $\mu_P^t = \mu_P^{gen}$ for almost all $t \in \mathbf{C}$. This fact is due to Broughton [**1**] (see also [**4**] for a simple direct proof). Hence the set

$$\Lambda(f) = \{t \in \mathbf{C} : \ \mu_P^t > \mu_P^{gen} \text{ for some } P \in C_\infty\}$$

is finite and the numbers

$$\lambda^t(f) := \sum_{P \in C_\infty} (\mu_P^t - \mu_P^{gen}) \text{ and } \lambda(f) := \sum_{t \in \mathbf{C}} \lambda^t(f)$$

are well defined. At any point $P \in C$ we consider the number $r_P(C)$ of branches of the curve C centered at P . We define the *number* $r_\infty(C)$ *of branches at infinity* of the curve C by putting

$$r_\infty(C) := \sum_{P \in C_\infty} r_P(C) \ .$$

It is known (see [**4**]) that the function

$$\mathbf{C} \setminus \Lambda(f) \ni t \to r_\infty(C^t) \in \mathbf{N}$$

is constant. Let $r_\infty(f) := r_\infty(C^t)$ for $t \in \mathbf{C} \setminus \Lambda(f)$. We call $r_\infty(f)$ the *generic number of branches at infinity.*

Let $\operatorname{supp} f = \{(\alpha, \beta) \in \mathbf{N}^2 : \ c_{\alpha\beta} \neq 0\}$. The Newton diagram at infinity $\Delta_\infty(f)$ is the convex hull of $\{(0,0)\} \cup \operatorname{supp} f$. For any f we define its global Newton number $\mu(\Delta_\infty(f))$ by putting

$$\mu(\Delta_\infty(f)) := 2 \,\text{Area}\, \Delta_\infty(f) - A - B + 1$$

where $A = \max\{\alpha \in \mathbf{N} : (\alpha, 0) \in \Delta_\infty(f)\}$ and $B = \max\{\beta \in \mathbf{N} : (0, \beta) \in \Delta_\infty(f)\}$. The *Newton polygon at infinity* $\partial\Delta_\infty(f)$ is the set of the faces of $\Delta_\infty(f)$ not included in the coordinate axes. We define the number

$$r(\Delta_\infty(f)) := \sum_{S \in \partial\Delta_\infty(f)} r(S)$$

where $r(S) =$ (number of integer points lying on the segment S) $- 1$. Hence the integer points divide S into $r(S)$ segments.

For any segment $S \in \partial\Delta_\infty(f)$ we let $\text{in}(f,S)(X,Y) =$ the sum of all monomials $c_{\alpha\beta}X^\alpha Y^\beta$ such that $(\alpha,\beta) \in S$. The polynomial f is *nondegenerate on* $S \in \partial\Delta_\infty(f)$ if the system of equations

$$\text{in}(f,S)(X,Y) = \frac{\partial}{\partial X}\text{in}(f,S)(X,Y) = \frac{\partial}{\partial Y}\text{in}(f,S)(X,Y) = 0$$

has no solution in $\mathbf{C}^* \times \mathbf{C}^*$ where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Our main result is the following:

Theorem 1.1.
Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial such that $\mu(f) < +\infty$. Suppose that the diagram $\Delta_\infty(f)$ has a nonempty interior. Then

(1) $\mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq r(\Delta_\infty(f)) - r_\infty(f) \geq 0$,
(2) *the equalities hold if f is nondegenerate on each segment $S \in \partial\Delta_\infty(f)$ not included in a line passing through the origin.*

We give the proof in Section 3. Our theorem implies the following estimation due to Cassou-Noguès:

Corollary 1.2 ([**2**], Theorem 10).
Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial such that $\mu(f) < +\infty$. Then
(1) $\mu(f) + \lambda(f) \leq \mu(\Delta_\infty(f))$,
(2) *the equality holds if f is nondegenerate on each segment $S \in \partial\Delta_\infty(f)$ not included in a line passing through the origin.*

Proof. If $\Delta_\infty(f)$ does not have interior points then $\deg f \leq 2$ (otherwise $\mu(f) = \infty$) and the result is easily seen. Therefore we can assume that $\Delta_\infty(f)$ has a nonempty interior and (1.2) follows from (1.1).

To give another application of our result let us put $\gamma(f) =$ the genus of the Riemann surface corresponding to the generic fiber $f^{-1}(t_{gen})$. Let $\gamma(\Delta_\infty(f))$ be the number of integer points lying inside $\Delta_\infty(f)$. □

Corollary 1.3. *With the assumptions given above we have*
(1) $\gamma(f) \leq \gamma(\Delta_\infty(f))$,
(2) *the equality holds if f is nondegenerate on each segment $S \in \partial\Delta_\infty(f)$ not included in a line passing through the origin.*

Proof. We may assume that $\Delta_\infty(f)$ has interior points. By Abhyankar-Sathaye's formula (see [**3**], Formula 4.4) we have

$$2\,\gamma(f) = \mu(f) + \lambda(f) - r_\infty(f) + 1.$$

On the other hand, by Pick's formula we get

$$2\,\gamma(\Delta_\infty(f)) = \mu(\Delta_\infty(f)) - r(\Delta_\infty(f)) + 1$$

and we obtain 1.3 directly from the main result. □

2. The Newton diagrams. Let $f(X,Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbf{C}[X,Y]$ be a nonzero polynomial of degree d. We say that the polynomial f is *quasi–convenient* if $c_{\alpha 0} \neq 0$ and $c_{0\beta} \neq 0$ for some integers $\alpha, \beta \geq 0$. If the above condition holds for some positive α, β, then f is called *convenient* polynomial. Let $\operatorname{supp} f = \{(\alpha,\beta) \in \mathbf{N}^2 : \ c_{\alpha\beta} \neq 0\}$. We define

$$\Delta(f) := \operatorname{convex}(\operatorname{supp} f) \text{ and } \Delta_\infty(f) := \operatorname{convex}(\{(0,0)\} \cup \operatorname{supp} f).$$

The polygons $\Delta(f)$ and $\Delta_\infty(f)$ are called respectively *Newton diagram* and *Newton diagram at infinity* of the polynomial f. For every quasi–convenient polynomial we consider additionaly its *Newton diagram at zero.* This polygon is the closure of the set $\Delta_\infty(f) \setminus \Delta(f)$. We denote it by $\Delta_0(f)$. If $a, b > 0$

are smallest integer numbers such that $(a,0)$, $(0,b) \in \operatorname{supp} f$, then $\Delta_0(f)$ is the polygon bounded by the segments joining the points $(0,0)$ with $(a,0)$ and $(0,0)$ with $(0,b)$ and by the faces of the diagram $\Delta(f)$ that separate it from the origin.

Obviously, $\Delta_\infty(f) = \Delta_0(f) \cup \Delta(f)$. If $(0,0) \in \operatorname{supp} f$, then $\Delta_\infty(f) = \Delta(f)$ and $\Delta_0(f) = \emptyset$. Similarly as in the definition of $\partial\Delta_\infty(f)$, we define for every quasi–convenient polynomial f its *Newton polygon at zero* $\partial\Delta_0(f)$ as the set of the faces of $\Delta_0(f)$ not included in the coordinate axes. By $\partial\Delta(f)$ we denote the set of all faces of $\Delta(f)$ and we call it *Newton polygon* of the polynomial f. If f is quasi–convenient, then $\partial\Delta_0(f)$, $\partial\Delta_\infty(f) \subset \partial\Delta(f)$. But if $f(0,0) \neq 0$, then $\partial\Delta_0(f) = \emptyset$ and $\partial\Delta_\infty(f) = \partial\Delta(f)$.

Newton Diagrams in affine systems of coordinates. If $U = (\vec{u}; \vec{e}_1, \vec{e}_2)$ is an affine system of coordinates of the real plane $\mathbf{R}^2$ (i.e. $\vec{u}$, $\vec{e}_1$, $\vec{e}_2 \in \mathbf{R}^2$ and $\vec{e}_1, \vec{e}_2$ are lineary independent), then we define the *support of the polynomial* $f(X,Y) \in \mathbf{C}[X,Y]$ *in the system* U: $\operatorname{supp}^U f := \{\vec{u} + \alpha\vec{e}_1 + \beta\vec{e}_2 : (\alpha,\beta) \in \operatorname{supp} f\}$ and *Newton diagram of the polynomial* $f(X,Y)$ *in the system* U: $\Delta^U(f) := \operatorname{convex}(\operatorname{supp}^U f)$. Similarly to the standard case we define $\Delta^U_\infty(f) := \operatorname{convex}(\{\vec{u}\} \cup \operatorname{supp}^U f)$ and if f is quasi–convenient we put $\Delta^U_0(f) :=$ closure of $\left(\Delta^U_\infty(f) \setminus \Delta^U(f)\right)$. If $f(0,0) \neq 0$, then $(0,0) \in \operatorname{supp} f$, hence $\vec{u} \in \operatorname{supp}^U f$ and then $\Delta^U_\infty(f) = \Delta^U(f)$ and $\Delta^U_0(f) = \emptyset$. Analogously to the standard case we define the polygons $\partial\Delta^U_0(f)$, $\partial\Delta^U(f)$ and $\partial\Delta^U_\infty(f)$ of the polynomial f in the system U. If

$$f(X,Y) = \sum_{(\alpha,\beta)\in \operatorname{supp} f} c_{\alpha\beta} X^\alpha Y^\beta \in \mathbf{C}[X,Y]$$

and $S \in \partial\Delta^U(f)$, then $\operatorname{in}^U(f,S)(X,Y)$ is the sum of all monomials $c_{\alpha\beta}X^\alpha Y^\beta$, such that $\vec{u} + \alpha\vec{e}_1 + \beta\vec{e}_2 \in S$. Let $(\alpha,\beta)_U := \vec{u} + \alpha\vec{e}_1 + \beta\vec{e}_2$. Write

$$\operatorname{in}^U(f,S)(X,Y) = \sum_{(\alpha,\beta)_U \in S} c_{\alpha\beta} X^\alpha Y^\beta.$$

If $U = (\vec{0}; \vec{\imath}, \vec{\jmath})$ where $\vec{\imath} = [1,0]$, $\vec{\jmath} = [0,1]$, then the notions introduced above correspond to the standard constructions presented above, i.e. $\Delta^U(f) = \Delta(f)$, $\Delta^U_\infty(f) = \Delta_\infty(f)$, $\operatorname{in}^U(f,S)(X,Y) = \operatorname{in}(f,S)(X,Y)$, etc.

Let $F(X,Y,Z) = Z^d f(X/Z, Y/Z)$, where $d = \deg f > 0$, be a homogenization of a polynomial $f(X,Y)$. The projective curve $F(X,Y,Z) = 0$ is the projective closure of the affine curve $f(X,Y) = 0$. It is natural to consider the affine curves $F(1,Y,Z) = 0$ and $F(X,1,Z) = 0$. If $f(X,0)f(0,Y) \neq 0$, then $F(X,Y,Z)$ is also the homogenization of $F(1,Y,Z)$ and $F(X,1,Z)$. The notion of the Newton diagram in an affine system of coordinates is useful while comparing the Newton diagrams of the polynomials $f(X,Y) = F(X,Y,1)$, $F(X,1,Z)$ and $F(1,Y,Z)$.

Lemma 2.1 (Main Lemma).
Let $U = (\vec{0};\vec{\imath},\vec{\jmath})$, $V = (d\vec{\imath};\vec{\jmath}-\vec{\imath},-\vec{\imath})$, $W = (d\vec{\jmath};\vec{\imath}-\vec{\jmath},-\vec{\jmath})$. *Then*

$$\operatorname{supp}^U F(X,Y,1) = \operatorname{supp}^V F(1,Y,Z) = \operatorname{supp}^W F(X,1,Z).$$

Proof. We prove the first equality. Denote $N = \operatorname{supp} f$. Hence if $f(X,Y) = \sum_{(\alpha,\beta)\in N} c_{\alpha\beta}X^\alpha Y^\beta$, then $F(X,Y,Z) = \sum_{(\alpha,\beta)\in N} c_{\alpha\beta}X^\alpha Y^\beta Z^{d-\alpha-\beta}$ and $F(1,Y,Z) = \sum_{(d-\beta-\gamma,\beta)\in N} c_{d-\beta-\gamma,\beta}Y^\beta Z^\gamma$. We have

$$\begin{aligned}
&\operatorname{supp}^V F(1,Y,Z) = \\
&= \{\beta(\vec{\jmath}-\vec{\imath}) + \gamma(-\vec{\imath}) + d\vec{\imath} : (\beta,\gamma) \in \operatorname{supp} F(1,Y,Z)\} = \\
&= \{\beta(\vec{\jmath}-\vec{\imath}) + \gamma(-\vec{\imath}) + d\vec{\imath} : \gamma = d-\alpha-\beta \text{ and } (\alpha,\beta)\in N\} = \\
&= \{(d-\beta-\gamma)\vec{\imath} + \beta\vec{\jmath} : \gamma = d-\alpha-\beta \text{ and } (\alpha,\beta)\in N\} = \\
&= \{\alpha\vec{\imath} + \beta\vec{\jmath} : (\alpha,\beta)\in N\} = N = \operatorname{supp}^U F(X,Y,1).
\end{aligned}$$

□

In the same way we prove that $\operatorname{supp}^U F(X,Y,1) = \operatorname{supp}^W F(X,1,Z)$.

Directly from the above lemma we get the following corollaries:

Corollary 2.2.

(1) $\Delta(f) = \Delta^U\big(F(X,Y,1)\big) = \Delta^V\big(F(1,Y,Z)\big) = \Delta^W\big(F(X,1,Z)\big)$.
(2) $\partial\Delta(f) = \partial\Delta^U(F(X,Y,1) = \partial\Delta^V(F(1,Y,Z) = \partial\Delta^W(F(X,1,Z)$.
(3) *If f is a quasi–convenient polynomial, then the polynomials $F(1,Y,Z)$ and $F(X,1,Z)$ are also quasi–convenient and the triangle with vertices at $(0,0)$, $(\deg f,0)$, $(0,\deg f)$ is the union of the polygons $\Delta_\infty(f)$, $\Delta_I(f)$ and $\Delta_{II}(f)$, whose interiors are disjoint, where*

$$\Delta_I(f) := \Delta_0^V(F(1,Y,Z)), \qquad \Delta_{II}(f) := \Delta_0^W(F(X,1,Z)).$$

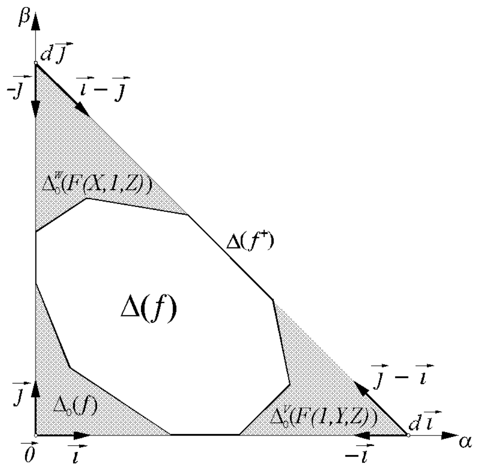
Suppose that the polynomial $f(X,Y)$ is quasi–convenient. We denote

$$\partial\Delta_I(f) := \partial\Delta_0^V(F(1,Y,Z)), \qquad \partial\Delta_{II}(f) := \partial\Delta_0^W(F(X,1,Z)).$$

The leading part $f^+(X,Y)$ is a homogeneous form and all points of its support lie on the line $\alpha+\beta = \deg f$. Hence the diagram $\Delta(f^+)$ is a segment or a point. Therefore the polygon $\partial\Delta(f^+)$ is the empty set or a one-element set. The segment $\Delta(f^+)$ is called the *main segment of the polynomial f*.

Corollary 2.3. *If f a is quasi–convenient polynomial, then*

$$\partial\Delta_\infty(f) = \partial\Delta_I(f) \cup \partial\Delta_{II}(f) \cup \partial\Delta(f^+).$$



Remark 2.4. The description of the Newton diagram at infinity by means of local diagrams was given by numerous authors [**2**], [**7**], [**8**]. Our version of this description allows us to give a simple proof of the main result.

Nondegeneracy. A nonzero polynomial f is nondegenerate on $S \in \partial\Delta(f)$ if the system of equations

$$\operatorname{in}(f,S)(X,Y) = \frac{\partial}{\partial X}\operatorname{in}(f,S)(X,Y) = \frac{\partial}{\partial Y}\operatorname{in}(f,S)(X,Y) = 0$$

has no solution in $\mathbf{C}^* \times \mathbf{C}^*$. We say that a quasi–convenient polynomial $f = f(X,Y)$ is *nondegenerate at zero (at infinity)* if it is nondegenerate on each segment $S \in \partial\Delta_0(f)$ ($S \in \partial\Delta_\infty(f)$). The introduced notions of nondegeneracy at zero and at infinity can be defined using the Newton diagram constructed at any affine system $U = (\vec{u}; \vec{e}_1, \vec{e}_2)$. Instead of the notions $\operatorname{in}(f,S)$, $\partial\Delta_0(f)$, $\partial\Delta_\infty(f)$ we consider their counterparts $\operatorname{in}^U(f,S)$, $\partial\Delta_0^U(f)$, $\partial\Delta_\infty^U(f)$). The nondegeneracy at zero (at infinity) does not depend on the choice of the system U because the diagram $\Delta^U(f)$ is the image of the diagram $\Delta(f)$ by the affine transformation of the real plane:

$$\mathbf{R}^2 \ni (\alpha,\beta) \to (\alpha,\beta)_U := \vec{u} + \alpha\vec{e}_1 + \beta\vec{e}_2 \in \mathbf{R}^2.$$

Proposition 2.5. *Let $f(X,Y) \in \mathbf{C}[X,Y]$ be a quasi–convenient polynomial of degree $d > 0$ and let $F(X,Y,Z)$ be its homogenization. Then $f(X,Y)$ is nondegenerate at infinity if and only if*

(1) *the polynomials $F(1,Y,Z)$, $F(X,1,Z)$ are nondegenerate at zero and*
(2) *the leading part $f^+(X,Y)$ is a homogeneous form without multiple factors of the form $\xi X - \eta Y$ where $\xi\eta \neq 0$.*

Proof. Let $U = (\vec{0}; \vec{\imath}, \vec{\jmath})$, $V = (d\vec{\imath}; \vec{\jmath} - \vec{\imath}, -\vec{\imath})$, $W = (d\vec{\jmath}; \vec{\imath} - \vec{\jmath}, -\vec{\jmath})$ and let $S \in \partial\Delta(f)$. We may consider the nondegeneracy of the polynomials $f(X,Y) =$

$F(X,Y,1)$, $F(1,Y,Z)$ and $F(X,1,Z)$ on S respectively in the systems U, V and W (see Corollary 3.2 (2)). By direct calculation we have

(a) $\mathrm{in}^V\big(F(1,Y,Z),S\big)(Y,Z) = Z^d \mathrm{in}(f,S)(1/Z,Y/Z)$,
(b) $\mathrm{in}^W\big(F(X,1,Z),S\big)(X,Z) = Z^d \mathrm{in}(f,S)(X/Z,1/Z)$.

We show that the following conditions are equivalent:

(1) $f(X,Y)$ is nondegenerate on S,
(2) $F(1,Y,Z)$ is nondegenerate on S,
(3) $F(X,1,Z)$ is nondegenerate on S.

We prove the equivalence (1) $\Leftrightarrow$ (2). The proof of (1) $\Leftrightarrow$ (3) runs analogously. We denote $g(X,Y) = \mathrm{in}(f,S)(X,Y)$ and $h(Y,Z) = \mathrm{in}^V(F(1,Y,Z),S)(Y,Z)$. We have to show that the system $\frac{\partial g}{\partial X}(X,Y) = \frac{\partial g}{\partial Y}(X,Y) = g(X,Y) = 0$ has a solution in $\mathbf{C}^* \times \mathbf{C}^*$ if and only if the system $\frac{\partial h}{\partial Y}(Y,Z) = \frac{\partial h}{\partial Z}(Y,Z) = h(Y,Z) = 0$ has a solution in $\mathbf{C}^* \times \mathbf{C}^*$. From (a) we get

$$h(Y,Z) = Z^d g\left(\frac{1}{Z},\frac{Y}{Z}\right),\ Z\frac{\partial h}{\partial Y}(Y,Z) = Z^d\frac{\partial g}{\partial Y}\left(\frac{1}{Z},\frac{Y}{Z}\right) \text{ and}$$

$$Z^2\frac{\partial h}{\partial Z}(Y,Z) = Z^d\left(dZg\left(\frac{1}{Z},\frac{Y}{Z}\right) - \frac{\partial g}{\partial X}\left(\frac{1}{Z},\frac{Y}{Z}\right) - Y\frac{\partial g}{\partial Y}\left(\frac{1}{Z},\frac{Y}{Z}\right)\right).$$

These equalities imply the above equivalence.

By Corollary 2.3 we have

$$\partial\Delta(f) = \partial\Delta_\infty(f) = \partial\Delta_I(f) \cup \partial\Delta_{II}(f) \cup \partial\Delta(f^+).$$

Note that f is nondegenerate on each segment $S \in \partial\Delta_I(f)$ ($S \in \partial\Delta_{II}(f)$) if and only if the polynomial $F(1,Y,Z)$ ($F(X,1,Z)$) is nondegenerate at zero. Let $\phi = \phi(X,Y)$ be a homogneous form of positive degree. It is easy to check that the sytem $\phi = \frac{\partial\phi}{\partial X} = \frac{\partial\phi}{\partial Y} = 0$ has no solution in $\mathbf{C}^* \times \mathbf{C}^*$ if and only if $\phi(X,Y) = X^m Y^n \phi_1(X,Y)$ for some $m,n \in \mathbf{N}$ where ϕ_1 is a reduced homogeneous form such that $\phi_1(X,0)\phi_1(0,Y) \neq 0$. Hence the polynomial f is nondegenerate on the main segment $S = \Delta(f^+)$ if and only if the homogeneous form $f^+(X,Y) = \mathrm{in}(f,S)(X,Y)$ has only single factors of the form $\xi X - \eta Y$ where $\xi\eta \neq 0$. The above observations complete the proof of our proposition. □

3. The Milnor numbers and number of branches. Let $f(X,Y) \in \mathbf{C}[X,Y]$ be a convenient polynomial without constant term and let $\Delta_0(f)$ be its Newton diagram at zero. We define the numbers

$$\mu(\Delta_0(f)) = 2\,\mathrm{Area}\ \Delta_0(f) - \mathrm{ord}\ f(X,0) - \mathrm{ord}\ f(0,Y) + 1;$$

$$r(\Delta_0(f)) := \sum_{S\in\partial\Delta_0(f)} r(S).$$

We denote by $r_0(f)$ the number of branches of the curve $f(X,Y)=0$ at zero. Let us recall the following:

Theorem 3.1 ([**9**], Theorem 1.2).
If $f(X,Y)\in \mathbf{C}[X,Y]$ is a convenient polynomial without constant term, then
(1) $\mu_0(f)-\mu(\Delta_0(f))\geq r(\Delta_0(f))-r_0(f)\geq 0$,
(2) *the equality holds if f is nondegenerate at zero.*

Theorem 3.2 (Cassou-Noguès' formula, [**2**], Proposition 12).
Let $c=\#C_\infty$. If $\mu(f)<+\infty$, then

$$\sum_{P\in C_\infty}\mu_P^{\mathrm{gen}}-c+\mu(f)+\lambda(f)-1=d(d-3).$$

A proof of the above formula without using Eisenbud-Neumann diagrams is given in [**3**].

Proof of the main result. Without loss of generality we can assume that the polynomial f is quasi-convenient with the generic fiber $f^{-1}(0)$. Otherwise we consider the polynomial $f^t=f-t$, where $t\in\mathbf{C}\setminus\Lambda(f)$ is such that $f^t(0,0)\neq 0$. Then
(a) $\mu(\Delta_\infty(f))=\mu(\Delta_\infty(f^t))$, $r(\Delta_\infty(f)=r(\Delta_\infty(f^t)$,
(b) $\mu(f)=\mu(f^t)$, $\quad\lambda(f)=\lambda(f^t)$ and $r_\infty(f)=r_\infty(f^t)$.
Moreover, if f satisfies the assumption of the second part of our theorem then we can choose $t\in\mathbf{C}\setminus\Lambda(f)$ such that f^t is nondegenerate at infinity.

Therefore, it is enough to check our theorem for a quasi-convenient polynomial f such that $0\notin\Lambda(f)$. Moreover, in the proof of (2) we may assume that f is nondegenerate at infinity.

Let $P_1=(1{:}0{:}0)$, $P_2=(0{:}1{:}0)\in\mathbf{P}^2$. We have the following cases:
(i) $P_1\in C_\infty$, $P_2\in C_\infty$
(ii) $P_1\in C_\infty$, $P_2\notin C_\infty$
(iii) $P_1\notin C_\infty$, $P_2\in C_\infty$
(iv) $P_1\notin C_\infty$, $P_2\notin C_\infty$

We give the proof in the case (i). In other cases the proof runs analogously. We prove the both parts of our therem paralelly. In the case under consideration $f^+(X,Y)=aX^pY^{d-p}+\cdots+bX^{d-q}Y^q$ where a, $b\in\mathbf{C}^*$ and p, q are integers such that $p,q>0$, $p+q\leq d$. Hence $c-2\leq d-p-q$. It is easily seen that $c=d-p-q+2$ if and only if the polynomial f is nondegenerate on the main segment $\Delta(f^+)$.

Let $A=\deg f(X,0)$ and $B=\deg f(0,Y)$. Let $F(X,Y,Z)$ be the homogeneous form corresponding to the polynomial $f(X,Y)$. Note that

$$\operatorname{Area}\Delta_0(F(1,Y,Z))=\operatorname{Area}\Delta_0^V(F(1,Y,Z))$$

and

$$\text{Area}\,\Delta_0(F(X,1,Z)) = \text{Area}\,\Delta_0^W(F(X,1,Z))$$

where $V = (d\vec{\imath}; \vec{\jmath} - \vec{\imath}, -\vec{\imath})$ and $W = (d\vec{\jmath}; \vec{\imath} - \vec{\jmath}, -\vec{\jmath})$. Hence

$$\mu(\Delta_0((F(1,Y,Z))) = 2\,\text{Area}\Delta_I(f) - (d-A) - q + 1$$

and

$$\mu(\Delta_0((F(X,1,Z))) = 2\,\text{Area}\Delta_{II}(f) - (d-B) - p + 1.$$

Recall that $\mu(\Delta_\infty(f)) = 2\,\text{Area}\Delta_\infty(f) - A - B + 1$. Therefore, by Corollary 2.2 (3) we get

$$(*) \qquad \mu(\Delta_0(F(1,Y,Z)) + \mu(\Delta_0(F(X,1,Z))) + \mu(\Delta_\infty(f)) =$$
$$= d(d-3) + d - p - q + 3 \geq d(d-3) + c + 1$$

and the equality holds if and only if the polynomial f is nondegenerate on the main segment. In the case under consideration the polynomials $F(1,Y,Z)$ and $F(X,1,Z)$ are convenient without constant term. By Theorem 3.1 we have

$$\mu_{P_1}^{gen} = \mu_{P_1}^0 = \mu_0(F(1,Y,Z)) \geq \mu(\Delta_0(F(1,Y,Z)))$$

and

$$\mu_{P_2}^{gen} = \mu_{P_2}^0 = \mu_0(F(X,1,Z)) \geq \mu(\Delta_0(F(X,1,Z))).$$

and the equalities hold if the polynomial f is nondegenerate on each segment $S \in \partial\Delta_I(f^t) \cup \partial\Delta_{II}(f^t)$ (see Corollary 2.3, Proposition 2.5 and Theorem 3.1 (2)). Using the above estimate, formula $(*)$ and Cassou-Noguès' formula we get

$$(**) \qquad \mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq [\mu_{P_1}^{gen} - \mu(\Delta_0(F(1,Y,Z)))] +$$
$$[\mu_{P_2}^{gen} - \mu(\Delta_0(F(X,1,Z)))] + \sum_{P \in C_\infty \setminus \{P_1,P_2\}} \mu_P^{\text{gen}}.$$

Applying Theorem 3.1 and the Main Lemma we get

$$\mu_{P_1}^{gen} - \mu(\Delta_0(F(1,Y,Z))) \geq \sum_{S \in \partial\Delta_I(f)} r(S) - r_{P_1}(C) \geq 0$$

and

$$\mu_{P_2}^{gen} - \mu(\Delta_0(F(X,1,Z))) \geq \sum_{S \in \partial\Delta_{II}(f)} r(S) - r_{P_2}(C) \geq 0.$$

On the other hand we have $\mu_P^{\text{gen}} = \mu_P^0 \geq \text{ord}\,_P F - 1$ for each $P \in C_\infty$. Thus

$$\sum_{P \in C_\infty \setminus \{P_1,P_2\}} \mu_P^{\text{gen}} \geq \sum_{P \in C_\infty \setminus \{P_1,P_2\}} (\text{ord}\,_P F - 1) = (d-p-q) - (c-2) \geq$$

$$r(\Delta(f^+)) - (c-2) \geq r(\Delta(f^+)) - \sum_{P \in C_\infty \setminus \{P_1, P_2\}} r_P(C).$$

It is clear that the equalities above we get if f is nondegenerate on the main segment. The above estimate and the inequality $(**)$ give

$$\mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq$$
$$\sum_{S \in \partial\Delta_I(f)} r(S) + \sum_{S \in \partial\Delta_{II}(f)} r(S) + r(\Delta(f^+)) - \sum_{P \in C_\infty} r_P(C).$$

The diagram $\Delta_\infty(f)$ has a nonempty interior. Hence by Corollary 2.3 we get

$$\mu(\Delta_\infty(f)) - (\mu(f) + \lambda(f)) \geq r(\Delta_\infty(f)) - \sum_{P \in C_\infty} r_P(C)\,.$$

This completes the proof of (1). To proof of (2) we use Proposition 2.5. $\square$

References

1. Broughton S.A., *Milnor numbers and the topology of polynomial hypersufaces,* Invent. Math. **92** (1988), 217–241.
2. Cassou-Noguès P., *Sur la généralisation d'un théorème de Kouchnirenko,* Compositio Math. **103** (1996), 95–121.
3. Gwoździewicz J., Płoski A., *Formulae for the singularities at infinity of plane algebraic curves,* IMUJ preprint 2000/22, Kraków (2000).
4. Krasiński T., *The level sets of polynomials in two variables and the Jacobian conjecture,* Acta Univ. Lodz. Folia Math., (1991) (in Polish).
5. Khovanski A.G., *Nevton Polyhedra and the genus of complete intersections,* Functional Anal. Appl. **12** (1978), 38–46.
6. Kouchnirenko A.G., *Polyèdres de Newton et nombres de Milnor,* Invent. Math. **32** (1976), 1–31.
7. Merle M., Teissier B., *Condition d'adjonction, d'aprés Du Val,* Séminaire sur les Singularités des Surfaces, Palaiseau, France (1976-1977), Lecture Notes in Math. Springer **777**, 230–245.
8. Némethi A., Zaharia A., *On the Bifurcation Set of a Polynomial Function and Newton Boundary,* Publ. RIMS. Kyoto Univ. **26** (1990), 681–689.
9. Płoski A., *Milnor number of a plane curve and Newton polygons,* Univ. Iagel. Acta Math., Fasc. XXXVII, Kraków (1999), 75–80.

Received April 18, 2001

Technical University
Department of Mathematics
Al. 1000 L PP 7
25-314 Kielce
Poland
e-mail: matmm@tu.kielce.pl