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# DELAYED VON FOERSTER EQUATION

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**Abstract.** In the paper the existence and uniqueness of a solution of an integro-differential with delayed argument in integral part is proved.

**1. Introduction.** The theory of first order partial integro-differential equations is interesting because of its applications of mathematics to biology. The most interesting problem is that, of the chaotic behaviour considered by Dawidowicz [**1**], [**2**], [**3**], Lasota [**7**], Rudnicki [**9**] and Łoskot [**8**]. To study this problem it is necessary to prove the existence and uniqueness of solutions. This problem has been studied in a lot of papers [**6**] In the present paper, the results of the paper [**4**] are generalized on the case of delayed argument for $z$.

**2. Formulation of theorems.** Let us consider the system of equations

$$\frac{\partial u}{\partial t} + c(x, z_t)\frac{\partial u}{\partial x} = \lambda(x, u, z_t) \tag{1}$$

$$z(t) = \int_0^\infty u(t, x)dx \tag{2}$$

where

$$z_t : [-r, 0] \to \mathbb{R}_+ \tag{3}$$

is defined by the formula

$$z_t(s) = z(t - s) \tag{4}$$

for $t \geq 0$ and $x \geq 0$.

The equation (1) is considered with the initial condition

$$u(0,x) = u_0(x) \tag{5}$$

Throughout the paper, the coefficients $c$ and $\lambda$ are assuming to satisfy the following assumptions

$(C_1)$
$$c : \mathbb{R}_+ \times C([-r;0];\mathbb{R}) \to \mathbb{R}_+$$

$(C_2)$ The coefficient $c$ is of class $C^1$ for $x \geq 0$

$(C_3)$
$$c(0,Z) = 0$$

$(C_4)$
$$|\frac{\partial c}{\partial x}| \leq \alpha$$

$(C_5)$
$$|c(x,Z) - c(x,\overline{Z})| \leq \gamma ||Z - \overline{Z}||$$

where
$$||Z|| = \sup_{-r \leq s \leq 0} |Z(s)|$$

$(C_6)$
$$|\frac{\partial c}{\partial x}(x,z)| \leq \mu(z)$$

where $\mu$ is continuous

$(\Lambda_1)$ The function $\lambda$ is of class $C^1$ for $x \geq 0, u \geq 0$

$(\Lambda_2)$
$$\lambda(x,0,\varphi) = 0$$

$(\Lambda_3)$
$$\frac{\partial \lambda}{\partial u} \leq \beta$$

$(\Lambda_4)$
$$|\frac{\partial \lambda}{\partial u}| \leq \beta(u,z)$$

where $\beta$ is continuous

$(\Lambda_5)$ $\exists \gamma'$
$$|\lambda(x,u,Z) - \lambda(x,u,\overline{Z})| \leq \gamma' ||Z - \overline{z}|| u$$

($\Lambda_6$)

$$|\frac{\partial\lambda}{\partial x}| \leq \nu(z,u)u$$

THEOREM 1. *Let $u_0$ be bounded and continuous on $(0,\infty)$, $u_0 \geq 0$ and let*

$$A = \int_0^\infty u_0(x)dx < \infty. \tag{6}$$

*Let*

$$z_0 \in C\big([-r,0]\big), z_0(0) = A$$

*Define*

$$z_t : [0,T] \to C([-r,0])$$

*by the formula*

$z_t(s) = z(t-s)$ *for* $t \geq s$

$z_t(s) = z_0(t-s)$ *for* $t < s$

*Then there exists exactly one non negative function $u$ which is a solution of* (1),(4),(5)

**3. The method of characteristics and construction of operator $\Theta$.** Let $C^+([0,T])$be the set of all continuous and non-negative function on the interval $[0,T]$

First we consider problem (1), (5) where $z \in C([-r,T])$ is a given function

Denote by $\psi(t,x,y) = \psi(t,x,y,z_t)$ and $\varphi(t,x) = \varphi(t,x,z_t)$

he characteristics of (1)

i.e. the solution of

$$\xi' = c(\xi, z_t),\ \xi(0) = x \tag{7}$$

and

$$\eta' = \lambda(\xi,\eta,z_t),\ \eta(0) = y \tag{8}$$

respectively, for $t \in [0,T]$

DEFINITION 1. The function $u : [0,T] \times [0,\infty)$ is a solution of (1), (5) if for every $t \in [0,T], x \geq 0$,

$$u(t,\varphi(t,x)) = \psi(t,x,v(x)) \tag{9}$$

PROPOSITION 1. *Under assumptions* $(C_1)$–$(C_3)$ *and* $(\Lambda_1)$–$(\Lambda_3)$ *if* $z \in C_+([0,T])$, *$v$ satisfies* (4) *and $u$ is the solution of* (1), (3), *then for* $t \geq 0$

$$\int_0^\infty u(t,x)dx < \infty \tag{10}$$

*and the function* $[0,T] \ni t \mapsto \int_0^\infty u(t,x)dx$ *is continuous.*

In fact, u depends on z (this dependence is omitted). For fixed $v \geq 0$ define $\Theta z$ by the formula

$$\Theta z(t) = \int_0^\infty u(t,x)dx \tag{11}$$

From proposition 1 there follows that $\Theta : C_+([0,T]) \to C_+([0,T])$

Definition 2. The function $u : [0,T] \times [0,\infty)$ is solution of (1), (2), (5) if u is the solution of(1), (5) for $z$ satisfying the condition

$$\Theta z = z \tag{12}$$

Remark 1. To prove the existence or uniquencess of the solution of (1), (2), (5) it is sufficient to prove the existence or uniqueness of the fixed point of operator $\Theta$.

**4. Proof of the Theorem.** We start with recalling the following lemmas proved in [**4**]

Lemma 1. *The $C^1$-function $\varphi$ is defined on $\Delta \times \mathbb{R}_+$, and $C^1$-function $\psi$ is defined on $\Delta \times \mathbb{R}_+ \times \mathbb{R}_+$. Moreover, for fixed t the function $x \to \varphi(t,x)$ is a bijection of $\mathbb{R}_+$ onto $\mathbb{R}_+$.*

The Lemma is a simple consequence of our assumption. Let

$$s(t,x,z) = s(t,x) = \frac{\partial}{\partial x}\varphi(t,x) \tag{13}$$

It is obvious that s satisfies the condition

$$\frac{\partial S}{\partial t} = \frac{\partial c}{\partial x}\big(\varphi(t,x), z_t\big)S,\ S(0,x) = 1 \tag{14}$$

Lemma 2. *The following inequalities hold*

$$0 \leq S(t,x) \leq e^{\alpha t},\ 0 \leq \varphi(t,x,y) \leq e^{\beta t}y \tag{15}$$

As in [**4**], from these Lemmas it follows that for u defined by (9)

$$\int_0^\infty u(t,x)dx \leq Ae^{(\alpha+\beta)t} < \infty. \tag{16}$$

Moreover, $\Theta z(t) = \int_0^\infty u(t,x)dx$ is a continuous function. This follows from [**4**] and the Lebesgue dominated convergence theorem.

Corollary 1. *From* [**4**] *it follows that*

$$\Theta z(t) = e^{(\alpha+\beta)t}A$$

Assume that $z$ satisfies the Lipschitz condition

Let us consider

$$H : [0, T] \times \mathbb{R}_+ \times C_+[0, T] \to \mathbb{R},\ T > 0$$

defined by the formula

$$H(t, x, z) = \psi(t, x, v(x), z)S(t, x, z). \tag{17}$$

Since $v$ is bounded, from lemma 2 it follows that $u$ also is bounded for $t \leq T$. Since $z$ is continuous, the set $\{z_t | t \in [0, T]\}$ is compact and in consequence there exists

$$B_T = \sup_{t\in[0,T]} \beta(u, z_t) \tag{18}$$

Hence, from $(\Lambda_4)$ it follows, that

$$\left|\frac{\partial \lambda}{\partial u}\right| \leq B_T \tag{19}$$

for $z \in X$ and $u$ satisfying (1), (5). Hence

$$\left|\frac{\partial H}{\partial t}\right| \leq \left|\frac{\partial}{\partial t}\psi(t, x, v(x), z_t)\right| S(t, x, z_t) + \psi(t, x, v(x), z_t) \left|\frac{\partial}{\partial t}S(t, x, z_t)\right|$$

$$\left|\frac{\partial H}{\partial t}\right| \leq (B_T + \alpha)e^{(\alpha+\beta)T}.v(x) \tag{20}$$

Thus

$$|\Theta z(t + h) - \Theta z(t)| \leq A(B_T + \alpha)e^{(\alpha+\beta)T}h \tag{21}$$

for $t,\ t + h \in [0, T]$.

In consequence, if $\Delta = [0, \infty]$ then the set

$$K \subset C(\Delta)$$

This set is relatively compact if and only if, for every $T > 0$, the set of restrictions

$$\{\ z_{|_{[0,T]}} : z \in K\}$$

is relatively compact.

We notice that the set $\overline{K}$ of all functions from $C_+(\Delta)$ bounded by $Ae^{(\alpha+\beta)t}$ and satisfying the Lipschitz condition with the constant

$$N(T) = A(B_T + \alpha)e^{(\alpha+\beta)T}$$

satisfies

$$\Theta(\overline{K}) \subset \overline{K} \tag{22}$$

To prove Theorem 1 we use the following

Proposition 2. *Under the assumptions of Theorem 1, for* $z$, $\overline{z} \in \overline{K}$, *the following inequality holds*

$$||\Theta z - \Theta\overline{z}||_T \le M(T)||z - \overline{z}||_T \tag{23}$$

*where* $\overline{K}$ *is defined in the previous section,* $||\cdot||_T$ *denotes the norm in* $C([0,T])$ *and*

$$\lim_{T\to 0} M(T) = 0 \tag{24}$$

To prove this proposition we shall prove some Lemmas.

Lemma 3. *Under the assumptions of Theorem 1,* $\psi$ *satisfies the inequality*

$$\int |\psi(t,x,v(x),z) - \psi(t,x,v(x),\overline{z})|_T dx \le M_1(T)||z-\overline{z}||_T \tag{25}$$

*or* $t \in [0,T]$ *and* $z$, $\overline{z} \in K$. *Moreover,*

$$\lim_{T\to 0} M_1(T) = 0.$$

Proof. Let $W(t,x) = \psi(t,x,v(x),z) - \psi(t,x,v(x),\overline{z})$.
Obviously, $W(0,x) = 0$.
We shall estimate $\frac{\partial W}{\partial t}(t,x)$.
We notice that, for $z$, $\overline{z} \in \overline{K}$, we have

$$z(t) \le Ae^{(\alpha+\beta)T}, \quad \overline{z}(t) \le Ae^{(\alpha+\beta)T} \tag{26}$$

$$\psi(t,x,v(x),z) \le \sup_{\xi\ge 0} v(\xi)e^{\beta T} \tag{27}$$

and, consequently, there exists a compact set $F$ such that

$$\big(z, \psi(t,x,v(x),z)\big) \in F,$$
$$\big(\overline{z}, \psi(t,x,v(x),\overline{z})\big) \in F.$$

There exists a finite number

$$\nu_0 = sup\{\nu(z,u) : (z,u) \in F\} \tag{28}$$

We estimate $\frac{\partial}{\partial t}\big(W(t,x)\big)$,

$$\frac{\partial}{\partial t}\big(W(t,x)\big) = I_1 + I_2 + I_3 \tag{29}$$

where

$$I_1 = \lambda(\varphi(z), \psi(z), z) - \lambda(\varphi(\overline{z}), \psi(z), z),$$
$$I_2 = \lambda(\varphi(\overline{z}), \psi(z), z) - \lambda(\varphi(\overline{z}), \psi(\overline{z}), z),$$
$$I_3 = \lambda(\varphi(\overline{z}), \psi(\overline{z}), z) - \lambda(\varphi(\overline{z}), \psi(\overline{z}), \overline{z}).$$

In the last formula

$$\varphi(z) = \varphi(t,x,z),\ \ \varphi(\overline{z}) = \varphi(t,x,\overline{z}) \text{ and}$$

$$\psi(z) = \psi(t,x,v(x),z),\ \ \psi(\overline{z}) = \psi(t,x,v(x),\overline{z})$$

From assumption $\Lambda_5$ and (29) it follows, that

$$|I_1| \leq \nu_0|\varphi(t,x,z) - \varphi(t,x,\overline{z})|\psi(t,x,v(x),\overline{z})$$

But

$$\frac{\partial}{\partial t}\big[\varphi(t,x,z) - \varphi(t,x,\overline{z})\big] = c(\varphi(t,x,z),z) - c(\varphi(t,x,\overline{z}),\overline{z})$$

From assumption $C_3$, $C_4$ and the Gronwall inequality [**10**] it follows, that

$$|\varphi(t,x,z) - \varphi(t,x,\overline{z})| \leq \overline{M}(T), \tag{30}$$

where

$$\lim_{T\to 0}\overline{M}(T) = 0,$$

and in consequence

$$|I_1| \leq \nu_0\overline{M}(T)v(x)e^{(\alpha+\beta)T},$$

$$|I_2| \leq B_T W(t,x).$$

($B_T$ is defined by (19))

$$|I_3| \leq \gamma'||z_t - \overline{z}_t||\psi(t,x,v(x),\overline{z}) \leq \gamma'||z-\overline{z}||_T e^{\beta T}v(x)$$

Therefore

$$\Big|\frac{\partial}{\partial t}\big(W(t,x)\big)\Big| \leq B_T|W(t,x)| + M(T)||z-\overline{z}||_T v(x) \tag{31}$$

where

$$\overline{\lim_{T\to 0}} M(T) < \infty.$$

From the Gronwall inequality [**10**] there follows

$$|W(t,x)| \leq M^1(T)v(x)||z-\overline{z}||_T B_T^{-1}(e^{B_T T} - 1). \tag{32}$$

Integrating (32), we obtain

$$\int_0^\infty W(t,x)dx \leq M^1(T)A||z-\overline{z}||_T B_T^{-1}(e^{B_T T} - 1). \tag{33}$$

Let $M_1 = M_1(T) = A||z-\overline{z}||_T B_T^{-1}(e^{B_T}T - 1)$. We obtain (26) since we may define $B_T = B_{T_0}$ for $T < T_0$ and some arbitrary $T_0$, formula

$$\lim_{T\to 0} M_1(T) = 0$$

is obvious. $\square$

Lemma 4. *Under assumption of Theorem 1, for $t \leq T$ and $z$, $\overline{z} \in \overline{K}$*

$$|s(t,x,z) - s(t,x,\overline{z})| \leq M_2(T)||z - \overline{z}||_T. \tag{34}$$

*Moreover*

$$\lim_{T \to 0} M_2(T) = 0 \tag{35}$$

Proof. There exists

$$\mu_0 = sup\{\mu(z_t) : z \in K, t \in [0,T]\} \tag{36}$$

We shall estimate $\sigma(t,x) = s(t,x,z) - s(t,x,\overline{z})$. From (14), we derive

$$\sigma(0,x) = 0, \text{ and} \tag{37}$$

$$\frac{\partial \sigma}{\partial t} = E_1 + E_2 + E_3 \text{ where} \tag{38}$$

$$E_1 = \left[\frac{\partial c}{\partial x}(\varphi(t,x,z),z_t) - \frac{\partial c}{\partial x}(\varphi(t,x,\overline{z}),z_t\right] s(t,x,z),$$

$$E_2 = \left[\frac{\partial c}{\partial x}(\varphi(t,x,\overline{z}),z_t) - \frac{\partial c}{\partial x}(\varphi(t,x,\overline{z}),\overline{z}_t)\right] s(t,x,\overline{z}),$$

$$E_3 = \frac{\partial c}{\partial x}(\varphi(t,x,\overline{z}),\overline{z}_t)\sigma.$$

By virtue of (30) $|E_1| \leq \eta_0 \overline{M}(T)e^{\alpha T}$. By (15), (36) and assumption $C_5$

$$|E_2| \leq \eta_0 ||z_t - \overline{z}_t||e^{\alpha T} \leq \eta_0 ||z - \overline{z}||_T e^{\alpha T}.$$

From (38) and assumption $C_3$

$$\left|\frac{\partial \sigma}{\partial t}\right| \leq M^{"}(T)||z - \overline{z}||_T + \alpha|\sigma|,$$

where

$$\overline{\lim_{T \to 0}} M^{"}(T) < \infty.$$

Hence, using the Gronwall inequality, from (37) and (39) we obtain

$$|\sigma(t)| \leq M^{"}(T)\alpha^{-1}(e^{\alpha T} - 1)||z - \overline{z}||_T.$$

Denoting $M_2(T) = M^{"}(T)\alpha^{-1}(e^{\alpha T} - 1)$ we shall prove Proposition 2.

For $t \leq T$,$z_t$, $\overline{z}_t \in \overline{K}$

$$\left|\Theta z(t) - \Theta \overline{z}(t)\right| =$$

$$= \left| \int_0^\infty \left[\psi(t,x,v(x),z)s(t,x,z) - \psi(t,x,v(x),\overline{z})s(t,x,\overline{z})\right] dx \right|. \tag{39}$$

This is not greater than $\lambda$.

$$\int_0^\infty |\psi(t,x,v(x),z) - \psi(t,x,v(x),\overline{z})| s(t,x,z)dx +$$
$$+ \int_0^\infty \psi(t,x,v(x),\overline{z})|\sigma(t,x)|dx \le M_1(T)e^{\alpha T}||z-\overline{z}||_T +$$
$$+ Ae^{\beta T}M_2(T)||z-\overline{z}||_T.$$

Setting

$$M(T) = M_1(T)e^{\alpha T} + Ae^{\beta T}M_2(T)$$

we obtain Proposition 2. □

Proof of Theorem 1. To prove Theorem 1 it remains to notice that for sufficiently small $T$ the operator
$\Theta : \overline{K}_T \to \overline{K}_T$ fulfil the assumption of the Banach fixed-point theorem,

$$\overline{K}_T = \{z_{|[0,T]} : z \in \overline{K}\}.$$

Hence the operator $\Theta$ has exactly one fixed point in $\overline{K}_T$. Since

$$\Theta\big(C_+(\Delta)\big) \subset \overline{K}$$

$\Theta$ has no fixed-point out of $\overline{K}$, and $\Theta$ has exactly one fixed point in $C_+([0,T])$. To prove $\Theta$ has exactly one fixed point in $C_+(\mathbb{R}_+)$ we notice that the problem (1), (2), (5) is time-independent, the Theorem 1 true in $\Delta = [t_0, T]$ with initial condition

$$u(t_0, x) = \overline{v}(x). \tag{40}$$

From this follows that the set of all $t_0 \in \mathbb{R}_+$ for which (1), (2), (5) has exactly one solution in $\mathbb{R}_+$ is closed. This completes the proof. □

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