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LINEAR FORMS ON MODULES OF PROJECTIVE DIMENSION ONE

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Let R be a noetherian ring and M an R–module which has a presentation

$$0 \to F \xrightarrow{\psi} G \to M \to 0$$

with finite free R–modules F and G of rank m and n. In [**2**] we proved:

Proposition 1. *Assume that $r = n - m > 1$ and that the first non-vanishing Fitting ideal of M has grade $r + 1$. Then the following conditions are equivalent.*

(1) *There is a $\chi \in M^* = \operatorname{Hom}_R(M, R)$ such that the ideal $\operatorname{Im}\chi$ has grade $r + 1$.*

(2) *There exists a submodule U of M with the following properties:*
   (i) $\operatorname{rank} U = r - 1$;
   (ii) *U is reflexive, orientable, and $U_\mathfrak{p}$ is a free direct summand of $M_\mathfrak{p}$ for all primes $\mathfrak{p}$ of R such that* $\operatorname{grade}\mathfrak{p} \le r$.

(3) *$m = 1$ and r is odd.*

The equivalence (1) ⇔ (2) can easily be proved directly (see Proof of Proposition 7 in [**2**]) while the equivalence (1) ⇔ (3) results from a description of the homology of the Koszul complex associated to a linear form on M (see Theorem 5 in [**2**]).

With the assumptions of Proposition 1, let $m = 1$ and $n \ge 4$ be even (which means that the rank r of M is odd). Fix a basis $e_1, \dots, e_n$ of G and let $\psi(1) = \sum_{i=1}^n (-1)^i x_i e_{n+1-i}$. The map $\varphi : \sum_{i=1}^n a_i e_i \mapsto \sum_{i=1}^n a_i x_i$ then obviously induces a linear form χ on M such that $\operatorname{grade}\operatorname{Im}\chi = n$. The submodule $U = \operatorname{Ker}\chi$ has properties (i) and (ii) of Proposition 1, and the (skewsymmetric) map $\rho : G \to G^*$ given by $\rho(e_i) = (-1)^i e^*_{n+1-i}$, $i = 1, \dots, n$, induces an isomorphism $\bar{\rho} : U \to U^*$. (Here as in the following $e_1^*, \ldots, e_n^*$ denotes the basis of G^* dual to $e_1, \ldots, e_n$.) So condition (2) in Proposition 1 may be replaced by

(2') There exists a submodule U of M with the following properties:
  (i) $\operatorname{rank} U = r - 1$;
  (ii) U is orientable, and $U_\mathfrak{p}$ is a free direct summand of $M_\mathfrak{p}$ for all primes $\mathfrak{p}$ of R such that $\operatorname{grade} \mathfrak{p} \leq r$;
  (iii) U is selfdual in a skewsymmetric way, i.e. there is an isomorphism $\rho : U \to U^*$ such that $\rho^* \circ h = -\rho$, $h : U \to U^{**}$ being the natural map.

The Koszul complex associated to φ induces an exact sequence

$$\bigwedge^3 G \xrightarrow{\tau} \bigwedge^2 G \xrightarrow{\sigma} \operatorname{Ker} \varphi \to 0, \tag{1}$$

and there is a map p from $\operatorname{Ker} \varphi$ onto U which has the kernel $\psi(1)$. So, in particular, U is minimally generated by $\binom{n}{2} - 1$ elements. The aim of this note is to give an explicit construction of U as a submodule of the free module $R^{\binom{n}{2}-1}$. Since

$$\operatorname{Ker}(p \circ \sigma) = R \cdot \sum_{i=1}^{n/2} (-1)^{i-1} e_i \wedge e_{n+1-i} \ + \ \operatorname{Ker} \sigma,$$

in view of (1) we obtain an exact sequence

$$R \oplus \bigwedge^3 G \xrightarrow{\widetilde{\tau}} \bigwedge^2 G \xrightarrow{p \circ \sigma} U \to 0,$$

where

$$\widetilde{\tau}(1,0) = \sum_{i=1}^{n/2} (-1)^{i-1} e_i \wedge e_{n+1-i} \quad \text{and} \quad \widetilde{\tau}(0,y) = \tau(y)$$

for all $y \in \bigwedge^3 G$. Dualizing yields the exact sequence

$$0 \to U^* \to \bigwedge^2 G^* \xrightarrow{\widetilde{\tau}^*} R \oplus \bigwedge^3 G^*,$$

where we used the natural isomorphims $\bigwedge^k G^* \cong (\bigwedge^k G)^*$. We shall explicitly represent $U^* = \operatorname{Ker} \widetilde{\tau}^*$ as a submodule of $\bigwedge^2 G^*$.

Proposition 2. *The elements*

$$r_{ij} = \varphi \wedge ((-1)^j x_i e_{n-j+1}^* + (-1)^{i+1} x_j e_{n-i+1}^*),$$

$i, j = 1, \ldots, n$, generate U^*.

Proof. Since $\varphi \wedge \eta$ vanishes on $\operatorname{Im} \tau$ for all $\eta \in G^*$, we have $r_{ij} \circ \tau = 0$. Moreover,

$$r_{ij}(\widetilde{\tau}(1,0)) = (x_j x_i (-1)^j e_j^* \wedge e_{n-j+1}^* + (-1)^{i+1} x_i x_j e_i^* \wedge e_{n-i+1}^*)(\widetilde{\tau}(1,0)) = 0.$$

So $r_{ij} \in U^*$ for all i, j.

Let $\alpha = \sum_{1 \le k < l \le n} a_{kl} e_k^* \wedge e_l^* \in U^*$. Then, in particular,

$$a_{1n} - a_{2,n-1} + \ldots + (-1)^{\frac{n}{2}+1} a_{\frac{n}{2},\frac{n}{2}+1} = 0. \tag{2}$$

Since $\alpha \circ \tau(e_k \wedge e_l \wedge e_m) = 0$ for all k, l, m, we have in addition, that $a_{kl} \in Rx_k + Rx_l$ for all k, l.

Next we claim that there is an element $\beta = \sum_{1 \le k < l \le n} b_{kl} e_k^* \wedge e_l^* \in \sum_{i,j} R \cdot r_{ij}$, such that $a_{kl} = b_{kl}$ for $k + l = n + 1$. To prove this let $1 \le k < n/2$, $k + l = n + 1$, and $a_{st} = 0$ if $s < k, s + t = n + 1$. We show that there is a β which satisfies $b_{st} = 0$ for $s < k, s + t = n + 1$, and $b_{kl} = a_{kl}$. Because of (2) this will prove our claim. First we deduce

$$\begin{aligned} a_{kl} &\in (Rx_k + Rx_l) \cap (Rx_{k+1} + \ldots + Rx_{l-1}) \\ &= Rx_k x_{k+1} + \ldots + Rx_k x_{l-1} + Rx_l x_{k+1} + \ldots + Rx_l x_{l-1}, \end{aligned}$$

since $x_1, \ldots, x_n$ is a regular sequence in R. Consider r_{kj}, r_{jl} for $j = k + 1, \ldots, l - 1$. Using the canonical isomorphism $G \to G^{**}$, we get

$$(e_s \wedge e_t)(r_{kj}) = \begin{cases} 0 & \text{if } 1 \le s < k,\ s + t = n + 1, \\ \pm x_k x_j & \text{if } (s, t) = (k, l), \end{cases}$$

and

$$(e_s \wedge e_t)(r_{jl}) = \begin{cases} 0 & \text{if } 1 \le s < k,\ s + t = n + 1, \\ \pm x_j x_l & \text{if } (s, t) = (k, l). \end{cases}$$

So we can find an appropriate $b \in \sum_{i,j} R \cdot r_{ij}$.

In proving the proposition, namely $\alpha \in \sum_{i,j} R \cdot r_{ij}$, we may now assume that $a_{kl} = 0$ whenever $k + l = n + 1$. We then show that there is an element $\gamma = \sum c_{kl} e_k^* \wedge e_l^* \in \sum_{i,j} R \cdot r_{ij}$ with $c_{kl} = 0$ for $k + l = n + 1$ and $c_{1l} = a_{1l}$ for $l = 2, \ldots, n/2$. Since

$$x_1 a_{ln} - x_l a_{1n} + x_n a_{1l} = 0 = x_1 a_{l,n-l+1} - x_l a_{1,n-l+1} + x_{n-l+1} a_{1l} \tag{3}$$

(which follows from $\alpha \circ \tau(e_1 \wedge e_l \wedge e_n) = 0 = \alpha \circ \tau(e_1 \wedge e_l \wedge e_{n-l+1})$), we obtain $a_{1l} \in Rx_1 x_l$. Obviously

$$(e_s \wedge e_t)(r_{l,n-l+1}) = \begin{cases} 0 & \text{if } s + t = n + 1 \text{ or } s = 1, t < l, \\ (-1)^{l+1} x_1 x_l & \text{if } s = 1, t = l. \end{cases}$$

So there is an appropriate $c \in \sum_{i,j} R \cdot r_{ij}$.

Finally suppose that $a_{kl} = 0$ for $k + l = n + 1$ and $a_{1l} = 0$ for $l = 2, \ldots, n/2$. Then, because of (3), $a_{1j} = 0$ for $j = 2, \ldots, n$. Let $1 < i < j \le n$. Since $x_1 a_{ij} - x_i a_{1j} + x_j a_{1i} = 0$, we get $a_{ij} = 0$. The proof is complete now. □

Proposition 3. *With the above notation, for $i, j = 1, \ldots, n$ and $1 \le k < l \le n$*

$$(*) \qquad (e_k \wedge e_l)(r_{ij}) = -(e_i \wedge e_j)(r_{kl})$$

holds. Furthermore, $r_{ii} = 0$, $r_{ij} = -r_{ji}$, and

$$(**) \qquad r_{1n} - r_{2n-1} + \ldots + (-1)^{\frac{n}{2}+1} r_{\frac{n}{2},\frac{n}{2}+1} = 0.$$

Consequently, U^* is minimally generated by the elements r_{ij} for which $i < j$ and $(i, j) \neq (1, n)$, and is represented by the skewsymmetric matrix

$$\big((e_k \wedge e_l)(r_{ij})\big), \quad 1 \le k < l \le n, 1 \le i < j \le n,\ (k, l) \neq (1, n) \neq (i, j).$$

Proof. Equation $(*)$ is obtained by a straightforward computation, and $(**)$ is a direct consequence of $(*)$ since for all $k, l,\ k < l$:

$$\begin{aligned}(e_k \wedge e_l) \sum_{\substack{i<j \\ i+j=n+1}} (-1)^{i+1} r_{ij} = -\Big(\sum_{\substack{i<j \\ i+j=n+1}} (-1)^{i+1} e_i \wedge e_j\Big)(r_{kl}) \\ = r_{kl}(\widetilde{\tau}(1, 0)) = 0.\end{aligned}$$

The remaining assertions follow from Proposition 2, the definition of the r_{ij}, and equations $(*)$,$(**)$. □

In the simplest case $n = 4$, the matrix representing $U^* \cong U$ is

$$\begin{pmatrix} 0 & x_1^2 & x_1x_2 & x_2^2 & -x_1x_4 + x_2x_3 \\ -x_1^2 & 0 & x_1x_3 & x_1x_4 + x_2x_3 & x_3^2 \\ -x_1x_2 & -x_1x_3 & 0 & x_2x_4 & x_3x_4 \\ -x_2^2 & -x_1x_4 - x_2x_3 & -x_2x_4 & 0 & x_4^2 \\ x_1x_4 - x_2x_3 & -x_3^2 & -x_3x_4 & -x_4^2 & 0 \end{pmatrix}.$$

Remarks 4. Suppose that $R = K[X_1, \ldots, X_n]$ is the polynomial ring in n indeterminates over a field K. In case $x_i = X_i$, the module U considered above seems to have already been studied in [**4**]; this is definitely true for $n = 4$. In this case it also coincides with the rank $n - 2$ module M_n constructed in [**5**] which likewise satisfies conditions (i) and (ii) of Proposition 1. For $n > 4$, the two modules are definitely different: from [**2**] we know that $\operatorname{projdim} U =$ ($\operatorname{projdim} U^* =$) $n - 2$, while $\operatorname{projdim} M_n^* = 2$ for all n.

Besides the fact that M_n is defined for arbitrary $n \geq 2$, its dual has, in contrast to U, the remarkable property to be "optimal" in view of the Evans-Griffith syzygy theorem (cf. [**1**], 9.5.6 for example) because M_n^* is an $(n-2)$–th syzygy of rank $n-2$. A concrete description of M_n^* similar to that we gave for U

in Proposition 3, seems to be more complicated. An attempt with SINGULAR [**3**] for $n = 5, 6$ leads to the supposition that the entries of a representing matrix are homogeneous of degree $n - 2$ if the characteristic of K is $\neq 2$.

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