UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIX
2001

ON THE WEIERSTRASS DIVISION

by Stanisław Łojasiewicz*, Tomasz Maszczyk and Kamil Rusek*

Abstract. We propose some abstract counterparts of the classical Weierstrass Division Theorem starting from simple properties of topological groups, rings and vector spaces together with rather elementary algebraic notions. It is shown how to deduce from these results the classical versions of the Weierstrass Division Theorem for formal and convergent power series.

1. Preliminaries. Let G be an abelian topological group which is separated and sequentially complete.

A sequence $\{E_n\}_{n\in\mathbb{N}}$ of subsets of G is said to be *summable* if

$$x_n \in E_n, \quad n = 0, 1, 2, \ldots \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} x_n \text{ converges.}$$

Then necessarily $\bigcap\limits_{n=0}^{\infty} E_n \subset \{0\}$.

Every decreasing sequence of subgroups of an abelian group G determines a unique topology in G for which it is a bases of neigbourhoods of zero and the group operation is continuous. Moreover, if this topology is sequentially complete then

$$\sum a_n \text{ converges} \qquad \Longleftrightarrow \qquad a_n \to 0.$$

In particular, each sequence of subsets of G which converges to zero (i.e. each neighbourhood of zero contains almost all its terms) is summable.

The following lemma, inspired by an idea from [**1**], is fundamental for our purposes.

1991 *Mathematics Subject Classification.* Primary 13J99, Secondary 13A18.
*Partially supported by KBN Grant 2P03A01314.

LEMMA 1.1. *Let G be a topological group which is separated and sequentially complete. Suppose that $f : G \longrightarrow G$ is a continuous endomorphism and $\{E_n\}$ is a summable sequence of subsets of G such that $E_0 = G$ and*

$$(+)\qquad (\mathrm{id}_G - f)(E_n) \subset E_{n+1}, \qquad n = 0, 1, 2, \ldots .$$

Then f is an automorphism of G.

PROOF. Let $x \in \ker f$. By (+) we have $x \in E_n$ for $n = 0, 1, 2, \ldots$. Hence $x = 0$. Thus f is injective.

To prove surjectivity, take $y \in G$ and define the sequence: $u_0 = y$, $u_{n+1} = u_n - f(u_n)$, $n = 1, 2, \ldots$. Then $u_n \in E_n$, $n = 0, 1, 2, \ldots$, so $\sum_0^\infty u_n$ converges. Putting $x := \sum_0^\infty u_n$ and using continuity of f we obtain $f(x) = \sum_0^\infty f(u_n) = \sum_0^\infty (u_n - u_{n+1}) = u_0 = y$. Therefore f is surjective and the lemma is proved. □

2. Weierstrass elements and division. Let A be a commutative ring and let H be its additive subgroup. An element ω of A which is not a zero divisor in A is said to be a Weierstrass element, briefly: WEL (with respect to H), if $A = A\,\omega \oplus H$.

Each Weierstrass element determines two group homomorphisms

$$Q_\omega : A \longrightarrow A \quad \text{and} \quad R_\omega : A \longrightarrow H$$

defined by the condition $x = Q_\omega(x)\,\omega + R_\omega(x)$ for $x \in A$.

Note that even if A is a Banach algebra these mappings may not be continuous.

EXAMPLE 2.1. Let A denote the Banach algebra of all bounded holomorphic functions on the unit disc with the supremum norm. Let $\omega = (1 - z)^2$ and let H be a linear supplement of $A\,\omega$. Consider the sequence $f_n(z) = (1 - z)^2(1 + \frac{1}{n} - z)^{-1}$ of elements of A. Then $Q_\omega(f_n)(z) = (1 + \frac{1}{n} - z)^{-1}$ and $f_n \longrightarrow f$ in A, where $f(z) = 1 - z$, but the sequence $Q_\omega(f_n)$ does not converge in A.

Our first version of the Weierstrass Division Theorem (briefly: WDT) is the following

THEOREM 2.2. *Let A be a commutative topological ring, separated and sequentially complete. Fix an additive subgroup H of A and a summmable sequence $\{E_n\}$ of subsets of A such that $E_0 = A$.*

Let ω be a WEL such that the mapping Q_ω is continuous. Then every $\omega' \in A$ satisfying

$$(*)\qquad (\omega - \omega')Q_\omega(E_n) \subset E_{n+1}, \quad n = 0, 1, 2, \ldots$$

is also a WEL.

Proof. By our assumptions both mappings $Q = Q_\omega : A \longrightarrow A$ and $R = R_\omega : A \longrightarrow H$ are continuous, which implies that the group isomorphism

$$f : A \times H \ni (u, v) \longrightarrow u\,\omega + v \in A$$

has a continuous inverse

$$f^{-1} : A \ni x \longrightarrow (Q(x), R(x)) \in A \times H.$$

Clearly, it suffices to prove that the continuous mapping

$$g : A \times H \ni (u, v) \longrightarrow u\,\omega' + v \in A$$

is an isomorphism of groups, provided that ω' satisfies (*).

So, note that the mapping $(f - g) \circ f^{-1} : A \longrightarrow A$ is a continuous endomorphism of the group A and

$$(f - g)(f^{-1}(x)) = (\omega - \omega')Q(x) \text{ for } x \in A.$$

Now (*) and the last equality imply that

$$(\mathrm{id}_A - g \circ f^{-1})(E_n) \subset E_{n+1}, n = 0, 1, 2, \dots .$$

Therefore, by Lemma 1.1, the mapping $g \circ f^{-1}$ is an automorphism and hence g is an isomorphism. □

3. Weierstrass division in local rings. In the sequel we will consider $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ as an additive monoid.

If A is an integral domain then a function $\mathbf{o} : A \longrightarrow \overline{\mathbb{N}}$ such that:

$$\mathbf{o}(a) = \infty \quad \iff \quad a = 0,$$
$$\mathbf{o}(ab) = \mathbf{o}(a) + \mathbf{o}(b),$$
$$\mathbf{o}(a + b) \geq \min(\mathbf{o}(a), \mathbf{o}(b))$$

is said to be an *order* on A.

From this definition we easily derive the following properties:

(i) if a is a unit in A then $\mathbf{o}(a) = 0$; if the inverse implication holds the order is said to be *strict*;
(ii) $\mathbf{o}(-a) = \mathbf{o}(a)$;
(iii) $\mathbf{o}(a) < \mathbf{o}(b) \quad \Longrightarrow \quad \mathbf{o}(a + b) = \mathbf{o}(a)$;
(iv) the relation $\sim$ defined on $A \setminus 0$ by

$$a \sim b \quad \iff \quad \mathbf{o}(a - b) > \mathbf{o}(a)$$

is an equivalence relation;
(v) $a \sim b \quad \Longrightarrow \quad \mathbf{o}(a) = \mathbf{o}(b)$.

Example 3.1. If $A = R[[X_1, \dots, X_n]]$, where R is an integral domain, the function $A \ni f \longrightarrow \mathrm{ord}(f) \in \overline{\mathbb{N}}$ is an order. It is strict if and only if R is a field.

EXAMPLE 3.2. If $(A, \mathfrak{m})$ is a regular local ring then the function $\mathbf{o}\colon A \to \overline{\mathbb{N}}$, where

$$\mathbf{o}(a) := \begin{cases} k, & \text{if } a \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1} \\ \infty, & \text{if } a = 0 \end{cases}$$

is a strict order on A (see [**4**]).

If an order $\mathbf{o}$ does not vanish identically on the set $A \setminus 0$ then $I_k = \{a : \mathbf{o}(a) \geq k\}$ are non-zero ideals and $A = I_0 \supset I_1 \supset I_2 \supset \dots$ is a filtration on A, i.e. $I_i I_j \subset I_{i+j}$. It defines a linear topology on A, called $\mathbf{o}$-*topology*, which makes A into a separated topological ring. Moreover, if the order $\mathbf{o}$ is strict then $\mathfrak{m} := I_1 = \{a : \mathbf{o}(a) > 0\}$ is a unique maximal ideal in A, i.e. $(A, \mathfrak{m})$ is a local ring. Since $\mathfrak{m}^k \subset I_k$ for each positive integer k, the $\mathbf{o}$-topology is not stronger than the $\mathfrak{m}$-adic topology. If A is noetherian and $\mathfrak{m}$-adically complete then both topologies coincide (Chevalley's theorem – see, for example, [**4**]).

Let us make the following obvious observation.

PROPOSITION 3.3. *Let H be an additive subgroup of an integral domain A and let $\mathbf{o}$ be an order on A.*

Then for any Weierstrass element $\omega \in A$ the following conditions are equivalent:

(α) $Q_\omega(I_{n+m}) \subset I_n$ *for* $n \in \mathbb{N}$ *where* $m := \mathbf{o}(\omega)$;

(β) $R_\omega(I_n) \subset I_n$ *for* $n \geq m$;

(γ) $\mathbf{o}(R_\omega(a)) \geq \mathbf{o}(a)$ *for all* $a \in A$.

□

A Weierstrass element with the above property is called *strong*, briefly: SWEL (with respect to H and $\mathbf{o}$).

Clearly, for such an element the mappings Q_ω and R_ω are continuous with respect to the $\mathbf{o}$-topology.

A unit is a trivial example of a SWEL (with respect to any subgroup and any order).

There is no distinction between WEL's and SWEL's in the ring $K[[X]]$, where K is a field (with respect to the standard order and any subgroup). Unfortunately, this is not true in general. The following simple observation will be useful for a counterexample.

Let an additive group A be a direct sum of two of its subgroups G and H and let $\varphi : H \longrightarrow G$ be a homomorphism of groups. Then, if we denote by $H' := \{h + \varphi(h) : h \in H\}$ the graph of φ, the group A is also the direct sum of G and H'.

Indeed, if $a = g + h$, $g \in G$, $h \in H$, then also $a = (g - \varphi(h)) + (h + \varphi(h))$ and $G \cap H' = \{0\}$.

Example 3.4. (given by K. Nowak). Let $A := K[[X,T]]$, where K is a field. We take $\omega := T$ and the subgroup $H := K[[X]]$. Evidently, ω is a SWEL with respect to H and the standard order on A.

Put $G := A\,\omega$ and define the homomorphism

$$\varphi : H \ni \sum_{n=0}^{\infty} a_n X^n \longrightarrow a_2 T \in G.$$

If H' denotes the graph of φ then according to the above remark the element ω is still a WEL with respect to H' but not a SWEL.

Indeed, for the element $X^2 \in H$ we have

$$R_\omega^{H'}(X^2) = R_\omega^H(X^2) + \varphi(R_\omega^H(X^2)) = X^2 + T$$

and hence

$$ordR_\omega^{H'}(X^2) = 1 < 2 = ord(X^2).$$

A counterpart of the WDT is the following (see also [**3**]).

Theorem 3.5. *Let A be an integral domain with an additive subgroup $H \subset A$ and with a strict order $\mathbf{o}$ such that the $\mathbf{o}$-topology is sequentially complete.*

If ω, ω' are two elements of $A \setminus 0$ such that $\omega \sim \omega'$, then

$$\omega \text{ is a SWEL} \quad \Longleftrightarrow \quad \omega' \text{ is a SWEL.}$$

Proof. Suppose that ω is a SWEL. If $\mathbf{o}(\omega) = 0$, then ω is a unit and so is ω'. So let $\mathbf{o}(\omega) = m > 0$.

The sequence $\{E_n\}$, defined by $E_0 = A$ and $E_n = I_{m+n}$ for $n \geq 1$ is summable and by Proposition 3.3 together with earlier mentioned properties of the order we verify that

$$(\omega - \omega')Q_\omega(E_n) \subset E_{n+1} \text{ for } n \in \mathbb{N}.$$

Therefore ω' is a WEL by Theorem 2.2 and we have the decomposition

$$\omega' = Q_\omega(\omega')\omega + R_\omega(\omega'). \tag{1}$$

Since ω is a SWEL and $R_\omega(\omega) = 0$, we get

$$\mathbf{o}(R_\omega(\omega')) = \mathbf{o}(R_\omega(\omega - \omega')) \geq \mathbf{o}(\omega - \omega') > \mathbf{o}(\omega') \tag{2}$$

Fix an $a \in A$ and, using (1), write the decomposition

$$a = Q_{\omega'}(a)\omega' + R_{\omega'}(a) = Q_{\omega'}(a)Q_\omega(\omega')\omega + Q_{\omega'}(a)R_\omega(\omega') + R_{\omega'}(a).$$

If $\mathbf{o}(Q_{\omega'}(a)\omega') < \mathbf{o}(R_{\omega'}(a))$, then $\mathbf{o}(a) = \mathbf{o}(Q_{\omega'}(a)\omega') < \mathbf{o}(R_{\omega'}(a))$. Assume that $\mathbf{o}(Q_{\omega'}(a)\omega') \geq \mathbf{o}(R_{\omega'}(a))$ and decompose

$$Q_{\omega'}(a)R_\omega(\omega') = q\,\omega + r,$$

where $r \in H$ and, by (2),

$$\begin{aligned} \mathbf{o}(r) &= \mathbf{o}(R_\omega(Q_{\omega'}(a)R_\omega(\omega'))) \geq \mathbf{o}(Q_{\omega'}(a)R_\omega(\omega')) \\ &> \mathbf{o}(Q_{\omega'}(a)\omega') \geq \mathbf{o}(R_{\omega'}(a)) \end{aligned} \tag{3}$$

Finally we obtain the formula

$$a = [Q_{\omega'}(a)Q_\omega(\omega') + q]\omega + r + R_{\omega'}(a),$$

where $r + R_{\omega'}(a) \in H$.
Therefore, in view of uniqueness of the decomposition,

$$R_\omega(a) = r + R_{\omega'}(a)$$

and by (3)

$$\mathbf{o}(R_\omega(a)) = \mathbf{o}(R_{\omega'}(a)) \geq \mathbf{o}(a).$$

Thus ω' is a SWEL and the proof of Theorem 3.5 is completed. □

We cannot replace in the statement of Theorem 3.5 SWEL's by WEL's. It is seen from the following

Example 3.6. (given by K. Nowak). Keeping notations from Example 3.4 let $\omega := T$ and $\omega' := T + X^2$.

Then $\omega \sim \omega'$ and ω is a WEL with respect to the subgroup H' while ω' is not.

Indeed, if ω' were a WEL then $X^2 = Q_{\omega'}(X^2)(T + X^2) + R_{\omega'}(X^2)$, where $R_{\omega'}(X^2) = a_0 + a_1X + a_2(T + X^2) + a_3X^3 + \ldots$, $a_j \in K$. Therefore $Q_{\omega'}(X^2) =: c \in K$ and $a_2 + c = 0$, a contradiction.

Finally, let us show how Theorem 3.5 implies the classical WDT in the formal case.

Theorem 3.7. *Let K be a field. If $F \in K[[X_1, \ldots, X_n, T]]$ is T-regular of order m (i.e. $F(0, \ldots, 0, T) = a_mT^m + a_{m+1}T^{m+1} + \ldots$ with $a_m \neq 0$), then for every $G \in K[[X_1, \ldots, X_n, T]]$ there exists exactly one pair $Q \in K[[X_1, \ldots, X_n, T]]$, $R \in K[[X_1, \ldots, X_n]][T]$ with $\deg_T R < m$, such that*

$$G = Q\,F + R.$$

Proof. Take some $q > m$ and consider the following monomorphism of the ring $K[[X_1, \ldots, X_n, T]]$:

$$h : U \longrightarrow U(X_1{}^q, \ldots, X_n{}^q,\ T).$$

As A we take the image of $K[[X_1, \ldots, X_n, T]]$, as H the additive subgroup of A of polynomials with respect to T of degree $< m$, and as $\mathbf{o}$ the restriction to A of the usual order. Then A is sequentially complete and separated. Evidently $\omega = a_mT^m$ is a SWEL. Since $\mathbf{o}(h(F) - \omega\ ) > m$ i.e. $h(F) \sim \omega$, the element $h(F)$ is a WEL according to Theorem 3.5.

This means that $h(G) = h(Q)\, h(F) + h(R)$ i.e. $G = Q\, F + R$, with a unique pair: $Q \in K[[X_1, \ldots, X_n, T]], R \in K[[X_1, \ldots, X_n]][T]$ with $\deg_T R < m$. $\square$

4. WDT for convergent power series. We need a slightly different versions of Lemma 1.1 and Theorem 2.2.

The proof of Lemma 1.1 does work for the following

LEMMA 4.1. *Let M be a vector space over a field $\mathbb{K}$ of real or complex numbers which is a topological additive group, separated and sequentially complete.*

Suppose that $f : M \longrightarrow M$ is a continuous linear mapping and $\{E_n\}$ is a summable sequence of subsets of G with E_0 absorbing[1]*, such that*

(++) $$(\mathrm{id}_G - f)(E_n) \subset E_{n+1}\ , \qquad n = 0, 1, 2, \ldots .$$

Then f is an automorphism of M.

The only difference is that in the surjectivity proof we get $E_0 \subset f(M)$, but this implies that $f(M) = M$.

Now, repeating the proof of Theorem 2.2, we obtain

THEOREM 4.2. *Let A be a commutative algebra which is a topological ring, separated and sequentially complete. Fix a vector subspace H of A and a summable sequence $\{E_n\}$ of subsets of A with E_0 absorbing.*

Let ω be a WEL such that the mapping Q_ω is continuous. Then every $\omega' \in A$ satisfying

(**) $$(\omega - \omega')Q_\omega(E_n) \subset E_{n+1}, \quad n = 0, 1, 2, \ldots$$

is also a WEL.

As an immediate consequence we get

THEOREM 4.3. *Let A be a Banach algebra with a vector subspace H. Let ω be a WEL such that the mapping Q_ω is continuous. Then every $\omega' \in A$ satisfying the inequality $\|(\omega - \omega')Q_\omega\| =: \Theta < 1$ is also a WEL.*

Moreover, both mappings $Q_{\omega'}$ and $R_{\omega'}$ are also continuous and

$$\max\{\|Q_{\omega'}\|, \|R_{\omega'}\|\} \leq \varepsilon\ (1 - \Theta)^{-1},$$

where ε is a positive constant, depending only on ω and on a norm on $A \times H$.

PROOF. For the first statement it suffices to apply Theorem 4.2 with $E_n = \{x \in A : \|x\| \leq \Theta^n\}$, $n = 0, 1, \ldots$.

To verify the second statement, we need some standard properties of linear endomorphisms of Banach spaces (see, for example, [**2**]).

Since ω' is a WEL for A with respect to H the mapping

$$g : A \times H \ni (u, v) \longrightarrow\ u\,\omega' + v \in A$$

[1] I.e. for every $x \in M$ there exists $t \in \mathbb{K} \setminus 0$ such that $t\,x \in E_0$.

is an isomorphism with the inverse

$$g^{-1} : A \ni u \longrightarrow (Q_{\omega'}(u), R_{\omega'}(u)) \in A \times H.$$

By the assumption, the mapping

$$f : A \times H \ni (u, v) \longrightarrow u\,\omega + v \in A$$

is also a continuous isomorphism with the continuous inverse

$$f^{-1} : A \ni u \longrightarrow (Q_{\omega}(u), R_{\omega}(u)) \in A \times H.$$

Thus

$$(f - g) \circ f^{-1} = \mathrm{id}_A - g \circ f^{-1} = (\omega - \omega')Q_{\omega}$$

is a continuous endomorphism of A and

$$\|(f - g) \circ f^{-1}\| = \Theta < 1.$$

Therefore, the mapping $g \circ f^{-1}$ is an automorphism of A and

$$\|(g \circ f^{-1})^{-1}\| \leq (1 - \|\,\mathrm{id}_A - g \circ f^{-1}\|)^{-1} = (1 - \Theta)^{-1}.$$

Since $g^{-1} = f^{-1} \circ (g \circ f^{-1})^{-1}$, we conclude the proof. □

In order to obtain WDT in the convergent series case as a consequence of Theorem 4.3 we shall use the Grauert-Malgrange norms.

Given $\rho > 0$, $\sigma > 0$ we define

$$(\#) \qquad \|F\|_{\rho,\sigma} = \sum |a_{pq}|\rho^{|p|}\sigma^{|q|}$$

for any formal power series $F = \sum a_{pq}X^pY^q \in \mathbb{K}[[X_1, \ldots, X_k, Y_1, \ldots, Y_l]]$. One checks easily that

$$\|F\,G\|_{\rho,\sigma} \leq \|F\|_{\rho,\sigma}\|G\|_{\rho,\sigma}.$$

Then

$$A(\rho, \sigma) = \{F \in \mathbb{K}[[X_1, \ldots, X_k, Y_1, \ldots, Y_l]] : \|F\|_{\rho,\sigma} < \infty\}$$

is a Banach algebra with the norm $\|\cdot\|_{\rho,\sigma}$ (the restriction of (#) to $A(\rho, \sigma)$). Observe that

$$(\#\#) \qquad \begin{cases} \rho' < \rho, \sigma' < \sigma & \longrightarrow \quad A(\rho, \sigma) \subset A(\rho', \sigma'), \\ \mathbb{K}\{X_1, \ldots, X_k, Y_1, \ldots, Y_l\} & = \quad \cup_{\rho>0,\sigma>0} A(\rho, \sigma), \end{cases}$$

where $\mathbb{K}\{X_1, \ldots, X_k, Y_1, \ldots, Y_l\}$ denotes the ring of convergent power series of the variables $X_1, \ldots, X_k, Y_1, \ldots, Y_l$ (see [**2**]).

In particular, we have the Banach algebra

$$A(\rho) = \{F \in \mathbb{K}[[X_1, \ldots, X_k]] : \|F\|_{\rho} < \infty\}$$

with the norm $\|\cdot\|_{\rho}$ which is the restriction of the function

$$\sum a_pX^p \longrightarrow \sum |a_p|\rho^{|p|}.$$

It is easy to verify that

$$\|F\|_{\rho,\sigma} \leq \sum \|A_q\|_\rho \, \sigma^{|q|}$$

for $F = \sum A_q Y^q \in \mathbb{K}\{X_1, \ldots, X_k, Y_1, \ldots, Y_l\}$ and $A_q \in \mathbb{K}\{X_1, \ldots, X_k\}$.

Now we shall show the classical WDT for convergent series.

THEOREM 4.4. *If $F \in \mathbb{K}\{X_1, \ldots, X_n, T\}$ is T-regular of order m, then for every $G \in \mathbb{K}\{X_1, \ldots, X_n, T\}$ there exists exactly one pair $Q \in \mathbb{K}\{X_1, \ldots, X_n, T\}$, $R \in \mathbb{K}\{X_1, \ldots, X_n\}[T]$ with $\deg_T R < m$, such that*

$$G = Q\,F + R.$$

Moreover, for every $\rho' > 0$, $\sigma' > 0$, there exist $\rho > 0$, $s \geq m+1$, $\rho^s \leq \rho'$, $\rho \leq \sigma'$, and a constant $\kappa_\rho > 0$ such that if $G \in A(\rho', \sigma')$ then

$$\max\{\|Q\|, \|R\|\} \leq \kappa_\rho \|G\|,$$

where $\|\cdot\| = \|\cdot\|_{\rho^s,\rho}$.

PROOF. We shall use Banach algebras $A = A(\rho^s, \rho)$ (with the norm $\|\cdot\| = \|\cdot\|_{\rho^s,\rho}$), where $\rho > 0$ and $s \geq m+1$. If ρ is sufficiently small, then $F = \sum c_k T^k \in A$ with $c_k \in A(\rho^s), c_\nu(0) = 0$ for $\nu < m$ and c_m invertible in $A(\rho^s)$. Then $C = c_m T^m$ is a WEL with respect to the subspace H of polynomials in T, of degree $< m$. Moreover, the mapping $Q_C : \sum_0^\infty a_\nu T^\nu \longrightarrow {c_m}^{-1} \sum_m^\infty a_\nu T^\nu$ is continuous and $\|Q_C\| \leq \lambda\,\rho^{-m}$, where $\lambda = \|{c_m}^{-1}\|$. Because $F - C = \sum_0^{m-1} c_\nu T^\nu + \sum_{m+1}^\infty c_\nu\, T^\nu$ and $s \geq m+1$, we have $\|F - C\| \leq \Theta\,\rho^{m+1}$ for some positive decreasing function Θ of ρ. Taking ρ such that $\Theta\lambda\rho < 1$, we get $\|(F - C)\,Q_C\| < 1$ and hence F is also a WEL by Theorem 4.3. Thus, in view of (##), the main statement of our theorem follows.

The second part of the theorem is an immediate consequence of the above argument and estimations from Theorem 4.3. □

Acknowledgements. The authors express their gratitude to Dr. Krzysztof Nowak for kindly communicating them Examples 3.4 and 3.6.

References

1. Bierstone, E., Milman, P.D., *The local geometry of analytic mappings,* Dip. Mat. Univ. Pisa, 1987.
2. Grauert, H., Remmert, R., *Analytische Stellenalgebren,* Springer, 1971.
3. Maszczyk, T., *Division with remainder in algebras with valuations,* (preprint).
4. Northcott, D.C., *Ideal Theory,* Cambridge Univ. Press, 1953.

Received March 29, 2001

St.Ł. & K.R.
Jagiellonian University
Institute of Mathematics
Reymonta 4
30–059 Kraków
Poland

T.M.
Warsaw University
Institute of Mathematics
Banacha 2
02–097 Warszawa
Poland