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THE LIE-ALGEBRAIC DISCRETE APPROXIMATION SCHEME FOR EVOLUTION EQUATIONS WITH DIRICHLET/NEUMANN DATA

by Mirosław Luśtyk

Abstract. In the paper we developed a new discrete approximation scheme for the Cauchy-Dirichlet and Cauchy-Neumann problems, based on the idea of including boundary conditions into polynomial interpolating functional spaces and subsequently calculating the corresponding Heisenberg-Weyl algebra quasi-representation. This discretization scheme being very convenient for practical applications, at the same time leads to a rather complicated final matrix expression suitable for computer calculations.

1. Introduction. This work is devoted to a new discrete approximation method based on the Lie-algebraic discrete approximation approach developed in [**1, 4**] and its generalizations. This method can be usefully applied to solving different problems of mechanical models using of both elliptic and parabolic equations, for instance, the elastostatics, elastic vibration, heat and mass transform. It is worth noticing from the very beginning that the rate of convergence of this discrete approximation method is pretty high, which stimulates development of such studies.

Below we shall briefly present a mathematical background of this Lie algebraic discrete approximation scheme. Consider for simplicity the following evolution equation in an open, connected, and bounded region $\Omega \subset \mathbf{R}^2$:

$$du/dt = A(t)u + f(t; x, y) \tag{1}$$

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with Cauchy-Dirichlet data

$$u|_{t=0} = g, \quad u|_{\Gamma} = 0, \tag{2}$$

where $\Gamma = \partial\Omega := \{(x,y) \in \mathbf{R^2} : \ \varphi(x,y) = 0\}$, $\varphi \in C^2(\mathbf{R^2};\mathbf{R})$, $u \in W_p^q \subset \mathbf{B}$, $\mathbf{B}$ is a Banach space, $g \in \mathbf{B} \cap \overline{W}_{p,0}^q(\Omega)$ and for all $t \in \mathbf{R}_+$, $f(t) \in \mathbf{B'}$ and the operator $A(t) : W_p^q(\Omega) \to \mathbf{B'}$ is given as

$$A := A(t) = \sum_{i+j \le q(A)} a_{ij}(t;x,y) \frac{\partial^{i+j}}{\partial x^i \partial y^j},$$

where $q(A) < \infty$ is the rank of operator A, with smooth enough coefficients, $\mathbf{B'}$ is a Banach space. Space $\overline{W}_{p,0}^q(\Omega)$ is a Sobolev space of the functions satisfying the boundary condition $u|_{\Gamma} = 0$.

Consider now the following differential operations $\partial_x := \frac{\partial}{\partial x}$, $\partial_y := \frac{\partial}{\partial y}$ and the multiplication $\widehat{x} := x$, $\widehat{y} := y$ by independent variables $x, y \in \mathbf{R}$, acting naturally in $W_p^q(\Omega) \subset \mathbf{B}$ satisfying the relationships

$$\begin{gathered} [x,y] = [x,x] = [y,y] \equiv 0, \quad [\partial_x, \partial_x] = [\partial_y, \partial_y] = [\partial_x, \partial_y] \equiv 0, \\ [\partial_x, x] = [\partial_y, y] = 1, \quad [\partial_y, x] = [\partial_x, y] = 0, \end{gathered} \tag{3}$$

where $[\cdot\ ,\ \cdot]$ is the usual commutator of operators in $\mathbf{B}$. It follows from (3) that the set of operators $\mathcal{G} := \{1, x, \frac{\partial}{\partial x}\} \oplus \{1, y, \frac{\partial}{\partial y}\}$ forms a closed finite-dimensional Heisenberg-Weyl algebra. Making use of the Lie-algebraic discrete approximations scheme [**2, 4**] one gets from (1) a sequence of discrete evolution systems of the type

$$du_n/dt = A_n(t)u_n + f_n(t) \tag{4}$$

with $u_n \in \mathbf{B}_n$, $f_n \in \mathbf{B}_n^{'}$, belonging to specially constructed approximated Banach spaces, and whose solution can be found for each $n \in N$ by some known numerical method.

Suitable quasi-representations of the basic Heisenberg-Weyl algebra are calculated exactly for a given (Lagrange or Hermite) boundary problem by means of interpolation schemes applied to our Banach space $\mathbf{B}$ in which the operator A acts. More detailed description of the method discussed can be found in [**4**]. In paper [**2**] there was analyzed an approach to the Lie algebraic discrete approximation scheme based on the idea of transforming boundary conditions into the structure of the operator A via its representation. We developed here a new discrete approximation of the Cauchy-Dirichlet problem [**2, 4**], based on the idea of including boundary conditions into the corresponding interpolating spaces and subsequently evaluating the corresponding Heisenberg-Weyl algebra quasi-representation. This in particular, means that the boundary condition of the Cauchy-Dirichlet problem (1) is included into this specially found quasi-representation, which is important for practical use of the method proposed. It is necessary to notice that a similar approach can be also developed for the Cauchy-Neumann problem or mixed Dirichet-Neumann problem.

2. Cauchy-Dirichlet problem. Consider the evolution equation (1)

$$du/dt = A(t)u + f(t; x, y)$$

in an open bounded region $\Omega \subset \mathbf{R}^2$ with Cauchy-Dirichlet data (2), i.e.,

$$u|_{t=0} = g, \quad u|_{\Gamma} = 0.$$

Having performer the standard space extensions $W_p^{q(A)}(\Omega) \subset W_p^{q(A)}(D)$, $W_p^q(\Omega) \subset W_p^q(D)$ where $D \supset \Omega$ is the minimum two-dimensional rectangle containing Ω, one can define projectors $P_n : \widetilde{\mathbf{B}}_0 \to \widetilde{\mathbf{B}}_{n,0}$ with $\widetilde{\mathbf{B}}_0 := \overline{W}^q_{p,0}(D)$ onto $\widetilde{\mathbf{B}}_{n,0} := \prod_{n,0}(D; \mathbf{R})$, $n \in N$, being interpolating polynomials of degree n satisfying a priori the boundary condition of (2). If spaces $\widetilde{\mathbf{B}}_{n,0}$, $n \in N$, are just chosen as Lagrangian interpolating ones with the nodes $\chi_n = \{(x_j,\ y_k) \in D : j = \overline{0, n_x}, \quad k = \overline{0, n_y}\}$, $n = n_x n_y$ is the dimension of the space $\widetilde{\mathbf{B}}_{n,0}$, then one obtains the following equality for each $u \in \widetilde{\mathbf{B}}_{n,0} \cap C(D; \mathbf{R})$

$$(P_n u)(x, y) := \sum_{\substack{i=\overline{0,n_x} \\ j=\overline{0,n_y}}} u(x_i, y_j) l_{ij}^{(n)}(x, y), \tag{5}$$

where by definition

$$l_{ij}^{(n)}(x, y) := l_i^{(n_x)}(x) \otimes l_j^{(n_y)}(y) \tag{6}$$

for $(x, y) \in D$, $i = \overline{0, n_x}$, $j = \overline{0, n_y}$ and

$$l_i^{(n_x)}(x) := \frac{\omega_{n_x}(x)}{(x - x_i)\omega'_{n_x}(x_i)}, \quad l_j^{(n_y)}(y) := \frac{\omega_{n_y}(y)}{(y - y_j)\omega'_{n_y}(y_j)}, \tag{7}$$

where $\omega_{n_x}(x) := \prod_{k=0}^{n_x} (x - x_k)$, $\omega_{n_y}(y) := \prod_{k=0}^{n_y} (y - y_k)$. Polynomials $l_i^{(n_x)}(x)$, $l_j^{(n_y)}(y)$ are called the Lagrange influence polynomials.

The condition $P_n u \in \widetilde{\mathbf{B}}_{n,0}$, $n \in N$, that is $P_n u|_{\Gamma} = 0$ will be fulfilled a priori when points $(x_j, y_k) \in D$, $j = \overline{0, n_x}$, $k = \overline{0, n_y}$, are chosen in the following way:

$$\begin{aligned} \{x_j : j = \overline{0, n_x}\} &= \{\widetilde{x}_j : j = \overline{0, \alpha_x}\} \cup \{\overline{x}_{j(k)} : j_{(k)} = \overline{0, \alpha_y + \beta_y}\}, \\ \{y_j : j = \overline{0, n_y}\} &= \{\widetilde{y}_j : j = \overline{0, \alpha_y}\} \cup \left\{\overline{y}_{j(k)} : j_{(k)} = \overline{0, \alpha_x + \beta_x}\right\} \end{aligned} \tag{8}$$

and $\varphi(\widetilde{x}_j, \overline{y}_{j(k)}) = 0 = \varphi(\widetilde{y}_i, \overline{y}_{i(k)})$ for all $j = \overline{0, \alpha_x}$, $\quad k = \overline{0, \alpha_y}$, $k \in N$.

Thus one gets numerical equalities $n_x = \alpha_x + \alpha_y + \beta_x + 1, \quad n_y = \alpha_x + \alpha_y + \beta_y + 1$, where by definition, points $\{x_j : j = \overline{0, \alpha_x}\}$ and $\{y_j : j = \overline{0, \alpha_y}\}$ are chosen, in general arbitrarily, for instance as zeros of ranks α_x and $\alpha_y \in N$ Chebyshev's polynomials. Now one can write down expression (5) in the following form

$$
\begin{aligned}
(P_n u)(x,y) = &\sum_{\substack{i=\overline{0,\alpha_x}\\ j=\overline{0,\alpha_y}}} u(\widetilde{x}_i, \widetilde{y}_j) l_{ij}^{(n)}(x,y) + \sum_{i+j=\overline{0,\alpha_x+\beta_x}} u(\widetilde{x}_i, \overline{y}_j) l_{ij}^{(n)}(x,y) + \\
+ &\sum_{j+i=\overline{0,\alpha_y+\beta_y}} u(\overline{x}_i, \widetilde{y}_j) l_{ij}^{(n)}(x,y) + \sum_{\substack{i=\overline{0,\alpha_y}\\ j=\overline{0,\alpha_x}}} u(\overline{x}_i, \overline{y}_j) l_{ij}^{(n)}(x,y).
\end{aligned} \tag{9}
$$

From this formula the fundamental basic quasi-representation of the Heisenberg-Weyl algebra $\mathcal{G} = \{1, x, \partial_x\} \oplus \{1, y, \partial_y\}$ can be obtained. Namely in the finite dimensional subspace $\mathbf{B}_n \cong \widetilde{\mathbf{B}}_{n,0}$, $n \in N$, the multiplication operators $x, y : W_{p,0}^q(D) \to W_{p,0}^q(D)$ and differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} : W_{p,0}^q(D) \to W_{p,0}^{q-1}(D)$ possess the following matrix quasi-representations

$$
\begin{aligned}
x &\to X_0^{(n)} = X^{(n_x)} \otimes 1^{(n_y)} - \overline{X}^{(n)}, \\
y &\to Y_0^{(n)} = 1^{(n_x)} \otimes Y^{(n_y)} - \overline{Y}^{(n)}, \\
\tfrac{\partial}{\partial x} &\to Z_x^{(n)} = Z_x^{(n_x)} \otimes 1^{(n_y)} - \overline{Z}_x^{(n)}, \\
\tfrac{\partial}{\partial y} &\to Z_y^{(n)} = 1^{(n_x)} \otimes Z_y^{(n_y)} - \overline{Z}_y^{(n)},
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
\overline{X}^{(n)} &= \left\{ \overline{X}_{sr,ik}^{(n)} = \delta_{si} x_i^{(n_x)} \delta_{ri} \delta_{ik} \right\}_{\substack{i,s=\overline{0,n_x}\\ k,r=\overline{0,n_y}}}, \\
\overline{Y}^{(n)} &= \left\{ \overline{Y}_{sr,ik}^{(n)} = \delta_{si} \delta_{ik} y_k^{(n_y)} \delta_{rk} \right\}_{\substack{i,s=\overline{0,n_x}\\ k,r=\overline{0,n_y}}}, \\
\overline{Z}_x^{(n)} &= \left\{ \overline{Z}_{x|sr,ik}^{(n)} = Z_{x|si}^{(n_x)} \delta_{ri} \delta_{ik} \right\}_{\substack{i,s=\overline{0,n_x}\\ k,r=\overline{0,n_y}}}, \\
\overline{Z}_y^{(n)} &= \left\{ \overline{Z}_{y|sr,ik}^{(n)} = \delta_{si} Z_{y|rk}^{(n_y)} \delta_{ik} \right\}_{\substack{i,s=\overline{0,n_x}\\ k,r=\overline{0,n_y}}},
\end{aligned} \tag{11}
$$

with matrices $X^{(n_x)}$, $Y^{(n_y)}$, $Z_x^{(n_x)}$, $Z_y^{(n_y)}$ being found in [**4**] and equal to the following expressions

$$
\begin{aligned}
&X^{(n_x)} = diag(x_1, x_2, ..., x_{n_x}) = [x_i\ \delta_{ji}^{(n_x)}],\\
&Y^{(n_y)} = diag(y_1, y_2, ..., y_{n_y}) = [y_k\ \delta_{sk}^{(n_y)}],\\
&Z_x^{(n_x)} = \left[\delta_{ji}^{(n_x)} \sum_{m\neq i}^{n_x} (x_i - x_m)^{-1} + (1-\delta_{ji}^{(n_x)})(x_i - x_j)^{-1}\right],\\
&Z_y^{(n_y)} = \left[\delta_{sk}^{(n_y)} \sum_{m\neq k}^{n_y} (y_k - y_m)^{-1} + (1-\delta_{sk}^{(n_y)})(y_k - y_s)^{-1}\right],
\end{aligned}
\tag{12}
$$

where $i, j = 1, ..., n_x$, and $s, k = 1, ..., n_y$. While computing matrices (11), one needs to take into account that (8) holds.

With respect to representation (10) under the Dirichlet boundary condition of (2), equation (1) takes the following discretized form in the space $\widetilde{\mathbf{B}}_n$:

$$du_n/dt = A_n(t)u_n + f_n(t), \tag{13}$$

where the finite dimensional matrix $A_n(t) : \mathbf{B}_n \to \mathbf{B}_n$, $\quad n \in N$, $\quad t \in \mathbf{R}_+$, is defined as

$$A_n(t) := A(X_0^{(n)}, Y_0^{(n)}, Z_x^{(n)}, Z_y^{(n)}; t) \tag{14}$$

with use of the representation (10). Equation (13) is considered with the Cauchy data

$$u_n|_{t=0} = g_n, \tag{15}$$

where the vector $g_n \in \mathbf{B}_n$, $n \in N$, and vector $f_n(t) \in \mathbf{B}_n$ in (13) are found by means of projection mapping (5) and representations (10). Solving Cauchy problem (13), (15) in the space $\mathbf{B}_n$, $n \in N$, one finds vector $u_n(t) \in \mathbf{B}_n$, $t \in \mathbf{R}_+$ and by means of the inverse mapping $\pi_n^{-1} u_n(t) = \widetilde{u}_n(x, y; t) \in \widetilde{\mathbf{B}}_{n.0} \subset \widetilde{\mathbf{B}}_0$, $n \in N$, an approximate solution to Cauchy-Dirichlet problem (1), (2).

From above and the approximation scheme convergence theorem given in [**6**] we have the following theorem.

Theorem 1. *The approximate solution of the problem (13), (15) exists and converge to exact solution (1), (2).*

A proof of this fact is analogous to the proof given in [**6**] (pp. 298–300) for the approximation scheme convergence theorem.

3. Cauchy-Neumann problem. Consider now equation (1) with the Cauchy-Neumann data

$$u|_{t=0} = g, \qquad \left.\frac{\partial u}{\partial \vec{n}}\right|_{\Gamma} = 0, \tag{16}$$

where $\vec{n} \in \mathbf{R^2}$ is the unit vector normal to a given boundary curve Γ, satisfying the nondegeneracy condition $\nabla\varphi(x,y)|_\Gamma \neq 0$.

If a function $u \in W_p^q(\Omega) \subset \mathbf{B}$ fulfils the equation (1) with conditions (16), then using the standard Sobolev space extension theorems [**3, 5**] one can find an extension $\widetilde{u} \in W_p^q(D)$, defined on the minimal rectangle $D \supset \Omega$ and fulfilling condition (16) on the boundary $\Gamma = \partial\Omega$. Define now a Banach space $\widetilde{B}_0 \subset W_p^q(D)$ of the functions, which satisfy the Neumann condition (16) on the curve Γ. Assume also that the Cauchy condition $g \in \widetilde{\mathbf{B}}_0$ correspondingly extended to a function $g \in \widetilde{\mathbf{B}}_0$ is such that the equation (1) possesses a solution $u \in W_p^q(D) \cap \widetilde{\mathbf{B}}_0$. This means that one can apply the Hermite interpolation method, described above, to the space $\widetilde{\mathbf{B}}_0$.

Let spaces $\widetilde{\mathbf{B}}_{n,0} \subset \widetilde{\mathbf{B}}_0$, $\;n \in N$, be finite-dimensional interpolating subspaces with respect to the usual Hermite scheme, that is for each function $u \in \widetilde{\mathbf{B}}_0$ there are defined the projection operators $\widetilde{P}_n : \widetilde{\mathbf{B}}_0 \to \widetilde{\mathbf{B}}_{n,0} \subset \widetilde{\mathbf{B}}_0$, $\;n \in N$, acting as

$$\begin{aligned}(\widetilde{P}_n u)(x,y) = &\\ &\sum_{i,j=0}^{n_x,n_y} u(x_i,y_j) l_{i,0}^{(n_x)}(x) l_{j,0}^{(n_y)}(y) + \sum_{i,j=0}^{n_x,n_y} \frac{\partial u}{\partial x}(x_i,y_j) l_{i,1}^{(n_x)}(x) l_{j,0}^{(n_y)}(y) + \\ &+ \sum_{i,j=0}^{n_x,n_y} \frac{\partial u}{\partial y}(x_i,y_j) l_{i,0}^{(n_x)}(x) l_{j,1}^{(n_y)}(y) + \sum_{i,j=0}^{n_x,n_y} \frac{\partial^2 u}{\partial x \partial y}(x_i,y_j) l_{i,1}^{(n_x)}(x) l_{j,1}^{(n_y)}(y).\end{aligned} \tag{17}$$

Assume now also that for each $u \in \widetilde{\mathbf{B}}_0$, $\;n \in N$, the following condition is fulfilled: $\langle \nabla\varphi, \nabla u \rangle|_\Gamma = 0$, that is

$$\frac{\partial \varphi}{\partial x}(x,y)\frac{\partial u}{\partial x}(x,y) + \frac{\partial \varphi}{\partial y}(x,y)\frac{\partial u}{\partial y}(x,y) = 0 \tag{18}$$

for all $(x, y) \in \Gamma$. Condition (18) can by naturally included into (17), yielding the elements $P_n u \in \widetilde{\mathbf{B}}_{n,0},\ n \in N$

(19)
$$
\begin{aligned}
&(\widetilde{P}_n u)(x,y) = \\
&\sum_{i,j=0}^{n_x,n_y} u(x_i,y_j) l_{i,0}^{(n_x)}(x) l_{j,0}^{(n_y)}(y) + \sum_{i,j=0}^{n_x,n_y} \frac{\partial u}{\partial x}(x_i,y_j) l_{i,1}^{(n_x)}(x) l_{j,0}^{(n_y)}(y) + \\
&+ \sum_{i,j=0}^{n_x,n_y} \frac{\partial u}{\partial y}(x_i,y_j) l_{i,0}^{(n_x)}(x) l_{j,1}^{(n_y)}(y) + \sum_{i,j=0}^{n_x,n_y} \frac{\partial^2 u}{\partial x \partial y}(x_i,y_j) l_{i,1}^{(n_x)}(x) l_{j,1}^{(n_y)}(y) - \\
&- \sum_{i=0}^{\alpha_x} \frac{\partial u}{\partial y}(\widetilde{x}_i, \overline{y}_{j(k)}) l_{i,0}^{(n_x)}(x) l_{i,1}^{(n_y)}(y) - \sum_{j=0}^{\alpha_y} \frac{\partial u}{\partial y}(\overline{x}_{j(k)}, \widetilde{y}_j) l_{j,0}^{(n_x)}(x) l_{j,1}^{(n_y)}(y) - \\
&- \sum_{i=0}^{\alpha_x} \frac{\partial u}{\partial x}(\widetilde{x}_i, \overline{y}_{i(k)}) \Big(\frac{\partial \varphi(\widetilde{x}_i, \overline{y}_{i(k)})}{\partial x} / \frac{\partial \varphi(\widetilde{x}_i, \overline{y}_{i(k)})}{\partial y} \Big) l_{i,0}^{(n_x)}(x) l_{i,1}^{(n_y)}(y) - \\
&- \sum_{j=0}^{\alpha_y} \frac{\partial u}{\partial x}(\overline{x}_j, \widetilde{y}_j) \Big(\frac{\partial \varphi(\overline{x}_{j(k)}, \widetilde{y}_j)}{\partial x} / \frac{\partial \varphi(\overline{x}_{j(k)}, \widetilde{y}_j)}{\partial y} \Big) l_{j,0}^{(n_x)}(x) l_{j,1}^{(n_y)}(y),
\end{aligned}
$$

where the following partition set was chosen:

$$
\begin{aligned}
\{x_j :\ j = \overline{0, n_x}\} &= \{\widetilde{x}_j :\ j = \overline{0, \alpha_x}\} \cup \{\overline{x}_{j(k)} :\ j(k) = \overline{0, \alpha_y + \beta_y}\}, \\
\{y_j :\ j = \overline{0, n_y}\} &= \{\widetilde{y}_j :\ j = \overline{0, \alpha_y}\} \cup \Big\{\overline{y}_{j(k)} :\ j(k) = \overline{0, \alpha_x + \beta_x}\Big\},
\end{aligned}
$$

$i = \overline{0, \alpha_x},\ \ j = \overline{0, \alpha_y}$ and $\varphi(\widetilde{x}_i, \overline{y}_{i(k)}) = 0 = \varphi(\overline{x}_{j(k)}, \widetilde{y}_j)$ for all $n_x = \alpha_x + \alpha_y + \beta_y + 1$, $n_y = \alpha_x + \alpha_y + \beta_x + 1$.

As in the above analysis for the Cauchy-Dirichlet problem, from (19) one gets the existence of such a representation for the base Heisenberg-Weyl algebra $\mathcal{G}$ in the space $\widetilde{\mathbf{B}}_{n,0},\ \ n \in N$

(20)
$$
\begin{aligned}
&x \to X^{(n)} = X^{(n_x)} \otimes 1^{(n_y)} - \overline{X}^{(n)}, \\
&y \to Y^{(n)} = 1^{(n_x)} \otimes Y^{(n_y)} - \overline{Y}^{(n)}, \\
&\tfrac{\partial}{\partial x} \to Z_x^{(n)} = Z_x^{(n_x)} \otimes 1^{(n_y)} - \overline{Z}_x^{(n)}, \\
&\tfrac{\partial}{\partial y} \to Z_y^{(n)} = 1^{(n_x)} \otimes Z_y^{(n_y)} - \overline{Z}_y^{(n)},
\end{aligned}
$$

where matrices $\overline{X}^{(n)}$, $\overline{Y}^{(n)}$, $\overline{Z}_x^{(n)}$ and $\overline{Z}_y^{(n)}$ are found directly from expression (19) in the tensor form analogously to the formula (11). Thus, having now applied the quasi-representations (18) to the equation (1) projected upon the subspace $\mathbf{B}_n \cong \widetilde{\mathbf{B}}_{n,0},\ \ n \in N$, one finds the following a finite dimensional evolution system of equations in the space $\mathbf{B}_n,\ \ n \in N$

$$du_n/dt = A_n(t)u_n + f_n(t), \tag{21}$$

where, by definition

$$A_n := A(X^{(n)}, Y^{(n)}, Z_x^{(n)}, Z_y^{(n)}; t) : \mathbf{B}_n \to \mathbf{B}_n. \tag{22}$$

The initial condition for the system (21) is found directly with use of (19) in the case when $g \in \widetilde{\mathbf{B}}_0$

$$u_n|_{t=0} = g_n, \tag{23}$$

where $g_n := \pi_n \widetilde{P}_n g \in \mathbf{B}_n,\ \ n \in N$. Having solved the Cauchy problem (21), (23) in the space $\mathbf{B}_n$, by means of the inverse mapping $\pi_n^{-1} u_n(t) = \widetilde{u}_n(x, y; t) \in \widetilde{\mathbf{B}}_{n,0} \subset \widetilde{\mathbf{B}}_0$ one gets an approximate analytical solution to the Cauchy-Neumann problem (1), (16).

Now we can formulate the following theorem.

THEOREM 2. *The approximate solution of the problem (21), (23) exists and converge to the exact solution (1), (16).*

A Proof of this fact is also analogous to the proof given in [**6**].

It is easy to see that the described approximation scheme devised above for the Cauchy-Dirichlete problem (1), (2) and the Cauchy-Neumann problem (1), (16) is much more convenient in use than that in [**4**], leading at the same time to a rather complicated matrix expression (21).

References

1. Calogero F., *Integrable dynamical systems & related mathematical results,* Lecture Notes in Phys., **189** (1983), 46–109.
2. Luśtyk M., *Lie-algebraic approach to discrete approximations of a Neumann boundary problem for evolution equations, Abstract,* The Intern. Research Conference "Modern problems of mechanics and mathematics", Lviv, Ukraine, 25–28 May 1998.
3. Marcinkowska H., *Dystrybucje, Przestrzenie Sobolewa, Równania Różniczkowe,* PWN, Biblioteka Matematyczna, **75**, Warszawa, 1993.
4. Mitropolsky Yu.A., Prykarpatsky A.K., Samoilenko V.H., *An algebraic scheme of discrete approximations for evolution equations of mathematical physics,* Ukr. Math. Journal, **40** (1988), Plenum Press Publisher, N.Y, 453–458.
5. Sobolev S.L., *Some Applications of the Functional Analysis in Mathematical Physics,* Moscow, 1987.
6. Trenogin V.A., *Functional Analysis,* in Russian, Nauka, Moscow, 1980.

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University of Mining and Metallurgy
Faculty of Applied Mathematics
al. Mickiewicza 30
30-059 Kraków, Poland.
e-mail: lustyk@mat.agh.edu.pl