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ENTIRE CURVES IN COMPLEMENTS OF CARTESIAN PRODUCTS IN $\mathbb{C}^N$

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Abstract. It is proved that if F is the Cartesian product of n closed subsets $F_1, F_2, \dots, F_n$ of $\mathbb{C}$ ($n \geq 2$) with $F_1 \neq \mathbb{C}$ and $F_2 \neq \mathbb{C}$, then for any two different points $a, b \in D := \mathbb{C}^n \setminus F$ there is a holomorphic mapping $f : \mathbb{C} \to D$ such that $f(0) = a$ and $f(1) = b$.

The purpose of this note is to prove the following

Proposition 1. *Let F be the Cartesian product of n closed subsets $F_1, F_2, \dots, F_n$ of $\mathbb{C}$ ($n \geq 2$) with $F_1 \neq \mathbb{C}$ and $F_2 \neq \mathbb{C}$. Then for any two different points $a, b \in D := \mathbb{C}^n \setminus F$ there is a holomorphic mapping $f : \mathbb{C} \to D$ such that $f(0) = a$ and $f(1) = b$.*

In the particular case when $a \in (\mathbb{C} \setminus F_1) \times \mathbb{C}^{n-1}$ and $b \in \mathbb{C} \times (\mathbb{C} \setminus F_2) \times \mathbb{C}^{n-2}$, this proposition has been proved in [**1**] and the authors raised the question if it still holds for any two different points $a, b \in D := \mathbb{C}^n \setminus F$.

Proof. It suffices to prove the proposition for points $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ such that $a_2, b_2 \in D_2 := \mathbb{C} \setminus F_2$. It is trivial if $a_2 = b_2$. Let $a_2 \neq b_2$. Since $D_1 := \mathbb{C} \setminus F_1$ and D_2 are nonempty open sets, after linear changes in the first two complex planes, we may assume that D_1 contains the unit disc $\Delta \subset \mathbb{C}$, $a_1, b_1 \notin \Delta$, $a_2 = 1, b_2 = -1$, and $D_2 \supset G := \{z : |z-1| < \varepsilon \text{ or } |z+1| < \varepsilon\}$ for some $\varepsilon > 0$. Let

$$g_1(z) := \frac{1 - \exp(-z^2)}{z^2}, \ g_2(z) := \frac{(1 - \exp(-z^2))^2}{z^3},$$
$$h_j(z) := z \int_0^\lambda g_j(zt)dt, \ j = 1, 2, \ \hat{f}_1(z) := \exp(2z^2 - 1)$$

(we shall choose the number $\lambda > 0$ later on). Note that the set $A := \{z \in \mathbb{C} : |\hat{f}_1(z)| \geq 1\}$ is the union of the sets $A_1 := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, 2Re(z^2) \geq 1\}$

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and $A_2 := \{z \in \mathbb{C} : \operatorname{Re}(z) < 0, 2\operatorname{Re}(z^2) \geq 1\}$. Then there exist numbers $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$ such that $\hat{f}_1(\alpha_1) = a_1$ and $\hat{f}_1(\alpha_2) = b_1$. Let $z \in A_1$. Since $g_1(u)$ and $g_2(u)$ are entire functions, we have

$$\begin{aligned}|h_1(z) - h_1(1)| = &\left|\int_\lambda^{\lambda z} g_1(u)du\right| < 2\frac{|z-1|}{\lambda}\int_0^1 \frac{dt}{|1+(z-1)t|^2} \\ \leq &2\frac{|z-1|}{\lambda}\int_0^1 \frac{dt}{(1+\operatorname{Re}(z-1)t)^2} = \frac{2|z-1|}{\lambda\operatorname{Re}(z)} < \frac{2\sqrt{2}}{\lambda},\end{aligned} \tag{1}$$

and

$$\begin{aligned}|h_2(z) - h_2(1)| = &\left|\int_\lambda^{\lambda z} g_2(u)du\right| < 4\frac{|z-1|}{\lambda^2}\int_0^1 \frac{dt}{|1+(z-1)t|^3} \\ \leq &4\frac{|z-1|}{\lambda^2}\int_0^1 \frac{dt}{(1+\operatorname{Re}(z-1)t)^3} = 2\frac{|z-1|\operatorname{Re}(z+1)}{(\lambda\operatorname{Re}(z))^2} < \frac{4}{\lambda^2}.\end{aligned} \tag{2}$$

Analogously, if $z \in A_2$, then

$$|h_1(z) - h_1(-1)| < \frac{2\sqrt{2}}{\lambda} \text{ and } |h_2(z) - h_2(-1)| < \frac{4}{\lambda^2}. \tag{3}$$

Note that
$$h_1(1) = -h_1(-1) \underset{\lambda\to\infty}{\longrightarrow} d_1 := \int_0^\infty g_1(t)dt > 0$$
and
$$h_2(1) = h_2(-1) \underset{\lambda\to\infty}{\longrightarrow} d_2 := \int_0^\infty g_2(t)dt > 0.$$

Now, it follows from (1), (2), (3), and the triangle inequality that for any $\lambda \gg 1$, we may find constants c_1 and c_2 ((c_1, c_2) tends to the solution of the system $d_1x_1 + d_2x_2 = 1$, $-d_1x_1 + d_2x_2 = -1$, when $\lambda \to \infty$) such that if $\hat{f}_2 = c_1h_1 + c_2h_2$, then $\hat{f}_2(\alpha_1) = 1$, $\hat{f}_2(\alpha_2) = -1$, $|\hat{f}_2(z) - 1| < \varepsilon$ for $z \in A_1$, and $|\hat{f}_2(z) + 1| < \varepsilon$ for $z \in A_2$. Set $l(z) = (\alpha_2 - \alpha_1)z + \alpha_1$, $f_j(z) = \hat{f}_j(l(z))$ for $j = 1, 2$, and $f_j(z) = (b_j - a_j)z + a_j$ for $3 \leq j \leq n$. Then the mapping $f := (f_1, f_2, \ldots, f_n)$ has the required properties. $\square$

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References

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