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SUPERFIXED POSITIONS IN THE GEOMETRY OF GOURSAT FLAGS

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Abstract. In the local classification of Goursat flags, far from being complete, there appear numerical invariants (moduli). First examples of them were given in 1997 ([**12**], [**7**]). For the first of them, c, appearing in flag's member D^9 of corank 9, we give a *geometric* interpretation of one of its non-zero values. In fact, after standard normalizations, the value $c_0 = 0.740625$ is geometrically discerned from all other values including 0 – it is more singular than the others (also more singular than the value 0). It has far-reaching consequences for the local classification of 4–step Goursat prolongations, say D^{13}, of such germs D^9: when $c \neq c_0$, classification is the simplest possible one within that fixed value of c, while for $c = c_0$ a new module appears, closely related to the position of D^{13} at the reference point p. Repeating in other terms: all possible positions of D^9 at p (corresponding to all values of the invariant c, plus the vertical position $c = \infty$) are *fixed* in the sense of [**6**], while the position corresponding to c_0 is also 'doubly' fixed due to the – then singular – behaviour of D^9.

1. Geometric classes, prolongations, and interesting distances.

Goursat flags are certain special nested sequences, say $\mathcal{F}$, of variable length r $(2 \leq r \leq n-2)$ of subbundles in the tangent bundle TM to a smooth (C^∞) or analytic (C^ω) n-dimensional manifold M: $D^r \subset D^{r-1} \subset \cdots \subset D^1 \subset D^0 = TM$. Namely, one demands, for $l = r, r-1, \ldots, 1$, that (a) $\operatorname{cork} D^l = l$, and (b) the Lie square of D^l be D^{l-1}. Every member of $\mathcal{F}$ save D^1 is called *Goursat distribution.*

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For the **local classification problem** (and also for a much rougher characterization of Goursat germs – basic geometry recalled below), only length r is important: there exists a local reduction, due to E. Cartan, to rank–2 Goursat distributions on $(r+2)$-dimensional manifolds. A turning point has been the work [**5**] putting forward a multi-parameter family of local writings (so-called *KR pseudo-normal forms*) for Goursat distributions, recalled in Thm. 1 below. Those forms are so important because they feature parameters only, and no functional moduli so common for general distribution germs. This can be perceived as a trade off: KR pseudo-normal forms *at the price of* highly restricting conditions defining Goursat objects. But the cornucopia of those forms, not only simplest among them, germs at $0 \in \mathbb{R}^n(x^1, x^2, \dots, x^{r+2}; x^{r+3}, \dots, x^n)$ of

$$\text{(C)} \qquad dx^2 - x^3dx^1 = dx^3 - x^4dx^1 = \cdots = dx^{r+1} - x^{r+2}dx^1 = 0\,,$$

believed by von Weber – the inventor of the Goursat condition – to have locally described all flags of length r (*cf.* Thm. VI in [**13**]).

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1.1. Geometric classes and their codes. As a matter of fact, Kumpera and Ruiz discovered *singularities* hidden in flags and not showing up in the family of generic models (C). The first version of a coordinate-free definition of them was given in [**1**], p. 455. In [**6**] first order singularities of Goursat flags were defined in a canonical way, using systematically the associated *subflag* of Cauchy characteristic subdistributions. (In general, to a distribution D there is associated the sheaf $L(D)$ of local Cauchy characteristic modules of v. f.'s which, for any member of a Goursat flag, is a regular codimension–2 subdistribution of that member.) Namely, excepting D^1 and D^2, D^k is in the basic singular position at a point p when it coincides with the Cauchy characteristics, $L(D^{k-2})$, of D^{k-2} at that point: $D^k(p) = L(D^{k-2})(p)$. $D^3, D^4, \dots, D^r$ can be in basic singular positions independently of one another giving rise to 2^{r-2} rough invariant *KR classes* of flag's germs. By analyzing singularities of higher orders, Montgomery and Zhitomirskii define finer *geometric classes* of germs of Goursat flags.[1] Since their collaboration in 1999, they have been labelling those classes by words GG… of length r over the alphabet $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$, with: letters S at places where basic singular positions hold at p, letters T meaning possible *tangent* positions of flag's members, and G – *generic* positions (*cf.* Sec. 1.3 of [**11**] for a detailed description). The only restrictions in

[1] Higher order singularities of flags are implicitly present already in [**4**] – see Rem. 1 below; in [**6**] they are explicitly called *tangent*.

the labels are: – there are two G's in the beginning, – a T never goes directly after a G.
The classes are, obviously, pairwise disjoint and invariant under the action of the local diffeomorphisms between manifolds of the same dimension $n \geq r+2$; the class GG…G with r letters G is, for fixed n, the fattest single orbit with the chained system (C) as a model.

Remark 1. The geometric classes have, basically, been created already in [**4**]. Jean considered a kinematic model of a car drawing a given number N of attached passive trailers, representing a rank–2 Goursat distribution, say D, on the configuration space $\Sigma^N = \mathbb{R}^2 \times (S^1)^{N+1}$. He precisely described a stratification of Σ^N by different 'regions' defined in terms of his critical angles $a_1 = \frac{\pi}{2}$, $a_{j+1} = \arctan(\sin(a_j))$, $j = 1, 2, 3, \ldots$, and proved that the germs of D at points of any fixed stratum have the same small growth vector. Jean's strata are nothing but the geometric classes of the germs of D at different points of Σ^N. They can be encoded — this is done in [**2**], Chap. 6 — with the words of length $N-1$ over $\{1, 2, 3\}$ s. t. a 1 never goes directly after a 2. The respective geometric class is obtained from the stratum word via the translation $1 \to \mathrm{T},\ 2 \to \mathrm{G},\ 3 \to \mathrm{S}$ and adding two extra G's on the left of the code.

Often, instead of 'belongs to a class $\mathcal{C}$', we will say 'has the *basic geometry* $\mathcal{C}$'. In this terminology, Jean's pioneering contribution can be described as finding, in the trigonometric presentation (or disguise) using strings of trailers, the natural stratification of Goursat germs by their different basic geometries.

The essence of the Kumpera-Ruiz construction, for the germs in a geometric class $\mathcal{C}$ (the concept, naturally, absent in [**5**]), is as follows. When $\mathcal{C}$ starts with s letters G, one puts $\overset{s+1}{Y} = p_1 + x^3\partial_2 + \cdots + x^{s+2}\partial_{s+1}$. When $s < r$, the $(s+1)$-th letter in $\mathcal{C}$ is S. More generally, if the m-th letter in $\mathcal{C}$ is S, and $\overset{m}{Y}$ is already defined, then

$$\overset{m+1}{Y} = x^{m+2}\overset{m}{Y} + \partial_{m+1}\,. \tag{1}$$

But there can also be T's or G's after an S. If the m-th letter in $\mathcal{C}$ is not an S, and $\overset{m}{Y}$ is already defined, then

$$\overset{m+1}{Y} = \overset{m}{Y} + \big(c^{m+2} + x^{m+2}\big)\partial_{m+1}\,. \tag{2}$$

The gist of KR pseudo-normal forms is that c^{m+2} is not absolutely free but

- equal to 0 when the m-th letter in $\mathcal{C}$ is T,
- not equal to 0 when the m-th letter is G going directly after a string ST…T (or after a short string S).

Remark. In the sequel we will write shortly $X^{m+2} = c^{m+2} + x^{m+2}$.
In this language the result of [**5**] reads

Theorem 1 ([**5**]). *Any Goursat germ D of corank r on a manifold of dimension n, sitting in a geometric class $\mathcal{C}$, can be put (in certain local coordinates) in a form $D = \big(\overset{r+1}{Y}, \partial_{r+2}; \partial_{r+3}, \ldots, \partial_n\big)$ understood as the germ at $0 \in \mathbb{R}^n$, with certain constants in the field $\overset{r+1}{Y}$ corresponding to the letters* G *past the first* S *in the code of $\mathcal{C}$.*

One easily recognizes (C) in this theorem – for $\mathcal{C} = \mathrm{GG}\ldots\mathrm{G}$ with r letters G.

The codimensions of Jean strata are easily computable: they are equal to the number of letters S and T in the relevant encoding words. Singularities of codimension 1 are simple; they have been classified, for all lengths r, in [**10**]. On manifolds of fixed dimension, the germs in each geometric class $\mathrm{G}_{k-1}\mathrm{SG}_{r-k}$, $3 \le k \le r$, are all mutually equivalent. (The subscript means in this context, **also in the sequel**, the number of repetitions of a letter in the code.) Certain, but not all, singularities of codimension 2 are simple, too ([**11**]). The present paper investigates one series of singularities of codimension 3.

1.2. Certain flags' members more involved than others.
In [**6**], Chap. 3 there is a systematization proposed concerning all theoretically possible manners of local prolongations of flags: from a given length r to length $r+1$. That systematization puts in proper places existing classification results for G. germs, such as [**5**], [**3**], [**2**], [**7**], [**8**], [**9**], [**12**], [**11**].
The authors have separated five distinctly different cases, Possibilities I – V, by considering a longer flag $D^1 \supset D^2 \supset \cdots \supset D^r \supset D^{r+1}$ and analyzing the properties of local symmetries ϕ of the shorter flag of D^r, around a point p preserved by ϕ. All such ϕ induce a subgroup $\Gamma \subset PGl(2)$ consisting of projectivities of $S^1 = \mathbb{P}\big(D^r(p)/L(D^r)(p)\big)$, always having one fixed point $K = L(D^{r-1})(p)/L(D^r)(p)$, sometimes having also a second fixed point L, and sometimes being just $\{\mathrm{id}\}$. (Note that this contribution was already recalled in [**9**]; see also Sect. 1.6 in [**11**].)

Example 1. It follows from [**10**] (*cf.* Rem. 4 there) that prolonging any germ having basic geometry $\mathrm{G}_{k-1}\mathrm{SG}_{r-k}$, $3 \le k < r$, is always Possibility I. Prolonging a germ in the class $\mathrm{G}_{r-1}\mathrm{S}$ is Poss. II, with the second fixed point L given by the tangent position of $D^{r+1}(p)$ wrt the locus of 1st order singularity.

The hard core of the classification problem resides in predicting how Possibilities I through V *can* interweave when prolongations are done one after another. In [**8**] *interesting distances* were defined that enrich existing patterns of successions of Possibilities. To exemplify, referring always to [**8**], the germs in the class G_3ST_2G prolong by Poss. II to the ones having the code extended by: S (fixed point K), G^0 (fixed point L), G^* (the complement $S^1 \setminus \{K,\ L\}$). Then, at the next step, as shown in [**12**], p. 165, $G_3ST_2GG^*$ prolongs by Poss. V to all pairwise non-equivalent positions of $D^9(\cdot)$. While $G_3ST_2GG^0$ prolongs by Poss. III to: S (point K), G^0 (point L), G^+, G^- (the two connected components of $S^1 \setminus \{K,\ L\}$). All the above information can be put in a single line as

$$G_3ST_2G \xrightarrow{\text{II}} \begin{cases} G_3ST_2GG^* \xrightarrow{\text{V}} G_3ST_2GG^*G \quad {}^2 \\ G_3ST_2GG^0 \xrightarrow{\text{III}} \end{cases}$$

What at the next step? $G_3ST_2GG^*G$ prolongs by Poss. I just to S and G, and even a stronger statement, plus two other akin to it, hold:

$$G_3ST_2G_3 \xrightarrow{\text{I}} G_3ST_2G_4 \xrightarrow{\text{I}} G_3ST_2G_5 \xrightarrow{\text{I}} G_3ST_2G_6\,.$$

But there is no continuation of this pattern! In fact, the subfamily

$$G_3ST_2GG^0G^0G_3 \subset G_3ST_2G_6 \tag{3}$$

prolongs by Poss. III to S, G^0, G^+, G^-.
After that the prolongations of the entire geometric classes $G_3ST_2G_j$, $j = 7, 8, 9$ are by Poss. I, but then there awaits one last surprise: the germs in the subfamily $G_3ST_2GG^0G^0G_3G^0G_3$ prolong again by Poss. III to: new S, G^0, G^+, and G^-. Eventually the regular pattern '$G_3ST_2G_j$ prolongs by Poss. I' returns forever – for $j \geq 11$.

Things being so, certain positions in the codes above are somehow particular: those i steps after, $i = 1, 2, 6, 10$ (or: at the distance i from) the block G_3ST_2G that starts the code. These are instances of *interesting distances* introduced in [**8**]. What is the underlying principle? For $j = 2$ (two T's in a row in the codes) and k such that $k + 2 = \#$ of G's in the beginning of the code, Thm. 4.1 in [**8**] says that i is interesting iff

$$i - 1 - 4k,\ i,\ i + 1 \notin \left(4,\ 5 + 4k\right)_{\mathbb{Z}^+}\,, \tag{4}$$

the semigroup of all nonnegative integer combinations of 4 and $5 + 4k$ (*cf.* Def. 4.2 in [**8**]). For $k = 0$ only $i = 2$ fulfils (4), for $k = 1$ — only already mentioned $i = 1, 2, 6, 10$; for $k = 2$ — only $i = 1, 2, 5, 6, 10, 14, 18$, etc.

All this is a necessary background before formulating Thm. 2 in the next chapter. We will only deal with $k = 1$, and it will turn out that the interesting distance $i = 6$ conceals a remarkable singular geometrical phenomenon.

[2] This last letter G after G^* means anything (any position) but the new S at the end. Analogous shorthand encoding convention will be applied when continuing the codes after G^0, G^+, G^-, too.

2. Infinitesimal symmetries of Goursat flags and Main Theorem.

By an automorphism of a distribution D on M we mean any diffeomorphism $g:\ M \hookleftarrow$ sending D to itself: $g_*D = D$. By *infinitesimal automorphism* (or: infinitesimal symmetry, in the sequel we write i. a. for short) of D we mean any vector field $\mathcal{Y}$ on M – of the same class of smoothness as M – whose flow, at least for times small in absolute value, preserves D. In other words, such that $[\mathcal{Y},\, X] \in D$ for all vector fields X with values in D.
We denote by $\mathcal{I}(D)$ the set of all i. a.'s of D. By definition, the *symmetry dimension* $\mathrm{SD}_p(D)$ of D at p is the linear dimension of $\mathcal{I}(D)$ at p.

Example 2. The splitting of the geometric class $\mathrm{G_3ST_2G_2}$ used in Chap. 1 and caused by the apparition of a second fixed point L not definable in the G, S, T language,

$$\mathrm{G_3ST_2G_2} = \mathrm{G_3ST_2GG^*} \cup \mathrm{G_3ST_2GG^0}\,,$$

stems from the behaviour of the symmetry dimension of the corank–8 G. germs. Namely, if it is, say, D^8 at $p \in M^n$, then, by Thm. 1, $D^8 = \big(\partial_n, \dots,\, \partial_{11};\, \partial_{10}, \overset{9}{Y}\big)$, with

$$(5)\qquad \overset{9}{Y} = \overset{8}{Y} + (b + x^{10})\partial_9\,, \qquad \text{where} \qquad \overset{8}{Y} = \overset{7}{Y} + (a + x^9)\partial_8 \qquad \text{with}$$

$$(6)\qquad a \neq 0$$

and $\overset{7}{Y} = x^6\overset{4}{Y} + \partial_5 + x^7\partial_6 + x^8\partial_7$, $\overset{4}{Y} = \partial_1 + x^3\partial_2 + x^4\partial_3 + x^5\partial_4$. How to interpret the parameter b? It is the second, in general non-zero, constant in this family D^8 of KR pseudo-normal forms. Precisely such situations are discussed in [**8**]. By the formulas (31) and (33) there, $\mathrm{SD}_p(D^8) = n - 4$ iff $b = 0$. The positions of $D^8(p)$ yielding this (smaller than typical) SD give those second fixed points L in the prolongation by Poss. II of $\mathrm{G_3ST_2G}$, and such germs D^8 sit in the singularity set $\mathrm{G_3ST_2GG^0}$ (see Sec. 1.2).
Whereas $\mathrm{SD}_p(D^8) = n - 3$ iff

$$(7)\qquad b \neq 0\,.^3$$

Assuming (7), i. e., avoiding the singular position L, is the very environment, encoded in Chap. 1 as $\mathrm{G_3ST_2GG^*}$, that gives rise to the module of [**12**] in the next prolongation step. Recalling, for $D^8 \in \mathrm{G_3ST_2GG^*}$ — a germ at $p \in M$, and D^9 – its prolongation, all possible positions of $D^9(p)$ are frozen in the sense of [**6**]: all points on the circle $\mathbb{P}\big(D^8(p)/L(D^8)(p)\big)$ are fixed because of Possibility V governing this prolongation.

[3] *Cf.* also Sec. 5.3 in [**11**] where similar computations were done for codimension–2 germs, not codimension–3 like here

2.1. Main Theorem.

THEOREM 2 (Main Theorem). *For every Goursat germ* $D^8 \in \mathrm{G_3ST_2GG^*}$ *at* $p \in M$ *of dimension* $n \geq 15$*, there exists precisely one (‘***superfixed***’) position* $N \subset T_pM$ *possible to be assumed by* D^9 *at* p, $N \neq L(D^7)(p)$*, such that*

$$\mathrm{SD}_p(D^{13}) = \begin{cases} n-5\,, & \text{when } D^9(p) = N\,, \\ n-4\,, & \text{when } D^9(p) \neq N \end{cases} \tag{8}$$

for every *local prolongation* D^{13} *of* D^8 *sitting in* $\mathrm{G_3ST_2GG^*G_5}$.
In fact, when D^9 *around* p *is given in a KR pseudo-normal form prolonging the visualisation* (5) *for* D^8*,*

$$\overset{10}{Y} = \overset{9}{Y} + (c + x^{11})\partial_{10}\,, \tag{9}$$

then that superfixed N *is defined by*

$$c = c(a,\, b) = 0.740625\, b^2 a^{-1} \tag{10}$$

(*cf.* (6), (7)) *and so corresponds to a* non-zero *value of the last constant* c*. In particular, when (after an appropriate rescaling of the KR coordinates)* a *and* b *are normalized to* 1*, then* N *is given by* $c_0 = 0.740625 = \frac{237}{320}$.

A proof will be given in Chapter 3.

2.2. Recursive formulas for the infinitesimal symmetries. Let us start with **any** KR pseudo-normal form for the flag of D^{13} around p in Thm. 2 **that extends** (9). We mean following (9) by $\overset{j+1}{Y} = \overset{j}{Y} + X^{j+2}\partial_{j+1}$, $j = 10, \ldots, 13$. The passive variables $x^{16}, \ldots, x^n$ are not visible. We will understand this D^{13} as a finite object in the vicinity of $0 \in \mathbb{R}^n$, not as the germ at 0, and will compute the i. a.’s of it near 0, especially those whose flows move 0 (they are important for the symmetry dimension at 0).
Every KR pseudo-normal form is a sequence of relatively simple extensions of a contact structure in Darboux local form $\omega = dx^2 - x^3dx^1 = 0$ on $\mathbb{R}^3$. It was observed by S. Lie that the infinitesimal automorphisms of $\omega = 0$ are generated by all C^∞ (or analytic, depending on the chosen category) functions $f(x^1,\, x^2,\, x^3)$. Those generating functions are called *contact Hamiltonians.* It turns out that the i. a.’s of KR pseudo-normal forms are sequences of relatively simple prolongations of the automorphisms of the Darboux structure, and inherit the fact of being locally 1–1 parametrized by C^∞ or C^ω functions f in three variables. We set apart the first three components of an i. a. $\mathcal{Y}_f$,

$$\mathcal{Y}_f = A\partial_1 + B\partial_2 + C\partial_3 + \sum_{l=4}^{15} F^l\partial_l + \sum_{l=16}^{n} F^l\partial_l\,, \tag{11}$$

as well as the last (passive, not depending on f, making – when $n > 15$ – the infinity of such v. f.'s associated to every one function f) group of components sitting in $L(D^{13})$. So A, B, C depend only on x^1, x^2, x^3, and the vector field $A\partial_1 + B\partial_2 + C\partial_3$ is an i. a. of $dx^2 - x^3 dx^1 = 0$. Hence the classical expressions

$$(12) \qquad A = -f_3\,, \qquad B = f - x^3 f_3\,, \qquad C = f_1 + x^3 f_2\,.$$

The formulas issuing, in the case of the present D^{13}, from [**8**] are as follows, using the shorthand notation $\overset{4}{Y} = y$ and $\overset{14}{Y} = Y$.

$$(13) \qquad F^4 = yC - x^4 yA\,, \qquad F^5 = yF^4 - x^5 yA\,.$$

Now, since x^6 comes in in the distinguished way (1), F^6 stands apart,

$$(14) \qquad F^6 = x^6\big(yA - YF^5\big)\,.$$

Later certain regular pattern reappears,

$$(15) \qquad F^7 = YF^6 - x^7\,YF^5\,, \qquad F^8 = YF^7 - x^8\,YF^5\,,$$

and, using the introduced brief notation for shifted variables,

$$(16) \qquad F^l = YF^{l-1} - X^l\,YF^5 \quad \text{for} \quad l = 9,\, 10, \ldots,\, 15.$$

Example 3. As a first illustration, we use both formulas in (13) to express F^5 by A and C (hence by f ultimately),

$$F^5 \;=\; yF^4 - x^5 yA \;=\; y(yC - x^4 yA) - x^5 yA \;=\; y^2 C - 2x^5 yA - x^4 y^2 A\,.$$

Secondly, we supply *certain* (not eventually simplified) formula for $F^{12}\,|0$, where the sign $|0$ means, **here and in the sequel**, the evaluation at 0. Via (16), F^{12} gets expressed by Y, F^{11} and F^5, then by Y, F^{10} and F^5, and so on. Remembering that upon evaluating at 0 many terms vanish, $F^{12}\,|0 =$

(17)

$$Y^6\big(x^6(yA - YF^5)\big) - 15X^9Y^4F^5 - 20X^{10}Y^3F^5 - 15X^{11}Y^2F^5 - 6X^{12}YF^5\,|0\,.$$

2.3. Multiplicities and abstract weights. In view of expressions like (17), in order to obtain tractable formulas for $F^{12}\,|0$ and the like, one should know *how many* consecutive derivations of x^6, x^5, $x^4, \ldots$ wrt Y yield a non-zero output. That question was settled, in more general context, in [**8**], Def. 6.1. Minimal such numbers were there called the *multiplicities*, $\mu(\cdot)$, of x^6, x^5, $x^4, \ldots$ So we just substitute $j = 2$ and $k = 1$ to that definition, hence put

$$\mu(x^5) = 1, \quad \mu(x^4) = 5, \quad \mu(x^3) = 9, \quad \mu(x^2) = 13, \quad \mu(x^1) = 4,$$
$$\mu(x^6) = 3, \quad \mu(x^7) = 2, \quad \mu(x^8) = 1, \quad \mu(X^9) = 0.$$

In fact, in [**8**] we went further, declaring (Def. 6.2) that the versors ∂_l (here $l = 1, \ldots, 9$) have *orders* $\nu(\partial_l) = \mu(x^l)$. Then further still (Def. 6.3), artificially extending the above definitions to $\mu(X^l) = \nu(\partial_l) = 9 - l$ for $l = 10,\, 11, \ldots,\, 15$

in order to have the entire vector field Y *homogeneous* of order 1. That is to say,

(18)
$$\nu(\partial_l) - \Big(\sum \text{multipl. of } x^j \text{ or } X^j \text{ being factors in } l\text{th component of } Y\Big) = 1$$

for $l = 1, 2, \ldots, 14$. Why is it useful? Look at any of A, B, C or any of F^l, $4 \le l \le 15$. It is a polynomial in $x^3, x^4, \ldots, x^l$ with coefficients – integer combinations of certain partials of f. Those coefficients are to be treated as purely formal expressions (that for certain simple f's can even identically vanish), and only an *abstract* weight is to be associated with them! As in [**8**], Def. 7.1, to any monomial $x^I X^J f_K$ we attach its **abstract weight**

$$w(x^I X^J f_K) = \sum_{k \in K} \nu(\partial_k) - \sum_{j \in J} \mu(X^j) - \sum_{i \in I} \mu(x^i) \tag{19}$$

(multiindices vary: I between 3 and 8, J between 9 and 15, K between 1 and 3). Then the derivative $Y\big(x^I X^J f_K\big)$ is a polynomial with all terms of abstract weight $1 + w(x^I X^J f_K)$, because the two quantities (18) and (19) just add.

The vector field y is also homogeneous of order $\nu(y) = \nu(Y) + \mu(x^6) = 4$. In consequence (*cf.* Obs. 7.2 in [**8**]) all polynomials A, B, C, $F^4, \ldots, F^{15}$ are homogeneous of abstract weights 9, 0, 4, 8, 12, 10, 11, 12, …, 19, respectively.

Observation 1. (i) *To the value at* 0 *of any of* A, B, C, F^l, $l = 4, \ldots, 8$, *there can contribute only those partials of* f *with abstract weight equal to that of the component function in question.*

(ii) *To the values at* 0 *of* F^l, $l = 9, \ldots, 15$, *there can contribute only those partials of* f *with abstract weight not greater than* $w(F^l)$.

Proof. The terms contributing at 0 are free of factors $x^3, \ldots, x^8$. Hence they are among the free terms in (i), and in (ii) – among the terms with only $X^9, \ldots, X^l$ as factors. These shifted variables have non-positive multiplicities that are *subtracted* in (19). □

Guided by Obs. 1, we ask, therefore, what partials of f have weights not exceeding $19 = w(F^{15})$. The answer, when the orders $\nu(\partial_1)$, $\nu(\partial_2)$, $\nu(\partial_3)$ are known, is almost immediate,

weight:	0	4	8	9	12	13	14	15	16	17	18	19
partials of f:	f	f_1	f_{11}	f_3	f_{111}	f_2			f_{1111}	f_{12}	f_{33}	
						f_{13}				f_{113}		

Example 4. The line of computation interrupted in (17) has the following continuation:

$$F^{12}\,|\,0 \;=\; -\,35(X^9)^2 f_{1111} - 15X^{12} f_{13}\,|\,0\,. \tag{20}$$

This answer illustrates Obs. 1, (ii). To the evaluation at 0 of F^{12} – a polynomial of weight 16 – there only contribute the partials: f_{1111} of weight 16 and f_{13} of weight 13 (the latter, because X^{12}, of multiplicity -3, stands by it).

3. Proof of Theorem 2.

It follows immediately from (12) and (13) that $A\,|\,0 = -\,f_3\,|\,0$, $B\,|\,0 = f\,|\,0$, $C\,|\,0 = f_1\,|\,0$, $F^4\,|\,0 = f_{11}\,|\,0$, $F^5\,|\,0 = f_{111}\,|\,0$. On the other hand, $F^6\,|\,0 = 0$ because, by (14), $F^6 \in (x^6)$, the ideal generated by the function x^6. Substituting the expression for F^6 to (15),

$$F^7 = x^7\big(yA - YF^5\big) + x^6\,Y\big(yA - YF^5\big) - x^7\,YF^5 \in (x^6,\, x^7)\,, \tag{21}$$

while substituting the middle expression in (21) to (15),

$$\begin{aligned} F^8 = x^8\big(yA - YF^5\big) + 2x^7Y\big(yA - YF^5\big) + x^6\,Y^2\big(yA - YF^5\big) + \\ -\,2x^8\,YF^5 - x^7\,Y^2F^5 \;\in\; (x^6,\, x^7,\, x^8)\,. \end{aligned}$$

Therefore, $F^7\,|\,0 = F^8\,|\,0 = 0$ (the necessary preservation of the stratum $\mathrm{G_3ST_2}$).

In the present stuation, $F^9\,|\,0$ and $F^{10}\,|\,0$ have already been computed in [**8**]:

$$F^9\,|\,0 \;=\; -\,a(4f_2 + 13f_{13})\,|\,0\,, \qquad F^{10}\,|\,0 \;=\; -\,b(5f_2 + 16f_{13})\,|\,0\,. \tag{22}$$

Passing to $F^{11}\,|\,0$, its weight is 15 and the table shows that there is no partial of this weight. Thus, by Obs. 1, this quantity is a combination of partials of weights $<$ 15, hence $\leq$ 13. At this point it is instrumental to introduce a uniform notation. We define

$$\begin{aligned} &p_1 = -\,f_3\,|\,0\,, \quad p_2 = f\,|\,0\,, \quad p_3 = f_1\,|\,0, \quad p_4 = f_{11}\,|\,0\,, \;\; p_5 = f_{111}\,|\,0, \\ &p_6 = -\,a(4f_2 + 13f_{13})\,|\,0\,, \quad p_7 = -\,b(5f_2 + 16f_{13})\,|\,0\,. \end{aligned}$$

Under (6) and (7), $f_2\,|\,0$ and $f_{13}\,|\,0$ are expressable by p_6 and p_7, and

$$F^{11}\,|\,0 \;=\; \text{a combination of } p_1,\, p_2, \ldots,\, p_7\,. \tag{23}$$

Now, putting $p_8 = -\,35a^2f_{1111}\,|\,0$, (20) assumes the form

$$F^{12}\,|\,0 \;=\; p_8 + \big(\text{a combination of } p_6,\, p_7\big)\,. \tag{24}$$

In $F^{13}\,|\,0$, being of abstract weight 17, some terms are pure powers of $X^9\,|\,0 = a$ times partials at 0 of weight 17, while the remaining have coefficients – combinations of partials at 0 of weights $\leq$ 16, hence are themselves combinations of $p_1, p_2, \ldots, p_8$. After a prolonged computation,

$$F^{13}\,|\,0 \;=\; p_9 + \big(\text{a combination of } p_1,\, p_2, \ldots,\, p_8\big)\,, \tag{25}$$

where $p_9 = -\,a^2\big(952f_{12} + 1547f_{113}\big)\,|\,0$.

As for F^{14}, its weight is $18 = w(f_{33})$, so that that new partial f_{33} may show up in the polynomial $F^{14}(f)$ in coefficients next to powers of X^9, while f_{12} and f_{113} from the previous step – in coefficients standing by X^{10} times powers of X^9 (because $w(f_{12}) - \mu(X^{10}) = 17 - (-1) = 18$). At first sight it seems reasonable, as previously, to focus on the new partial alone, putting all lower weights to the queue terms. But presently the horizon of action is different: there is no partial of weight 19 (see the table). In consequence, we will need to know the precise interplay among $f_{12}\,|\,0$, $f_{113}\,|\,0$, $f_{33}\,|\,0$ in the analysis of the last important component F^{15}, and that would be meaningless when forgetting about $f_{12}\,|\,0$ and $f_{113}\,|\,0$ in $F^{14}\,|\,0$. So we are looking for the three coefficients with which these three partials appear in $F^{14}\,|\,0$. After rather lengthy computations,

$$F^{14}\,|\,0 \;=\; p_{10} + \big(\text{a combination of } p_1,\, p_2, \dots,\, p_8\big)\,, \tag{26}$$

where $p_{10} = -\,a\big(6272\,af_{33} + 3906\,bf_{12} + 6300\,bf_{113}\big)\,|\,0$.

As for the last important component $F^{15}\,|\,0$, the outcome of similar computations (made twice, separated by a long period of time) reads

$$\begin{aligned} F^{15}\,|\,0 = -\; & 28644\,abf_{33} - (7284ac + 4410b^2)f_{12} - (11694ac + 7056b^2)f_{113}\,|\,0 \\ & + \big(\text{a combination of } p_1,\, p_2, \dots,\, p_8\big)\,. \end{aligned} \tag{27}$$

This time one can only ask *whether*

$$q_{11} = -\,28644\,abf_{33} - (7284ac + 4410b^2)f_{12} - (11694ac + 7056b^2)f_{113}\,|\,0 \tag{28}$$

is a parameter independent of p_9 and p_{10}. Remembering the definitions of p_9 and p_{10}, it is iff

$$\begin{vmatrix} 0 & 952\,a^2 & 1547\,a^2 \\ 6272\,a^2 & 3906\,ab & 6300\,ab \\ 28644\,ab & 7284\,ac + 4410\,b^2 & 11694\,ac + 7056\,b^2 \end{vmatrix} \neq 0\,.$$

Expanding this determinant, it vanishes precisely for $c = c(a,\, b)$ defined by (10). For all other values of c (including the value 0) it is non-zero.

Now we can sum up our long computations. By the (short) computation of the first eight components and formulas (22) – (27), for $c \neq c(a,\, b)$, upon putting $p_{11} \overset{\text{def}}{=} q_{11}$, the component $A(f)\partial_1 + B(f)\partial_2 + C(f)\partial_3 + \sum_{l=4}^{15} F^l(f)\partial_l$ of $\mathcal{Y}_f\,|\,0$ equals $\sum_{j=1}^{11} p_j e_j$, where p_j are free real parameters and e_j are linearly independent vectors. So, when a generating function f varies freely in its allowed class of smoothness, these components span the 11−dimensional subspace $\big(e_1,\, e_2, \dots,\, e_{11}\big) \subset \big(\partial_1, \dots,\, \partial_{15}\big)$. And of course the remainder $\sum_{l=16}^{n} F^l\partial_l\,|\,0 \in L(D^{13})(0)$ has no relation to the former summand – it can be an arbitrary vector in $\big(\partial_{16}, \dots,\, \partial_n\big)$ independently of f. This justifies that except for the case (10), $\mathrm{SD}_0(D^{13}) \;=\; 11 + n - 15 \;=\; n - 4$.

What does happen in the case (10)? Then q_{11} given by (28) is not a parameter independent of p_9 and p_{10}, but to the contrary – is a fixed linear combination of p_9 and p_{10}. In that case, (27) boils down to

$$F^{15}\,|\,0 \;=\; \text{a combination of } p_1,\, p_2, \ldots,\, p_{10}\,. \tag{29}$$

There appears neither p_{11} nor e_{11} in the previous dimension count, while $e_1, \ldots,$ e_{10} get slightly perturbed, but still independent. In the outcome, in that case (10), $\mathrm{SD}_0(D^{13}) = 10 + n - 15 = n - 5$. Theorem 2 is proved. □

4. Consequences for the local classification.

Corollary 1. (i) *When $c \neq c(a, b)$ (that is, $D^9(p) \neq N$, the superfixed position), the germs in the class* $\mathrm{G_3ST_2GG^*G_4}$ *prolong by Possibility* I *(cf.* Sec. 1.2*). Or, in the KR language: c^{15} can be annihilated.*

(ii) *When $c = c(a, b)$ (D^9 in the superfixed position at p), the constant c^{15} in the description of D^{13} is important. In fact, the prolongation from D^{12} to D^{13} is either according to Possibility* IV *or* V*.*

Proof. Part (i): in view of the machinery developed in Chap. 3, for such c there are Hamiltonians f s. t.

$$A,\; B,\; C,\; F^4, \ldots,\; F^{14} \text{ vanish at 0, and } F^{15}\,|\,0 \neq 0\,. \tag{30}$$

Moreover, these conditions clearly imply $f_2\,|\,0 = f_{13}\,|\,0 = 0$ and hence $F^{15}(0, \ldots, 0,\, x^{15}) = F^{15}\,|\,0 \quad \forall\; x^{15}$ by Prop. 4 in [**10**]. Then, by the technique of [**11**], Chap. 9, the last constant c^{15} can be driven to 0 keeping the KR description of the preceding part of the flag.

Part (ii): For $c = c(a, b)$ one is in the situation – see (29) – opposite to (30). That is, the vanishing of $p_1, \ldots, p_{10}$ *implying* $F^{15}\,|\,0 = 0$ whatever the reference value of c^{15}. Therefore, that constant cannot be changed, keeping the description of the preceding part of the flag, by flag's symmetries embeddable in flows. This means that the germs prolong either by Poss. IV or V (the relevant orbits are discrete: either $|c^{15}|$ or c^{15} is an invariant). □

Added in proof. After the submission of the text we better understood a single mechanism responsible for the superfixed position from Thm. 2 and for an infinite series of other such positions, always in codimension 3. Directly generalizing from $k = 1$, for any $k \geq 1$ fixed, in the geometric class GG…GSTTGG*G with $k + 2 \geq 3$ letters G in the beginning, there resides a module of the local classification. It issues from Possibility V; accordingly – under automorphisms of D^{k+7} that keep a point p – all positions of $D^{k+8}(p)$ are fixed. One of them is *superfixed*, implying a singular behaviour (smaller symmetry dimension at p)

of the $4k$ steps' prolongation D^{5k+8} of D^{k+8}. We obtain this result by carefully analizing the abstract weights as recalled in Sec. 2.3, and the gaps in the semigroup $\big(4,\ 5+4k\big)_{\mathbb{Z}+}$ (*cf.* (4); for $k=1$ we have had the table preceding Ex. 4 and the semigroup $\big(4,\ 9\big)_{\mathbb{Z}+}$).

Normalizing the positions of D^{k+6} and D^{k+7} at p as in the text for $k=1$, that superfixed value $\mathrm{sf}(k)$ can be computed algorithmically. The value $\mathrm{sf}(1)=\frac{237}{320}$ was obtained by hand. The remaining quantities $\mathrm{sf}(k)$, $k\geq 2$ are not yet found; they need the use of a computer (the computation is straightforward if very long already for $k=2$).

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