UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XL
2002

ON INVARIANTS OF CONTINUOUS SUBGROUPS OF THE GENERALIZED POINCARÉ GROUP P(1,4)

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Abstract. For all continuous subgroups of the group $P(1,4)$ the invariants in the five-dimensional Minkowski space $M(1,4)$ have been constructed. The obtained invariants are one-, two-, three and four-dimensional.
Based on the obtained invariants, the nonsingular manifolds in the spaces $M(1,3)\times R(u)$ and $M(1,4)\times R(u)$, invariant under nonconjugate subgroups of the group P(1,4), have been described.
The invariant manifolds have been used for the symmetry reduction of some important equations of theoretical physics in the spaces $M(1,4)\times R(u)$ and $M(1,3)\times R(u)$.

Introduction. The knowledge of the nonconjugate subgroups of the local Lie groups of point transformations and construction of the invariants of these subgroups in explicit form is important in solving numerous problems of mathematics. In particular, in mathematical physics the subgroup structure of the invariance groups of partial differential equations (PDEs) and the invariants of these subgroups allow us to solve many problems. Let me mention some of them.
1. The symmetry reduction of PDEs to differential equations with fewer independent variables (see, for example, [**1, 2, 3**]).
2. The construction of systems of coordinates in which the linear PDEs which are invariant under given groups admit partial or full separation of variables [**4, 5**].

The development of theoretical and mathematical physics has required various extensions of four-dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. One extension of the group $P(1,3)$ is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is a group of rotations and translations of five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [**6, 7**].

The purpose of the present paper is to give a survey of results obtained in [**8**]–[**17**] as well as of some new results.

1. The subgroup structure of the group $P(1,4)$. The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$

$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

For this Lie algebra, we choose the following representation in the space $M(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3},$$

$$P'_4 = -\frac{\partial}{\partial x_4}, \quad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

In the following we will use new basis elements

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$

$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),$$

$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1,2,3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [**18**]. Continuous subgroups of the group $P(1,4)$ have been found in [**8**]–[**10**].

One of the nontrivial consequences of the description of subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [**7**], i.e. it naturally unites the Lie algebras of the symmetry groups of relativistic and nonrelativistic quantum mechanics.

2. The invariants of the continuous subgroups of the group $P(1,4)$. In this paragraph we say something about the invariants of subgroups of the group $P(1,4)$. For all continuous subgroups of the group $P(1,4)$, we have constructed the invariants in the five-dimensional Minkowski space. Some of these invariants have been presented in **[11, 12]**. Among the invariants obtained, there are one-, two-, three- and four-dimensional ones.

PROPOSITION 1. *In the space $M(1,4)$ the nonconjugate subgroups of the group $P(1,4)$ have jointly 38 one-dimensional invariants.*

Some examples

1. $\langle G, L_3, P_1, P_2, X_3\rangle : \omega = \left(x_4^2 + x_2^2 + x_1^2 - x_0^2\right)^{1/2};$
2. $\langle G + a_3X_3, P_1, P_2, X_1, X_2, X_4, (a_3 \neq 0)\rangle : \omega = \ln(x_0 + x_4) - \dfrac{x_3}{a_3};$
3. $\langle G + aX_3, L_3, X_0, X_1, X_2, (a \neq 0)\rangle : \omega = \ln(x_0 - x_4) + \dfrac{x_3}{a};$
4. $\langle P_1 + \mu_2X_2 + \mu_3X_3, P_2 + X_4 - X_0, X_1, X_4, (\mu_2 > 0)\rangle :$
   $\omega = \mu_3x_2 - \mu_2x_3 + \mu_3\dfrac{(x_0 + x_4)^2}{2};$

PROPOSITION 2. *The nonconjugate subgroups of the group $P(1,4)$ have 111 two-dimensional invariants.*

Some examples

1. $\langle G, L_1, L_2, L_3\rangle : \omega_1 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \ \omega_2 = \left(x_0^2 - x_4^2\right)^{1/2};$
2. $\langle L_3 + dG, P_3, X_3, X_4(d > 0)\rangle : \omega_1 = \arcsin \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}} - \dfrac{1}{d}\ln(x_0 + x_4),$
   $\omega_2 = \left(x_1^2 + x_2^2\right)^{1/2};$
3. $\langle L_3, P_1, P_2, X_3\rangle : \omega_1 = x_0 + x_4, \ \omega_2 = x_2^2 + x_1^2 - 2(x_0 + x_4)x_0;$
4. $\langle P_1 + \gamma X_3 + X_4, P_1 + \gamma X_3, P_2 + X_2 + \delta X_3(\gamma > 0)\rangle : \omega_1 = x_0 + x_4,$
   $\omega_2 = \dfrac{\gamma x_1}{x_0 + x_4} + \dfrac{\delta x_2}{x_0 + x_4 - 1} + x_3;$

PROPOSITION 3. *The nonconjugate subgroups of the group $P(1,4)$ have 84 three-dimensional invariants.*

Some examples

1. $\langle L_3 + eG, X_3(e > 0)\rangle : \omega_1 = \arcsin \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}} - \dfrac{1}{e}\,\mathrm{arch}\,\dfrac{x_0}{\sqrt{x_0^2 - x_4^2}},$
   $\omega_2 = \left(x_1^2 + x_2^2\right)^{1/2}, \ \omega_3 = \left(x_0^2 - x_4^2\right)^{1/2};$

2. $\langle L_3 + cG - \alpha X_3, X_4 (c > 0, \alpha \neq 0) \rangle : \omega_1 = \left(x_1^2 + x_2^2\right)^{1/2}$,
$\omega_2 = \dfrac{x_3}{\alpha} + \arcsin \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}}$, $\omega_3 = \dfrac{c}{\alpha} x_3 + \ln(x_0 + x_4)$;

3. $\langle P_3 + L_3 + \alpha(X_4 - X_0), X_4 (\alpha \neq 0) \rangle : \omega_1 = \left(x_1^2 + x_2^2\right)^{1/2}$,
$\omega_2 = \dfrac{x_0 + x_4}{\alpha} - \arcsin \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}}$, $\omega_3 = \alpha x_3 + \dfrac{1}{2}(x_0 + x_4)^2$;

4. $\langle P_1 + X_3, P_2 + \gamma X_2 + \beta X_3 (\beta, \gamma > 0) \rangle : \omega_1 = x_0 + x_4$,
$\omega_2 = \dfrac{x_1}{x_0 + x_4} + \dfrac{\beta x_2}{x_0 + x_4 - \gamma} + x_3$,
$\omega_3 = x_1^2 + x_2^2 + x_4^2 - x_0^2 + \dfrac{\gamma\left(x_0^2 - x_1^2 - x_4^2\right)}{x_0 + x_4}$;

PROPOSITION 4. *The nonconjugate subgroups of the group $P(1,4)$ have 28 four-dimensional invariants.*

Some examples

1. $\langle P_3 + C_3 + eL_3 + s_0 X_0 (e, s_0 \neq 0) \rangle : \quad \omega_1 = (x_1^2 + x_2^2)^{1/2}$,
$\omega_2 = \dfrac{2e}{s_0} x_0 - \arcsin \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}}$, $\omega_3 = x_4^2 + x_3^2 - \dfrac{s_0}{2} x_3$,
$\omega_4 = \arcsin \dfrac{x_3 - s_0/4}{\sqrt{(x_3 - s_0/4)^2 + x_4^2}} - \dfrac{4}{s_0} x_0$;

2. $\langle P_3 + X_1 \rangle : \omega_1 = x_2$, $\omega_2 = x_0 + x_4$, $\omega_3 = (x_0 + x_4) x_1 + x_3$,
$\omega_4 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}$;

3. $\langle L_3 + X_0 \rangle : \omega_1 = x_3$, $\omega_2 = x_0 - x_4$, $\omega_3 = \left(x_1^2 + x_2^2\right)^{1/2}$,
$\omega_4 = x_0 - \dfrac{1}{2} \arcsin \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}}$;

4. $\langle P_3 + X_0 + hX_1 (h \neq 0) \rangle : \omega_1 = x_2$, $\omega_2 = x_1 + h(x_0 + x_4)$,
$\omega_3 = \dfrac{1}{2}(x_0 + x_4)^2 - x_3$, $\omega_4 = \dfrac{1}{3}(x_0 + x_4)^3 - x_3(x_0 + x_4) + \dfrac{1}{2}(x_0 - x_4)$;

These propositions have been proved with use of the group-analysis methods.

In the following we will consider the application of the subgroup structure of the group $P(1,4)$ and the invariants of these subgroups for construction of nonsingular manifolds which are invariant under the group $P(1,4)$ or under its continuous subgroups.

3. The nonsingular invariant manifolds in the space $M(1,4) \times R(u)$. In the space $M(1,4) \times R(u)$, the nonsingular manifolds which are invariant under the continuous subgroups of the group $P(1,4)$ can be written in the following form:

$$F\left(\omega_1(x), \ldots, \omega_k(x), u(x)\right) = 0,$$

where $\omega_1(x), \ldots, \omega_k(x),\ u(x), (k = 1, 2, 3, 4)$ are invariants of these subgroups and F is an arbitrary smooth function.

Since the invariants of the continuous subgroups of the group $P(1,4)$ have already been constructed, the corresponding invariant manifolds have also been described.

The manifolds obtained provide us with the ansatzes which reduce those PDEs in the space $M(1,4) \times R(u)$ which are invariant under the group $P(1,4)$ or under continuous subgroups of this group to differential equations with k independent variables $\omega_1(x), ..., \omega_k(x) (k = 1, 2, 3, 4)$.

4. The nonsingular invariant manifolds in the space $M(1,3) \times R(u)$. In the space $M(1,3) \times R(u)$, the nonsingular manifolds which are invariant under the nonconjugate subgroups of the group $P(1,4)$ can be written in the form:

$$F\left(\omega_1(x,u), \ldots, \omega_k(x,u)\right) = 0,$$

where $\omega_1(x,u), \ldots, \omega_k(x,u), (k = 1, 2, 3, 4)$, are invariants of these subgroups and F is an arbitrary smooth function.

Since the invariants of the continuous subgroups of the group $P(1,4)$ in the space $M(1,3) \times R(u)$ have already been found, the considered invariant manifolds have also been constructed. These manifolds play an important role in studying the symmetry reduction of differential equations in the space $M(1,3) \times R(u)$ which are invariant under the group $P(1,4)$ or under its continuous subgroups. The manifolds considered give us ansatzes which reduce these equations to differential equations with k-1 independent variables $\omega_1(x,u), \ldots, \omega_k(x,u) (k = 1, 2, 3, 4)$.

5. Differential equations in the space $M(1,4) \times R(u)$. Let us consider some applications of the invariants of the continuous subgroups of the group $P(1,4)$ as well as the nonsingular invariant manifolds in the space $M(1,4) \times R(u)$ to the study of some important equations of theoretical physics.

Below, we present some of the results obtained.

5.1. *The nonlinear five-dimensional wave equation.* Let us consider the equation

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = F(u), \tag{5.1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3, x_4)$ and F is a sufficiently smooth function. The invariance group of the equation (5.1) is the generalized Poincaré group

$P(1,4)$. Using the invariants of subgroups of the group $P(1,4)$ we have constructed ansatzes which reduce the investigated equation to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. We have obtained one-, two-, three- and four- dimensional reduced equations. Taking into account the solutions of the reduced equations, some classes of exact solutions of the investigated equation have been found. The majority of these results have been published in **[11]**–**[13]**.

5.2. *The linear five-dimensional wave equation.* Let us consider the equation

$$\Box u = -\kappa^2 u, \tag{5.2}$$

where

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2},$$

$u(x) = u(x_0, x_1, x_2, x_3, x_4)$ is a scalar $\mathbf{C}^2$ function and $\kappa = const$. Equation (5.2) is invariant under the group $P(1,4)$. Using the four-dimensional Abelian subalgebras of the Lie algebra of the group $P(1,4)$ and their invariants, we have achieved full separations of variables in the considered equation. Using three-, two- and one-dimensional Abelian subalgebras of the considered Lie algebra and their invariants, we have achieved partial separations of variables. As a result of partial (or full) separation of variables, we have obtained PDEs with fewer independent variables (or ordinary differential equations (ODEs)) which replace the original equation. Some exact solutions of the five-dimensional wave equation have been constructed. More details about these results can be found in **[14]**.

5.3. *The Dirac equation in* $M(1,4)$. Let us consider the equation

$$\left(\gamma_k P^k - m\right) \psi(x) = 0, \tag{5.3}$$

where $x = (x_0, x_1, x_2, x_3, x_4)$, $P_k = i\dfrac{\partial}{\partial x_k}$, $k = 0, 1, 2, 3, 4$; γ_k are (4×4) – Dirac matrices.

The equation (5.3) is invariant under the group $P(1,4)$. Following **[19, 20]** and using the subgroup structure of the group $P(1,4)$ as well as the invariants of these subgroups, the ansatzes which reduce the equation (5.3) to systems of differential equations with a fewer number of independent variables were constructed. The corresponding symmetry reduction has been done. Among these systems of reduced equations, there are one-, two-, three- and four-dimensional ones. In order to obtain these results we have used one-, two-, three- and four-dimensional Abelian subalgebras of the considered Lie algebra as well as the invariants of these subalgebras. Let me note that in the study of this equation we have also used some non-Abelian subalgebras of the Lie algebra of the group

$P(1,4)$ and the invariants of these subalgebras. Some of the results obtained were presented in [**15**].

6. Differential equations in the space $M(1,3)\times R(u)$. Now, we begin to use the invariants of the continuous subgroups of the group $P(1,4)$ as well as the nonsingular invariant manifolds in the space $M(1,3)\times R(u)$ to investigate some equations in the space $M(1,3)\times R(u)$ important for theoretical physics. Below we present some of the results received.

6.1. *The Eikonal equation.* We consider the equation

$$u^{\mu}u_{\mu} \equiv (u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1, \tag{6.1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu} \equiv \dfrac{\partial u}{\partial x_{\mu}}$, $u^{\mu} = g^{\mu\nu}u_{\nu}$, $\mu,\nu = 0,1,2,3$.

From the results of [**19**] it follows that the symmetry group of equation (6.1) contains the group $P(1,4)$ as a subgroup. Using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups we have constructed ansatzes which reduce the investigated equation to differential equations with fewer independent variables, and the corresponding symmetry reduction has been carried out. We have obtained one-, two- and three-dimensional reduced equations. Among the reduced equations there are also linear ODEs. Having solved some of the reduced equations, we have found classes of exact solutions of the investigated equation. Some of these results were given in [**11, 12, 16**].

6.2. *The Euler-Lagrange-Born-Infeld equation.* Let us consider the equation

$$\Box u\,(1 - u_{\nu}u^{\nu}) + u^{\mu}u^{\nu}u_{\mu\nu} = 0, \tag{6.2}$$

where $u = u(x), x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu} \equiv \dfrac{\partial u}{\partial x^{\mu}}$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x^{\mu}\partial x^{\nu}}$, $u^{\mu} = g^{\mu\nu}u_{\nu}$, $g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu = 0,1,2,3$, $\Box$ is the d'Alembert operator.

The symmetry group [**19**] of equation (6.2) contains the group $P(1,4)$ as a subgroup. Based on the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, the symmetry reduction of the investigated equation to differential equations with a fewer number of independent variables has been done. We have obtained one-, two- and three-dimensional reduced equations. In numerous cases the reduced equations are linear ODEs. Taking into account the solutions of the reduced equations, we have found multiparameter families of exact solutions of the considered equation. It should be noted that among these solutions there are ones which contain arbitrary smooth functions of invariants of subgroups of the group $P(1,4)$. Some of these results can be found in [**16**].

6.3. *The multidimensional homogeneous Monge-Ampère equation.* Let us consider the equation

$$\det(u_{\mu\nu}) = 0, \tag{6.3}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $\mu, \nu = 0, 1, 2, 3$.

The symmetry group of equation (6.3) was found in [**19**].

We have achieved the symmetry reduction of the investigated equation to differential equations with fewer independent variables, using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups. We have received one-, two-, and three-dimensional reduced equations. Among the reduced equations there are linear ODEs. Having solved some of the reduced equations, we have obtained classes of exact solutions of the investigated equation. These classes contain the solutions with arbitrary smooth functions of invariants of subgroups of the group $P(1,4)$. Some of these results are presented in [**16**].

6.4. *The multidimensional inhomogeneous Monge-Ampère equation.* Let us consider the equation

$$\det(u_{\mu\nu}) = \lambda\left(1 - u_\nu u^\nu\right)^3, \quad \lambda \neq 0, \tag{6.4}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u^\nu = g^{\nu\alpha} u_\alpha$, $u_\alpha \equiv \dfrac{\partial u}{\partial x_\alpha}$, $g_{\mu\nu} = (1, -1, -1, -1)\delta_{\mu\nu}$, $\mu, \nu, \alpha = 0, 1, 2, 3$.

Equation (6.4) is invariant [**19**] under the group $P(1,4)$.

We have constructed ansatzes which reduce the investigated equation to differential equations with a fewer number of independent variables, using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups. The corresponding symmetry reduction has been done. We have found one-, two-, and three-dimensional reduced equations. Some classes of exact solutions of the considered equation have been found. The majority of these results are published in [**17**].

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Received January 2, 2002

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