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EXPONENTIAL STABILITY OF SOLUTIONS OF THE CAUCHY PROBLEM FOR A DIFFUSION EQUATION WITH ABSORPTION WITH A DISTRIBUTION INITIAL CONDITION

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Abstract. We establish the estimate of the L^1 norm of a solution of a diffusion equation with absorption with an initial condition given by a distribution with compact support.

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u - V(x)u \tag{1}$$

$$u(t_0) = \Lambda \tag{2}$$

where $0 \leq V \in L^1_{loc}(\mathbb{R}^n)$, $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}\Lambda$ is compact. Denote $X = (0,\infty)\times\mathbb{R}^n$.

We call a function $u \in L^1(X)$ a solution of (1) iff (1) holds in the sense of distributions i. e. for all $\psi \in \mathcal{D}(X)$

$$\int_0^\infty \int_{\mathbb{R}^n} u(t,x)\Big(\frac{\partial \psi}{\partial t}(t,x) + \Delta\psi(t,x) - V(x)\psi(t,x)\Big)dxdt = 0.$$

We say that the solution of (1) satisfies the initial condition (2) if for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\lim_{t\to t_0} \int_{\mathbb{R}^n} u(t,x)\varphi(x)dx = \Lambda(\varphi).$$

When Λ is a Dirac distribution, then a solution of $\{(1), (2)\}$ is called a fundamental solution of (1).

By $(T(t))_{t\geq 0}$ we denote the Gaussian semigroup on $L^1(\mathbb{R}^n)$ given by

$$(T(t)f)(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)e^{-\frac{|x-y|^2}{4t}}dy.$$

Note that $(T(t))$ is a holomorphic contraction semigroup. Δ is the generator of $(T(t))$ defined on its domain $D(\Delta) = \{f \in L^1 \,:\, \Delta f \in L^1\}$ meant in the sense of distributions.

For $0 \leq V \in L^1_{loc}(\mathbb{R}^n)$ we define an operator $\Delta - V$ as follows: let $D(A_{min}) = \mathcal{D}(\mathbb{R}^n)$ (the test functions on $\mathbb{R}^n$) and $A_{min}f = \Delta f - Vf$. Then A_{min} is closable in $L^1(\mathbb{R}^n)$ and we set $\Delta - V = \overline{A_{min}}$ in $L^p(\mathbb{R}^n)$. Then $\Delta - V$ generates a holomorphic semigroup $(S(t))_{t\geq 0}$ on $L^1(\mathbb{R}^n)$.

We say that $G \subset \mathbb{R}^n$ contains arbitrary large balls if for any $r > 0$ there exists $x \in \mathbb{R}^n$ such that the ball $\mathbb{B}(x, r) := \{y \in \mathbb{R}^n \,:\, |x - y| < r\}$ is included in G. By $\mathcal{G}$ we denote the set of all open subsets of $\mathbb{R}^n$ which contain arbitrary large balls. In [**1**] W. Arendt and Ch. Batty proved the theorem on stability of a solution of equation (1).

THEOREM 1. *Let $0 \leq V \in (L^1 + L^\infty)(\mathbb{R}^n)$. If for each $G \in \mathcal{G}$*

$$\int_G V(x)dx = \infty \tag{3}$$

then

$$\inf\{\omega \in \mathbb{R} \,:\, \sup_{t\geq 0} e^{-\omega t}||S(t)|| < \infty\} < 0.$$

So now, we can get an easy

COROLLARY 2. *Let $0 \leq V \in (L^1 + L^\infty)(\mathbb{R}^n)$. If for each $G \in \mathcal{G}$ (3) holds, then there exist constants $M, \omega > 0$ such that for all $f \in L^1(\mathbb{R}^n)$ and for any initial time $t_0 \in \mathbb{R}$ a distribution solution $u(t, x)$ of the Cauchy problem $\{(1), u(t_0, x) = f(x)\}$ satisfies*

$$||u(t, \cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}||f||_{L^1(\mathbb{R}^n)}.$$

PROOF. For $t \geq t_0$ define a holomorphic semigroup $S_0(t) = S(t-t_0)$. Then for any $f \in L^1(\mathbb{R}^n)$ the function $u(t, \cdot) = S_0(t)f$ is a solution in the sense of distributions of the problem $\{(1), u(t_0, \cdot) = f\}$, so by Theorem 1 there exists $\omega > 0$ such that

$$M := \sup_{t\geq 0} e^{\omega t}||S(t)|| < \infty.$$

Consequently,

$$sup_{t\geq t_0} e^{\omega(t-t_0)}||S_0(t)|| = sup_{t\geq t_0} e^{\omega(t-t_0)}||S(t - t_0)|| = M,$$

so

$$||u(t, \cdot)||_{L^1(\mathbb{R}^n)} = ||S_0(t)f||_{L^1(\mathbb{R}^n)} \leq ||S_0(t)|| \cdot ||f||_{L^1(\mathbb{R}^n)}$$

which completes the proof. □

Our main result is

Theorem 3. *Let* $0 \leq V \in (L^1 + L^\infty)(\mathbb{R}^n)$*. Let* $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$*,* $\operatorname{supp}\Lambda$ *is compact. Let* $t_0 \in \mathbb{R}$*. Let* u *be a solution of the Cauchy problem* $\{(1), (2)\}$*. If for each* $G \in \mathcal{G}$ (3) *holds, then there exist constants* $M, \omega > 0$ *such that*

$$||u(t,\cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}|\Lambda(1)|$$

Proof. Consider a function $h \in \mathcal{D}(\mathbb{R}^n)$ such that $h \geq 0$, $||h||_{L^1(\mathbb{R}^n)} = 1$ and define

$$h_\nu(x) := \nu^n h(\nu x).$$

Then $h_\nu \star \Lambda \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(h_\nu \star \Lambda) \subset \operatorname{supp} h_\nu + \operatorname{supp}\Lambda$. Moreover $[h_\nu \star \Lambda] \to \Lambda$, where by $[f]$ we denote a distribution generated by a function f.

Consider a sequence of the Cauchy problems $\{$ (1), $u(t_0) = h_\nu \star \Lambda$ $\}$. Denote by u_ν solutions in the sense of distributions of these problems. Thanks to Corollary 2 we have

$$||u_\nu(t,\cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}||h_\nu \star \Lambda||_{L^1(\mathbb{R}^n)},$$

Since Λ has a compact support, it can be uniquely extended to a continuous linear functional on $\mathcal{C}^\infty(\mathbb{R}^n)$. Moreover, let $\Lambda^+ = \sup\{\Lambda, 0\}, \Lambda^- = \sup\{-\Lambda, 0\}$, then

$$||h_\nu \star \Lambda^+||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} 1 \cdot (h_\nu \star \Lambda^+)(x)dx = \big(1 \star (h_\nu \star \Lambda^+)\big)(0) =$$
$$= \big((1 \star (h_\nu) \star \Lambda^+\big)(0) = \Lambda^+(1 \star \breve{h_\nu}) = \Lambda^+(1)$$

where $\breve{v}(x) = v(-x)$, and similarly

$$||h_\nu \star \Lambda^-||_{L^1(\mathbb{R}^n)} = \Lambda^-(1)$$

so

$$||h_\nu \star \Lambda||_{L^1(\mathbb{R}^n)} = |\Lambda(1)|.$$

Hence

$$||u_\nu(t,\cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}|\Lambda(1)|. \tag{4}$$

Moreover, we have

$$||u_\nu||_{L^1(X)} \leq \frac{M}{\omega}|\Lambda(1)|,$$

so the sequence u_ν is bounded in X, and so is $-Vu_\nu$.

Let $0 < \tau < \infty$, denote $Q_\tau := (0,\tau) \times \mathbb{R}^n$. Now, we need the following lemma which can be found in [**2**].

Lemma 4. *Consider the mapping* K *defined by*

$$K : L^1(\mathbb{R}^n) \times L^1(Q_\tau) \ni (u_0, f) \mapsto u = T(t)u_0 + \int_0^t T(t-\tau)f(\tau)d\tau \in L^1(Q_\tau),$$

i. e. u *is the solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} - \Delta u = f$$
$$u(0,x) = u_0(x)$$

Then K *is a compact operator.*

Obviously,

$$u_\nu = K(l_\nu, -Vu_\nu),$$

so by Lemma 4 there exist a subsequence still denoted by u_ν and a function $u_\tau \in L^1(Q_\tau)$ such that $u_\nu \to u_\tau$ in $L^1(Q_\tau)$. Let $u = \bigcup_{\tau>0} u_\tau$. Since u_ν are the solutions of (1), for any $\psi \in \mathcal{D}(X)$

$$\left|\int_0^\infty \int_{\mathbb{R}^n} u_\nu\Big(\frac{\partial\psi}{\partial t} + \Delta\psi - V\psi\Big)dxdt - \int_0^\infty \int_{\mathbb{R}^n} u\Big(\frac{\partial\psi}{\partial t} + \Delta\psi - V\psi\Big)dxdt\right| \le$$

$$\left|\int_0^\infty \int_{\mathbb{R}^n} |u_\nu - u|\Big(\frac{\partial\psi}{\partial t} + \Delta\psi - V\psi\Big)dxdt\right| \le C\int_0^\infty \int_{\mathbb{R}^n} |u_\nu - u|dxdt \to 0,$$

so u is a solution of (1) in the sense of distributions.

Moreover, by Riesz-Fischer theorem there exists a subsequence still denoted by u_ν which converges pointwise almost everywhere to u so

$$||u_\nu(t,\cdot)||_{L^1(\mathbb{R}^n)} \to ||u(t,\cdot)||_{L^1(\mathbb{R}^n)}.$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Denote $\Sigma = \sup\{|\varphi(x)| \,:\, x \in \mathbb{R}^n\}$. Let $\varepsilon > 0$. Then there exists N such that $|[h_N \star \Lambda](\varphi) - \varphi(0)| \le \frac{\varepsilon}{3}$ and

$$\int_{\mathbb{R}^n} |u_N(t,x) - u(t,x)|dx \le \frac{\varepsilon}{3\Sigma}.$$

For N there exists $\delta > 0$ such that if $0 < t - t_0 < \delta$

$$\int_{\mathbb{R}^n} |u_N(t,x) - (h_N \star \Lambda)(x)|dx \le \frac{\varepsilon}{3\Sigma}.$$

Then

$$\left|\int_{\mathbb{R}^n} u(t,x)\varphi(x)dx - \Lambda(\varphi)\right| \le \int_{\mathbb{R}^n} |u(t,x) - u_N(t,x)|\cdot|\varphi(x)|dx +$$

$$\int_{\mathbb{R}^n} |u_N(t,x) - (h_N \star \Lambda)(x)|\cdot|\varphi(x)|dx + |[h_N \star \Lambda](\varphi) - \Lambda(\varphi)| \le \varepsilon,$$

so u is the solution of the Cauchy problem for $\{(1), (2)\}$, which completes the proof. □

References

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