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THE DIRICHLET PROBLEM FOR A PARABOLIC SEMILINEAR DIFFERENTIAL EQUATION IN AN UNBOUNDED DOMAIN

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Abstract. The aim of the paper is to give theorems about the uniqueness and existence of a weak solution for a parabolic semilinear differential equation in an unbounded domain. The uniqueness and existence do not depend on the behaviour of the solution for $|x| \to +\infty$.

It is generally known that the uniqueness of the solutions of the Dirichlet problem for parabolic semilinear differential equations in unbounded domains take place in the class of the functions increasing for $|x| \to +\infty$ not faster than $e^{\lambda |x|^2}$, where λ depends on the coefficients of the equation.

It turned out ([**1**], [**2**]) that for some semilinear parabolic equations the uniqueness and existence of the solution of the Dirichlet problem did not depend on the behaviour of the solution for $|x| \to +\infty$.

In this paper the analogous result was obtained for a parabolic equation with a nonlinear derivative with respect to time.

Let $\Omega \subset \mathbf{R}^n$ be unbounded domain. We will denote by $Q_T = \Omega \times (0,T)$ for $T > 0$ and $S_T = \partial\Omega \times (0,T)$. We consider the following equation in Q_T

$$(1)\quad |u_t|^{p-2}u_t - \sum_{i,j=1}^{n} \Big(a_{ij}(x)u_{x_i}\Big)_{x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u = f(x,t), \quad p > 2$$

with the boundary condition

$$(2)\qquad u|_{S_T} = 0$$

and with the initial condition

$$(3)\qquad u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega.$$

The following assumptions will be needed throughout the paper:

(i) $\Omega \cap B_R$ is a domain for every $R > 0$, where $B_R \subset \mathbf{R}^n$ is the open ball of radius R and the centre at the origin of coordinates, $\partial\Omega \cap B_R = \Gamma_R \in C^1$;

(ii) $a_{ij} \in L^\infty(\Omega); b_i, c, c_t \in L^\infty(Q_T)$; functions a_{ij} satisfy inequality

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu \sum_{i=1}^{n} \xi_i^2 \quad \forall \xi \in R^n,\ \nu = \text{const} > 0,\ x \in \Omega,$$

$$a_{ij}(x) = a_{ji}(x) \quad \text{for almost every } x \in \Omega,$$

and

$$c(x,t) \geq c_0 > 0 \quad \text{for almost every } (x,t) \in Q_T;$$

(iii) $f \in L^q((0,T); L^q_{loc}(\bar{\Omega})); u_0 \in H^1_{0,loc}(\bar{\Omega})$, where

$$L^r_{loc}(\bar{\Omega}) = \Big\{u : u \in L^r(\Omega \cap B_R) \text{ for every } R > 0\Big\},\ 1 \leq r < +\infty,$$

$$H^1_{0,loc}(\bar{\Omega}) = \Big\{u : u \in H^1_0(\Omega \cap B_R) \text{ for every } R > 0\Big\},$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Definition. A function $u \in L^\infty((0,T); H^1_{0,loc}(\bar{\Omega}))$ such that $u_t \in L^p_{loc}(\bar{Q}_T)$ is said to be a *weak solution* of the equation (1), if u satisfies the integral equality

$$\int_{Q_T} \Big(|u_t|^{p-2}u_t v + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i}v$$
$$+c(x,t)uv - f(x,t)v\Big)dxdt = 0$$

for every $v \in C^\infty([0,T]; C_0^\infty(\Omega))$ and u satisfies the initial condition (3).

We can now formulate our result.

Theorem 1. *If assumptions (i)–(iii) hold and p satisfies inequalities*

$$\begin{cases} 2 < p < \frac{2n}{n-2}, & \text{for } n > 2, \\ p > 2, & \text{for } n = 1, 2, \end{cases}$$

then problem (1)–(3) *has no more than one weak solution.*

Proof. We prove that if problem (1)–(3) has a weak solution, then this solution is unique. On the contrary, suppose that there are two weak solutions u_1, u_2 of problem (1)–(3).
Let $u = u_1 - u_2$. By the definition of weak solution we have

$$\int_{Q_T} \Big(|u_{kt}|^{p-2}u_{kt} v + \sum_{i,j=1}^{n} a_{ij}(x)u_{kx_i}v_{x_j} + \sum_{i=1}^{n} b_i(x,t)u_{kx_i}v$$
$$+c(x,t)u_k v - f(x,t)v\Big)dxdt = 0$$

for $k = 1, 2$. Subtracting above the equation for $k = 1$ from the equation for $k = 2$ we obtain

$$\int_{Q_T} \Big((|u_{1t}|^{p-2}u_{1t} - |u_{2t}|^{p-2}u_{2t})v + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}v_{x_j} + \\ + \sum_{i=1}^{n} b_i(x,t)u_{x_i}v + c(x,t)uv \Big) dxdt = 0. \tag{4}$$

Let v be a function defined by the formula

$$v = \Big((\theta_m u_t \varphi^\beta e^{-\frac{\lambda t}{2}}) * \rho_l * \rho_l \Big) \theta_m e^{-\frac{\lambda t}{2}}$$
$$= \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} (\theta_m(\tau)u_t(x,\tau)\varphi^\beta(x)e^{-\frac{\lambda\tau}{2}})\rho_l(s-\tau)d\tau \Big) \times \rho_l(t-s)ds\theta_m(t)e^{-\frac{\lambda t}{2}},$$

where

$$\varphi(x) = \begin{cases} \frac{1}{R}(R^2 - |x|^2), & |x| \le R, \\ 0, & |x| > R; \end{cases}$$

$\beta \ge 1, \lambda > 0, R > 0$.
For fixed $\tau_0, \tau \in (0,T)$ such that $\tau_0 < \tau$ let θ_m be the function defined in $[0,T]$ as follows:

$\theta_m(t) = 1$, if $\tau_0 + \frac{2}{m} < t < \tau - \frac{2}{m}$,
$\theta_m(t) = 0$, if $t > \tau - \frac{1}{m}$ or $t < \tau_0 + \frac{1}{m}$,
θ_m is linear and continuous on $(\tau_0 + \frac{1}{m}, \tau_0 + \frac{2}{m})$ and $(\tau - \frac{2}{m}, \tau - \frac{1}{m})$;
ρ_l satisfy the condition $\rho_l(t) = \rho_l(-t)$, $\int_{-\infty}^{+\infty} \rho_l(t)dt = 1$ and
supp $\rho_l \subset [-\frac{1}{l}, \frac{1}{l}], l \in \mathbf{N}$.

Using the properties of operation $*$ and letting $l \to +\infty$ we can rewrite (4) as follows

$$\int_{Q_T} \Big(|u_{1t}|^{p-2}u_{1t} - |u_{2t}|^{p-2}u_{2t} \Big) \theta_m^2 u_t \varphi^\beta e^{-\lambda t} dxdt$$
$$- \sum_{i,j=1}^{n} \int_\Omega \int_{-\infty}^{+\infty} a_{ij}(x)\theta_m(t)\theta_m'(t)\varphi^\beta(x)e^{-\lambda t}u_{x_i}(x,t)u_{x_j}(x,t)dtdx$$
$$+ \frac{\lambda}{2} \sum_{i,j=1}^{n} \int_\Omega \int_{-\infty}^{+\infty} a_{ij}(x)\varphi^\beta(x)\theta_m^2(t)u_{x_i}(x,t)u_{x_j}(x,t)e^{-\lambda t}dtdx$$
$$+ \beta \sum_{i,j=1}^{n} \int_\Omega \int_{-\infty}^{+\infty} a_{ij}(x)\varphi^{\beta-1}\varphi_{x_j}(x)\theta_m^2(t)u_{x_i}(x,t)u_t(x,t)e^{-\lambda t}dtdx$$
$$+ \int_{Q_T} \sum_{i=1}^{n} b_i(x,t)u_{x_i}\theta_m^2 u_t \varphi^\beta e^{-\lambda t}dxdt + \int_{Q_T} c(x,t)u\theta_m^2 u_t \varphi^\beta e^{-\lambda t}dxdt = 0.$$

Letting $m \to +\infty$ we have

(5)

$$\int_{Q_{\tau_0,\tau}} \Big(|u_{1t}|^{p-2}u_{1t} - |u_{2t}|^{p-2}u_{2t}\Big)u_t\varphi^\beta e^{-\lambda t}dxdt$$
$$+\sum_{i,j=1}^{n}\int_\Omega \frac{1}{2}a_{ij}(x)\varphi^\beta(x)\Big(u_{x_i}(x,\tau)u_{x_j}(x,\tau)e^{-\lambda\tau} - u_{x_i}(x,\tau_0)u_{x_j}(x,\tau_0)e^{-\lambda\tau_0}\Big)dx$$
$$+\ \frac{\lambda}{2}\sum_{i,j=1}^{n}\int_\Omega\int_{\tau_0}^{\tau} a_{ij}(x)\varphi^\beta(x)u_{x_i}(x,t)u_{x_j}(x,t)e^{-\lambda t}dtdx$$
$$+\ \beta\sum_{i,j=1}^{n}\int_\Omega\int_{\tau_0}^{\tau} a_{ij}(x)\varphi^{\beta-1}\varphi_{x_j}(x)u_{x_i}(x,t)u_t(x,t)e^{-\lambda t}dtdx$$
$$+\ \int_{Q_{\tau_0,\tau}}\sum_{i=1}^{n} b_i(x,t)u_{x_i}u_t\varphi^\beta e^{-\lambda t}dxdt\ +\ \int_{Q_{\tau_0,\tau}} c(x,t)uu_t\varphi^\beta e^{-\lambda t}dxdt = 0.$$

Prolong the functions u and f by zero for $\tau < 0$,while b_i and c by $b_i(x,0)$, $c(x,0)$ for $\tau < 0$. We claim that (5) is true for almost every $\tau_0, \tau_0 < \tau, \tau \in (0,T), \tau_0 \in (-T,0)$. From (5) we thus get the equality

$$\int_{Q_\tau}\Big(|u_{1t}|^{p-2}u_{1t} - |u_{2t}|^{p-2}u_{2t}\Big)u_t\varphi^\beta e^{-\lambda t}dxdt$$
$$-\frac{1}{2}\sum_{i,j=1}^{n}\int_{\Omega_\tau} a_{ij}(x)\varphi^\beta(x)u_{x_i}(x,\tau)u_{x_j}(x,\tau)e^{-\lambda\tau}dx$$
$$+\frac{\lambda}{2}\sum_{i,j=1}^{n}\int_{Q_\tau} a_{ij}(x)\varphi^\beta(x)u_{x_i}(x,t)u_{x_j}(x,t)e^{-\lambda t}dxdt$$
$$+\beta\sum_{i,j=1}^{n}\int_{Q_\tau} a_{ij}(x)\varphi^{\beta-1}\varphi_{x_j}(x)u_{x_i}(x,t)u_{x_j}(x,t)e^{-\lambda t}dxdt$$
$$+\int_{Q_\tau}\sum_{i=1}^{n} b_i(x,t)u_{x_i}u_t\varphi^\beta e^{-\lambda t}dxdt + \int_{Q_\tau} c(x,t)uu_t\varphi^\beta e^{-\lambda t}dxdt = 0.$$

Now we can obtain the following estimates:

$$I_1 = \int_{Q_T}\Big(|u_{1t}|^{p-2}u_{1t} - |u_{2t}|^{p-2}u_{2t}\Big)\Big(u_{1t} - u_{2t}\Big)\varphi^\beta e^{-\lambda t}dxdt$$
$$\geq 2^{2-p}\int_{Q_T}|u_t|^p\varphi^\beta e^{-\lambda t}dxdt;$$

$$-I_2^2 \geq \frac{\nu}{2}\int_{\Omega_T}\sum_{i=1}^{n} u_{x_i}^2\varphi^\beta e^{-\lambda T}dx$$

$$-\sum_{i,j=1}^{n}\int_\Omega \frac{1}{2}a_{ij}(x)\varphi^\beta(x)u_{x_i}(x,s_0)u_{x_j}(x,s_0)e^{-\lambda s_0}dx;$$

$$I_2^3 \geq \frac{1}{2}\lambda\,\nu\int_{Q_T}\sum_{i=1}^{n} u_{x_i}^2\;\varphi^\beta e^{-\lambda t}dxdt;$$

$$I_2^4 = \beta\int_{Q_T}\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_t\varphi^{\beta-1}\varphi_{x_j}e^{-\lambda t}dxdt \leq \frac{1}{2}\beta n\delta\mu^2\int_{Q_T}\sum_{i,j=1}^{n} u_{x_i}^2\varphi^\beta e^{-\lambda t}dxdt$$

$$+\frac{1}{p}\beta n^2\delta\int_{Q_T}|u_t|^p\varphi^\beta e^{-\lambda t}dxdt + \frac{\beta}{\varrho(\delta)}\int_{Q_T}\sum_{i,j=1}^{n}\varphi^{\beta-\frac{2p}{p-2}}|\varphi_{x_j}|^{\frac{2p}{p-2}}e^{-\lambda t}dxdt,$$

where $\kappa = \frac{(p-2)\beta}{2p} - 1$, $|a_{ij}(x)| \leq \mu$, $\delta > 0$;

$$I_3 = \int_{Q_T}\sum_{i=1}^{n} b_i(x,t)u_{x_i}u_t\varphi^\beta e^{-\lambda t}dxdt \leq \frac{1}{2}\mu_2^2\delta_1\int_{Q_T}\sum_{i=1}^{n} u_{x_i}^2\varphi^\beta e^{-\lambda t}dxdt$$

$$+\;\frac{n\delta_1}{p}\int_{Q_T}|u_t|^p\varphi^\beta e^{-\lambda t}dxdt\;+\;\frac{n}{\zeta(\delta_1)}\int_{Q_T}\varphi^\beta e^{-\lambda t}dxdt,$$

where $|b_i(x,t)| \leq \mu_2$ for $i = 1,\ldots,n$ and $\delta_1 > 0$;

$$I_4 = \int_{Q_T} c(x,t)uu_t\varphi^\beta e^{-\lambda t}dxdt \geq \frac{1}{2}c_0\int_{\Omega_T} u^2\varphi^\beta e^{-\lambda T}dx$$

$$-\frac{1}{2}\int_{Q_T} c_t(x,t)u^2\varphi^\beta e^{-\lambda t}dxdt + \frac{1}{2}\lambda c_0\int_{Q_T} u^2\varphi^\beta e^{-\lambda t}dxdt.$$

Summarizing, we have

$$\Big(2^{2-p} - \frac{\beta n^2\delta}{p} - \frac{n\delta_1}{p}\Big)\int_{Q_T}|u_t|^p\varphi^\beta e^{-\lambda t}dxdt$$

$$+\Big(\frac{1}{2}\lambda\nu - \frac{1}{2}n\beta\delta\mu^2 - \frac{1}{2}\mu_2^2\delta_1\Big)\int_{Q_T}\sum_{i=1}^{n} u_{x_i}^2\varphi^\beta e^{-\lambda t}dxdt$$

$$-\frac{\beta}{\varrho(\delta)}\int_{Q_T}\sum_{i,j=1}^{n}\varphi^{\beta-\frac{2p}{p-2}}|\varphi_{x_j}|^{\frac{2p}{p-2}}e^{-\lambda t}dxdt - \frac{n}{\zeta(\delta_1)}\int_{Q_T}\varphi^\beta e^{-\lambda t}dxdt \tag{6}$$

$$+\Big(\frac{1}{2}\lambda c_0 - \frac{1}{2}c_1\Big)\int_{Q_T} u^2\varphi^\beta e^{-\lambda t}dxdt \leq 0.$$

We can choose the constants δ, δ_1 so that the expression $\left(2^{2-p} - \frac{1}{p}\beta n^2\delta - \frac{1}{p}n\delta_1\right)$ is positive. Then we choose the constant λ so that the expression $\Big(\frac{1}{2}\lambda\nu - \frac{1}{2}n\beta\delta\mu^2 - \frac{1}{2}\mu_2^2\delta_1\Big)$ is greater than 0. Addtionally, we note that

$$\int_\Omega \varphi^{\beta-\frac{2p}{p-2}}|\varphi_{x_j}|^{\frac{2p}{p-2}}dx \le 2^{\frac{2p}{p-2}}\int_\Omega \varphi^{\beta-\frac{2p}{p-2}}dx \le 2^{\frac{2p}{p-2}}\int_{\Omega_R}(R+|x|)^{\beta-\frac{2p}{p-2}}dx$$
$$\le 2^{\frac{2p}{p-2}}(2R)^{\beta-\frac{2p}{p-2}}\int_{B_R}dx = 2^\beta R^{\beta-\frac{2p}{p-2}+n}.$$

Then from (6) it follows that

$$\int_{Q_T^R} u^2\varphi^\beta e^{-\lambda t}dxdt \le CR^{\beta-\frac{2p}{p-2}+n}.$$

It is easily seen that

$$\int_{Q_T^R} u^2\varphi^\beta e^{-\lambda t}dxdt = \int_{Q_T^{R_0}} u^2\varphi^\beta e^{-\lambda t}dxdt + \int_{Q_T^{R-R_0}} u^2\varphi^\beta e^{-\lambda t}dxdt$$
$$\ge \int_{Q_T^{R_0}} u^2\varphi^\beta e^{-\lambda t}dxdt$$

and

$$\int_{Q_T^{R_0}} u^2 e^{-\lambda t}(R-R_0)^\beta dxdt \le CR^{\beta-\frac{2p}{p-2}+n},$$

where $R_0 < R$. Therefore

$$\int_{Q_T^{R_0}} u^2 e^{-\lambda t}dxdt \le C\Big(\frac{R}{R-R_0}\Big)^\beta R^{n-\frac{2p}{p-2}}.$$

Let ε be any small positive number. Then for R large enough and $n < \frac{2p}{p-2}$ we obtain

$$\int_{Q_T^{R_0}} u^2 e^{-\lambda t}dxdt < \varepsilon.$$

Thus

$$\int_{Q_T^{R_0}} u^2 e^{-\lambda t}dxdt = 0 \quad \text{and} \quad u = 0 \quad \text{in} \quad Q_T^{R_0}.$$

As R_0 is an arbitrary number, $u = 0$ in Q_T. This completes the proof. □

THEOREM 2. *Let assumptions (i)–(iii) hold; futhermore we assume that the functions* $b_i \equiv 0$ *for* $i = 1, \ldots, n$. *Then there exists a weak solution of problem* (1)–(3).

PROOF. We apply the Galerkin method. Set $\Omega_R = \Omega \cap B_R$.
Let

$$u_0^R(x) = \begin{cases} u_0(x), & x \in \Omega_{R-1}, \\ s(x)u_0(x), & x \in \Omega_R \setminus \Omega_{R-1}, \\ 0, & x \in \Omega \setminus \Omega_R, \end{cases}$$

where the function $s \in C^1(\mathbf{R}^n)$, $0 \le s(x) \le 1$ in $B_R \setminus B_{R-1}$,

$$s(x) = \begin{cases} 1, & x \in B_{R-1}, \\ 0, & x \in \mathbf{R}^n \setminus B_R; \end{cases}$$

while

$$f^R(x,t) = \begin{cases} f(x,t), & (x,t) \in Q_T^R, \\ 0, & (x,t) \in Q_T \setminus Q_T^R. \end{cases}$$

Let a function $u^N(x,t)$ be of the form

$$u^N(x,t) = \sum_{k=1}^{N} c_k^N(t)\varphi^k(x), \quad N = 1,2,\ldots$$

where $c_k^1(t), \ldots, c_k^N(t)$ are the solutions of the following Cauchy problem

(7)
$$\int_{\Omega_R} \Big(\varepsilon u_{tt}^N \varphi^l + |u_t^N|^{p-2} u_t^N \varphi^l + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^N \varphi_{x_j}^l + c(x,t) u^N \varphi^l - f^R \varphi^l\Big) dx = 0$$

(8)
$$c_k^N(0) = u_{0,k}^N \, , c_{kt}^N(0) = 0, \;\; l,k = 1,\ldots,N$$

$$\text{and } u_0^{N,R} = \sum_{k=1}^{N} u_{0,k}^N \varphi^k(x), \quad u_0^{N,R} \to u_0^R \text{ in } H_0^1(\Omega_R); \varepsilon > 0.$$

Multiplying equation (7) by $c_{kt}^N(t)$, summing over l, for $l = 1,\ldots,N$ and integrating over $t \in (0,\tau)$, we obtain

(9)
$$\int_{Q_\tau^R} \Big(\varepsilon u_{tt}^N u_t^N + |u_t^N|^{p-2} u_t^N u_t^N + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^N u_{x_j t}^N + c(x,t) u^N u_t^N - f^R u_t^N\Big) dx dt = 0.$$

It is easy to get the following estimates

$$J_1 = \int_{Q_\tau^R} \varepsilon u_{tt}^N u_t^N dxdt = \frac{1}{2}\varepsilon \int_{\Omega_{\tau,R}} (u_t^N)^2 dx;$$

$$J_2 = \int_{Q_\tau^R} |u_t^N|^{p-2}(u_t^N)^2 dxdt = \int_{Q_\tau^R} |u_t^N|^p dxdt;$$

$$J_3 = \int_{Q_\tau^R} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N u_{x_j t}^N dxdt$$

$$= \frac{1}{2}\int_{\Omega_{\tau,R}} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N u_{x_j}^N dx - \frac{1}{2}\int_{\Omega_{0,R}} \sum_{i,j=1}^n a_{ij}(x)(u_{x_i}^N(x,0))^2 dx,$$

$$J_4 = \int_{Q_\tau^R} c(x,t) u^N u_t^N dxdt \geq \frac{1}{2} c_0 \int_{\Omega_{\tau,R}} (u^N)^2 dx$$

$$- \frac{1}{2}\int_{\Omega_{0,R}} c(x,0)(u_0^{N,R})^2 dx - \frac{1}{2}\int_{Q_\tau^R} c_t(x,t)(u^N)^2 dxdt,$$

$$J_5 = \int_{Q_\tau^R} f^R(x,t) u_t^N dxdt \leq \frac{\delta_1}{p}\int_{Q_\tau^R} |u_t^N|^p dxdt + \frac{1}{q\delta_1^{\frac{p}{q}}}\int_{Q_\tau^R} |f^R|^q dxdt,$$

where $\delta_1 > 0$. So we have

$$\frac{1}{2}\varepsilon\int_{\Omega_{\tau,R}} (u_t^N)^2 dx + \Big(1 - \frac{\delta_1}{p}\Big)\int_{Q_\tau^R} |u_t^N|^p dxdt + \frac{1}{2}a_0\int_{\Omega_{\tau,R}} |u_x^N|^2 dx + \frac{1}{2}c_0\int_{\Omega_{\tau,R}} (u^N)^2 dx$$

$$\leq \frac{1}{2}c_1 \int_{Q_\tau^R} |u^N|^2 dxdt + c_2\|u_0\|_{W^{1,2}(\Omega_{0,R})} + c_3\|f\|_{L^q(Q_\tau^R)}$$

and taking into account the Gronwall–Bellman lemma we obtain

$$\int_{\Omega_R} (u^N)^2 dx \leq \sigma + \frac{1}{2}c_1 \int_{Q_\tau^R} (u^N)^2 dxdt = \sigma + \frac{1}{2}c_1 \int_0^\tau \Big(\int_{\Omega_R} (u^N)^2 dx\Big) dt.$$

It follows that

$$\int_{\Omega_{\tau,R}} \Big(|u^N|^2 + |u_x^N|^2\Big) dx \leq \mu,$$

$$\int_{Q_\tau^R} |u_t^N|^p dxdt \leq \mu, \tag{10}$$

$$\varepsilon \int_{\Omega_{\tau,R}} (u_t^N)^2 dx \leq \mu,$$

where μ is a constant independent of N.

Let $\varepsilon_k = \frac{1}{k}$. Then $\varepsilon_k u_t^k \to 0$ $*$-weakly in $L^\infty\Big((0,T); H_0^1(\Omega_R)\Big)$. From the sequence $\{u^N\}$ we can select a subsequence $\{u^k\}$ such that

$$\begin{aligned}
&u^k \longrightarrow u^R \ \ *\text{-weakly in } L^\infty((0,T); H_0^1(\Omega_R)),\\
&u_t^k \longrightarrow u_t^R \text{ weakly in } L^p(Q_T^R),\\
&|u_t^k|^{p-2}u_t^k \longrightarrow z^R \text{ weakly in } L^q(Q_T^R),
\end{aligned}$$

as $k \to +\infty$.

We show that $z^R = |u_t^R|^{p-2}u_t^R$. Consider the sequence $\{X_k\}$, where

$$\begin{aligned}
0 \le X_k = &\int_{Q_T^R} \Big(|u_t^k|^{p-2}u_t^k - |v|^{p-2}v\Big)\Big(u_t^k - v\Big)dxdt = \int_{Q_T^R} |u_t^k|^{p-2}(u_t^k)^2 dxdt\\
&- \int_{Q_T^R} |u_t^k|^{p-2}u_t^k v dxdt - \int_{Q_T^R} |v|^{p-2}v(u_t^k - v)dxdt,
\end{aligned} \tag{11}$$

for every $v \in L^p(Q_T^R)$. From (9) we have

$$\begin{aligned}
\int_{Q_T^R} |u_t^k|^p dxdt &= \int_{Q_T^R} \Big(f^R u_t^k - \varepsilon_k u_{tt}^k u_t^k - \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^k u_{x_j t}^k - c(x,t)u^k u_t^k\Big)dxdt\\
&= \int_{Q_T^R} f^R u_t^k dxdt - \frac{1}{2}\varepsilon_k \int_{\Omega_{T,R}} (u_t^k)^2 dx - \frac{1}{2}\int_{\Omega_{T,R}} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^k u_{x_j}^k dx\\
&+ \frac{1}{2}\int_{\Omega_{0,R}} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^k u_{x_j}^k dx - \frac{1}{2}\int_{\Omega_{T,R}} c(x,t)(u^k)^2 dx\\
&+ \frac{1}{2}\int_{\Omega_{0,R}} c(x,t)(u^k)^2 dx + \frac{1}{2}\int_{Q_T^R} c_t(x,t)(u^k)^2 dxdt.
\end{aligned}$$

Hence

$$\begin{aligned}
0 \le X_k = &\int_{Q_T^R} \bigg(f^R u_t^k - |u_t^k|^{p-2}u_t^k v - |v|^{p-2}v(u_t^k - v)\bigg)dxdt\\
&- \frac{1}{2}\varepsilon_k \int_{\Omega_{0,R}} |u_t^k|^2 dx - \frac{1}{2}\int_{\Omega_{T,R}} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^k u_{x_j}^k dx - \frac{1}{2}\int_{\Omega_{T,R}} c(x,T)(u^k)^2 dx\\
&+ \frac{1}{2}\int_{\Omega_{0,R}} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^k u_{x_j}^k dx + \frac{1}{2}\int_{\Omega_{0,R}} c(x,0)(u^k)^2 dx + \frac{1}{2}\int_{Q_T^R} c_t(x,t)(u^k)^2 dxdt.
\end{aligned}$$

Thus ([**4**])

$$
(12)\qquad
\begin{aligned}
0 \leq sup\ lim X_k \leq & \int_{Q_T^R} \Big(f^R u_t^R - z^R v - |v|^{p-2} v (u_t^R - v) \Big) dxdt \\
& - \frac{1}{2} \int_{\Omega_{T,R}} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^R u_{x_j}^R dx - \frac{1}{2} \int_{\Omega_{T,R}} c(x,T)(u^R)^2 dx \\
& + \frac{1}{2} \int_{\Omega_{0,R}} \sum_{i,j=1}^{n} a_{ij}(x) u_{0,x_i}^R u_{0,x_j}^R dx + \frac{1}{2} \int_{\Omega_{0,R}} c(x,0)(u_0^R)^2 dx \\
& + \frac{1}{2} \int_{Q_T^R} c_t(x,t)(u^R)^2 dxdt.
\end{aligned}
$$

We have (from (7), (8))

$$
\int_{Q_T^R} \Big(z^R v + \sum_{i,j=1}^{n} a_{ij} u_{x_i}^R v_{x_j} + c u^R v - f^R v \Big) dxdt = 0
$$

for every $v \in L^p(Q_T^R) \cap L^2\Big((0,T); H_0^1(\Omega_R)\Big)$. Analogously as in the proof of the uniqueness we can receive $v = u_t$. Hence

$$
(13)\qquad
\begin{aligned}
0 \leq & \frac{1}{2} \int_{\Omega_{T,R}} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^R u_{x_j}^R dx + \frac{1}{2} \int_{\Omega_{T,R}} c(x,T)(u^R)^2 dx \\
& - \frac{1}{2} \int_{\Omega_{0,R}} \sum_{i,j=1}^{n} a_{ij}(x) u_{0,x_i}^R u_{0,x_j}^R dx - \frac{1}{2} \int_{\Omega_{0,R}} c(x,0)(u_0^R)^2 dx \\
& + \int_{Q_T^R} \Big(z^R u_t^R - \frac{1}{2} c_t(x,t)(u^R)^2 - f^R u_t^R \Big) dxdt
\end{aligned}
$$

Adding (12) and (13) we obtain

$$
\int_{Q_T^R} \Big(- z^R v - |v|^{p-2} v (u_t^R - v_t) + z^R u_t^R \Big) dxdt \geq 0
$$

and hence

$$
\int_{Q_T^R} \Big(z^R - |v|^{p-2} v \Big) \Big(u_t^R - v \Big) dxdt \geq 0.
$$

Let $v = u_t^R - \lambda \omega_t,\ \lambda > 0, \omega \in L^p(Q_T^R)$. Therefore

$$
\int_{Q_T^R} \Big(z^R - |v|^{p-2} v \Big) \omega_t dxdt \geq 0.
$$

Letting $\lambda \to 0$

$$\int_{Q_T^R} \Big(z^R - |u_t^R|^{p-2} u_t^R \Big) \omega_t dxdt = 0.$$

This gives $z^R = |u_t^R|^{p-2} u_t^R$.

From (10) $u^k(\cdot, 0) \to u^R(\cdot, 0)$ weakly in $L^2(\Omega_{0,R})$. On the other hand $u^k(\cdot, 0) \to u_0^R$ in $H_0^1(\Omega_R)$. The result is $u^{R,\varepsilon_k}(x, 0) = u_0^R(x)$.

If R receives values $1, 2, 3, \dots$ than we have a sequence $\{u^m(x,t)\}$. Prolong every function u^m by zero beyond the domain Q_T^m. Then for the elements of the sequence $\{u^m\}$ we have the equality (for fixed R):

$$(14) \quad \int_{Q_\tau^R} \Big(|u_t^m|^{p-2} u_t^m v + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^m v_{x_j} + c(x,t) u^m v - f(x,t) v \Big) dxdt = 0$$

for every $v \in L^2((0,T); H_0^1(\Omega_R)) \cap L^p(Q_T^R)$. Let

$$u^k - u^m = u^{k,m} \text{ for } k, m > R.$$

We have

$$\int_{Q_\tau^R} \Bigg(|u_t^k|^{p-2} u_t^k v + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^k v_{x_j} + c(x,t) u^k v - f(x,t) v \Bigg) dxdt = 0$$

and hence

$$\int_{Q_\tau^R} \Bigg(\Big(|u_t^k|^{p-2} u_t^k - |u_t^m|^{p-2} u_t^m \Big) v + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{k,m} v_{x_j} + c(x,t) u^{k,m} v \Bigg) dxdt = 0.$$

Analogously as in the proof of Theorem 1, we can prove that the above equation is satisfied by $v = u_t^{k,m} \varphi^\beta$.

We estimate respective expressions

$$\begin{aligned} I_1 &= \int_{Q_\tau} \Big(|u_t^k|^{p-2} u_t^k - |u_t^m|^{p-2} u_t^m \Big) u_t^{k,m} \varphi^\beta dxdt \\ &\geq 2^{2-p} \int_{Q_\tau} |u_t^{k,m}|^p \varphi^\beta dxdt; \\ I_2 &= \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{k,m} v_{x_j} dxdt = \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{k,m} u_{tx_j}^{k,m} \varphi^\beta dxdt \\ &+ \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{k,m} u_t^{k,m} \beta \varphi^{\beta-1} \varphi_{x_j} dxdt = I_2^1 + I_2^2, \end{aligned}$$

where

$$
\begin{aligned}
I_2^1 &= \int_{Q_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^{k,m} u_{tx_j}^{k,m} \varphi^\beta dxdt \\
&= \frac{1}{2} \int_{\Omega_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^{k,m} u_{x_j}^{k,m} \varphi^\beta dx \\
&\quad - \frac{1}{2} \int_{\Omega_0} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^{k,m}(x,0) u_{x_j}^{k,m}(x,0) \varphi^\beta dx, \\
I_2^2 &= \beta \int_{Q_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^{k,m} u_t^{k,m} \varphi^{\beta-1} \varphi_{x_j} dxdt \\
&\leq \frac{1}{2} \beta n \delta_1 \mu^2 \int_{Q_\tau} \sum_{i,j=1}^{n} \left(u_{x_i}^{k,m}\right)^2 \varphi^\beta dxdt + \frac{1}{p} \beta n^2 \delta_1 \int_{Q_\tau} |u_t^{k,m}|^p \varphi^\beta dxdt \\
&\quad + \frac{\beta}{\varrho(\delta_1)} \int_{Q_\tau} \sum_{i,j=1}^{n} \varphi^{\beta - \frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} dxdt,
\end{aligned}
$$

where $\kappa = \frac{p-2}{2p}\beta - 1$;

$$
\begin{aligned}
I_3 &= \int_{Q_\tau} c(x,t) u^{k,m} v dxdt = \int_{Q_\tau} c(x,t) u^{k,m} u_t^{k,m} \varphi^\beta dxdt \\
&= \frac{1}{2} \int_{\Omega_\tau} c(x,\tau) \left(u^{k,m}\right)^2 \varphi^\beta dx - \frac{1}{2} \int_{\Omega_0} c(x,0) \left(u_0^{k,m}\right)^2 \varphi^\beta dx \\
&\quad - \frac{1}{2} \int_{Q_\tau} c_t(x,t) \left(u^{k,m}\right)^2 \varphi^\beta dxdt.
\end{aligned}
$$

Summarizing we obtain

$$
\begin{gathered}
\left(2^{2-p} - \frac{\beta n^2 \delta_1}{p}\right) \int_{Q_\tau} |u_t^{k,m}|^p \varphi^\beta dxdt + \frac{1}{2} \nu \int_{\Omega_\tau} \sum_{i=1}^{n} \left(u_{x_i}^{k,m}\right)^2 \varphi^\beta dx \\
+ \frac{1}{2} c_0 \int_{\Omega_\tau} \left(u^{k,m}\right)^2 \varphi^\beta dx \leq \frac{1}{2} \beta n \delta_1 \mu^2 \int_{Q_\tau} \sum_{i=1}^{n} \left(u_{x_i}^{k,m}\right)^2 \varphi^\beta dxdt \\
+ \frac{\beta}{\varrho(\delta_1)} \int_{Q_\tau} \sum_{i,j=1}^{n} \varphi^{\beta - \frac{2p}{p-2}} |\varphi_{x_j}|^{\frac{2p}{p-2}} dxdt + \frac{1}{2} c_1 \int_{Q_\tau} \left(u^{k,m}\right)^2 \varphi^\beta dxdt.
\end{gathered}
$$

Let $\varepsilon > 0$. Then, analogously as in the proof of the uniqueness, from the above inequality, we may obtain the following estimates

$$\int_{Q_\tau^{R_0}} \left(|u_t^{k,m}|^p + \sum_{i=1}^{n} \left(u_{x_i}^{k,m} \right)^2 \right) dxdt < \varepsilon,$$

$$\int_{\Omega_{\tau,R_0}} |u^{k,m}(x,t)|^2 dx < \varepsilon, \ \ \tau \in [0,T],$$

where R_0 is an arbitrary positive number.

We have proved that the sequence $\{u^k\}$ satisfies the Cauchy condition. It follows that this sequence is convergent. Observe that
$u_t^k \to u_t$ in $L^p_{loc}(Q_T)$, $u^k \to u$ in $L^2\Big((0,T); H^1_{0,loc}(\Omega)\Big)$,
$u^k \to u$ in $C\Big([0,T]; L^2_{loc}(\Omega)\Big)$.

On the other hand $u_0^k \to u_0$ in $H^1_{0,loc}(\Omega)$. Thus u satisfies the initial condition. Letting $m \to +\infty$ in (14) and by above convergences we conclude that u is a weak solution of problem (1)–(3). The proof of the theorem is complete. □

References

1. Brezis H., *Semilinear equations in* $\boldsymbol{R}^n$ *without condition at infinity,* Appl. Math. Optim. 1984, Vol.**12**, 271–282.
2. Bokalo M.M., *On the existence and uniqueness of a solution of the problem without conditions at infinity for semilinear parabolic equations in unbounded domain,* Sibirsky Mathematichesky Zhurnal, **34** (1993), N4 (in Russian).
3. ———, *Boundary value problem for semilinear parabolc equations without conditions at infinity in unbounded domain,* Sibirsky Mathematichesky Zhurnal, **37** (1996), N5, 977–985 (in Russian).
4. Lions J.L., *Quelques methodes de résolution des problémes aux limites non linéaires,* Dunod Gautier – Villars, Paris, 1969.

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