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BEHAVIOR OF THE CARATHÉODORY METRIC NEAR STRICTLY CONVEX BOUNDARY POINTS

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Abstract. The behavior of the Carathéodory metric near strictly convex boundary points of smooth bounded pseudoconvex domains in $\mathbb{C}^n$ is studied.

1. Introduction. Let D be a domain in $\mathbb{C}^n$. Let $\mathcal{O}(D,\varDelta)$ (resp. $\mathcal{O}(\varDelta,D)$) denote the space of all holomorphic mappings from D into the unit disc $\varDelta\subset\mathbb{C}$ (resp. from $\varDelta$ into D). The Carathéodory and Kobayashi metrics are defined by

$$C_D(a;X)=\sup\{|f'(a)X|:f\in\mathcal{O}(D,\varDelta)\},$$

$$K_D(a;X)=\inf\{\lambda>0:\exists_{f\in\mathcal{O}(\varDelta,D)},\ f(0)=a,\ f'(0)=X/\lambda\},\qquad a\in D,\ X\in\mathbb{C}^n.$$

Recall that $C_D(a;X)\le K_D(a;X)$.

Bedford and Pinchuk [**1**] proved that if D is convex and

$$d(a;X):=\inf\{\lambda>0:z+\frac{X}{\alpha}\in D\text{ if }|\alpha|>\lambda\},$$

then

$$\frac{d(a;X)}{2}\le C_D(a;X)=K_D(a;X)\le d(a;X),\quad a\in D,\ X\in\mathbb{C}^n. \tag{1}$$

Similar estimates are obtained by Chen [**2**] (see also [**5**]) near finite-type convex boundary points of smooth bounded pseudoconvex domains.

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Assume that D is a domain which is convex near a point $a_0 \in \partial D$, and ∂D does not contain any germ of a complex line through a_0. Since a localization result holds for the Kobayashi metric of D (cf. [**7**]), inequalities (1) imply that

$$\frac{1}{2} \le \liminf_{a \to a_0} \frac{K_D(a;X)}{d(a;X)} \le \limsup_{a \to a_0} \frac{K_D(a;X)}{d(a;X)} \le 1 \tag{2}$$

uniformly in $X \in \mathbb{C}^n \setminus \{0\}$. On the other hand, Graham [**3**] obtained a localization result for the Carathéodory metric of strongly pseudoconvex domains.

The main purpose of this note is to extend Graham's result and to get inequalities (analogous to (2)) for the Carathéodory metric.

Theorem 1. *Let a_0 be a boundary point of a $\mathcal{C}^\infty$-smooth bounded pseudoconvex domain $D \subset \mathbb{C}^n$. Assume that there exist a neighborhood of a_0 and a biholomorphic mapping $\Phi : U \longrightarrow \mathbb{C}^n$ such that $\Phi(D \cap U)$ is a convex domain whose boundary does not contain any segment with endpoint at $\Phi(a_0)$. Then for any neighborhood V of a_0 such that $D \cap V$ is connected, we have*

$$\lim_{a \to a_0} \frac{C_{D \cap V}(a;X)}{C_D(a;X)} = 1$$

uniformly in $X \in \mathbb{C}^n \setminus \{0\}$.

In particular, if $\Phi = \mathrm{Id}$, then

$$\frac{1}{2} \le \liminf_{a \to a_0} \frac{C_D(a;X)}{d(a;X)} \le \limsup_{a \to a_0} \frac{C_D(a;X)}{d(a;X)} \le 1$$

uniformly in $X \in \mathbb{C}^n \setminus \{0\}$.

Remarks. (i) If the conclusion of Theorem 1 holds, then ∂D obviously does not contain any germ of a complex line through a_0. There is a conjecture that the theorem still holds under this weaker assumption.

(ii) The constants $\frac{1}{2}$ and 1 in the above inequalities are the best possible for $n \ge 2$. For example, let $\mathbb{B}_n \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ be the unit ball ($n \ge 2$), $t \in (0,1)$, $a(t) := (t, 0')$, $X := (1, 0')$, and $Y := (0', 1)$; then

$$C_{\mathbb{B}_n}(a(t);Y) = d(a(t);Y) \text{ and } \frac{C_{\mathbb{B}_n}(a(t);X)}{d(a(t);X)} = \frac{1}{1+t} \underset{t \to 1-}{\longrightarrow} \frac{1}{2}.$$

On the other hand, we have the following

Proposition 2. *If a_0 is a $\mathcal{C}^1$-smooth boundary point of a plane domain D, then*

$$\lim_{a \to a_0} C_D(a;1)\,\mathrm{dist}(a;\partial D) = \lim_{a \to a_0} K_D(a;1)\,\mathrm{dist}(a;\partial D) = \frac{1}{2}.$$

Note that the assumption of smoothness is essential as the example of a quater-plane shows.

2. Proofs.

Proof of Theorem 1. It suffices to prove only the inequality

$$\limsup_{a\to a_0}\frac{C_{D\cap V}(a;X)}{C_D(a;X)}\le 1. \tag{3}$$

We apply ideas from [**8**] and [**6**]: We may assume that $\Phi(a_0)=0$, $V\subset\subset U$, $E:=\Phi(D\cap V)$ is a convex domain which is contained in

$$\Pi:=\{z\in\mathbb{C}^n:\operatorname{Re}z_1<0\},$$

and $\overline{E}\cap\partial\Pi=\{0\}$. Note that there exists a convex neighborhood $U_1\subset\Phi(V)$ of 0 such that for any point $b\in G:=E\cap U_1$ there exists the unique point $\widehat{b}\in\partial E\setminus\partial\Phi(V)$ with $\|b-\widehat{b}\|=\operatorname{dist}(b,\partial E)$, and for any $\alpha>1$, the domain E contains the image $G_{\alpha,b}$ of G under the translation $z\longrightarrow z+(b-\widehat{b})(1-1/\alpha)$ that maps the point $\frac{b-\widehat{b}}{\alpha}$ into $(b-\widehat{b})$ (use the fact that b lies on the inward normal to ∂E at $\widehat{b}$ and a continuity argument). Put

$$F_{\alpha,b}=\{z\in\mathbb{C}^n:\widehat{b}+\frac{z-\widehat{b}}{\alpha}\in G\}.$$

Since G is convex and $\overline{G}\cap\partial\Pi=\{0\}$, there exist neighborhoods $U_3\subset\subset U_2\subset\subset U_1$ such that for any $b\in G\cap U_3$ and any $\alpha>1$, we have $\widehat{b}\in\partial G\setminus\partial U_1$, $G\subset F_{\alpha,b}$, and $\operatorname{dist}(G\setminus U_2,\partial F_{\alpha,b})\ge\delta(\alpha)>0$, where $\delta(\alpha)$ does not depend on b.

Let χ be a smooth cut-off function with $\chi\equiv 0$ on $\mathbb{C}^n\setminus U_1$ and $\chi\equiv 1$ on U_2. Fix an $\alpha>1$. Let $a\in D$ with $b:=\Phi(a)\in G\cap U_3$, $X\in\mathbb{C}^n\setminus\{0\}$, and let f be an extremal function for $C_{F_{\alpha,b}}(b;Y)$, where $Y:=\Phi'(a)X$. Put $p(z):=\exp(z_1)$. For any positive integer m let

$$\widetilde{h}:=\begin{cases}(\chi f p^m)\circ\Phi & \text{on } D\cap V\\ 0 & \text{on } D\setminus V\end{cases},$$

$\widetilde{g}=\sum_{j=1}^n\widetilde{g}_j d\overline{z}_j:=\overline{\partial}\widetilde{h}$; $\widetilde{g}$ is a $\overline{\partial}$-closed smooth $(0,1)$ form on $\overline{D}$.

By Kohn's global regularity result [**4**] and Sobolev's Lemma, there exists a smooth function h on D with $\overline{\partial}h=\widetilde{g}$ and

$$\|h\|_{\mathcal{C}^1(D)}\le C\|\widetilde{g}\|_{\mathcal{C}^{n+1}(D)} \tag{4}$$

for some C which depends only on D, where

$$\|h\|_{\mathcal{C}^k(D)}:=\max_{|\mu|+|\nu|\le k}\sup_D|D_z^\mu D_{\overline{z}}^\nu h|,\quad \|\widetilde{g}\|_{\mathcal{C}^k(D)}:=\max_{j=1,\dots,n}\|\widetilde{g}_j\|_{\mathcal{C}^k(D)}.$$

Note that if $g:=fp^m\overline{\partial}\chi$ on G, then

$$\|\widetilde{g}\|_{\mathcal{C}^{n+1}(D)}\le C_n\|g\|_{\mathcal{C}^{n+1}(G)}\|\Phi\|_{\mathcal{C}^{n+1}(V)} \tag{5}$$

with a C_n depending only on n. Using the Leibniz formula, we obtain

$$\|g\|_{\mathcal{C}^{n+1}(G)} \leq 4^{n+1}\|\overline{\partial}\chi\|_{\mathcal{C}^{n+1}(\mathbb{C}^n)}\|f\|_{\mathcal{C}^{n+1}(G\setminus U_2)}\|p^m\|_{\mathcal{C}^{n+1}(G\setminus U_2)}. \tag{6}$$

The Cauchy inequalities show that

$$\|f\|_{\mathcal{C}^{n+1}(G\setminus U_2)} \leq \frac{(n+1)!}{\delta^{n+1}(\alpha)}. \tag{7}$$

Note that

$$\|p^m\|_{\mathcal{C}^{n+1}(G\setminus U_2)} = m^{n+1}\exp(-m\,\mathrm{dist}(G\setminus U_2, \partial\Pi)). \tag{8}$$

It follows from inequalities (4) – (8) that for any $\varepsilon > 0$ we may find a positive integer m which does not depend on a and X, and such that

$$\|h\|_{\mathcal{C}^1(D)} \leq \varepsilon.$$

Then $\widetilde{f} = \widetilde{h} - h$ is a holomorphic function on D and $\sup_D|\widetilde{f}| \leq 1+\varepsilon$. Recall that $f(b) = 0$ and $\chi \equiv 1$ on $U_3 \ni b$. Hence

$$(1+\varepsilon)C_D(a;X) \geq |\widetilde{f}'(a)X| \geq \exp(m\,\mathrm{Re}\,b_1)|f'(b)Y| - \varepsilon\|X\|.$$

Since the domains $F_{\alpha,b}$ and G are linearly equivalent, and $G_{\alpha,b} \subset E = \Phi(D\cap V)$, we have

$$|f'(b)Y| = C_{F_{\alpha,b}}(b;Y) = C_G(\widehat{b} + \frac{b-\widehat{b}}{\alpha}; \frac{Y}{\alpha})$$
$$= \frac{1}{\alpha}C_{G_{\alpha,b}}(b;Y) \geq \frac{1}{\alpha}C_{D\cap V}(a;X).$$

Thus

$$(1+\varepsilon)C_D(a;X) \geq \frac{\exp(m\,\mathrm{Re}\,b_1)}{\alpha}C_{D\cap V}(a;X) - \varepsilon\|X\|.$$

Finally, letting $a \longrightarrow a_0$, $\varepsilon \longrightarrow 0+$, and $\alpha \longrightarrow 1+$, we obtain inequality (3). □

Proof of Proposition 2. It suffices to show that

$$\liminf_{a\to a_0} C_D(a;1)\,\mathrm{dist}(a;\partial D) \geq \frac{1}{2} \tag{9}$$

and

$$\limsup_{a\to a_0} K_D(a;1)\,\mathrm{dist}(a;\partial D) \leq \frac{1}{2}.$$

The last inequality follows from [**6**]. Using a similar idea, we prove (9): We may assume that $a_0 = 0$. Note that for any point $a \in D$ close to a_0 there exists a point $\widehat{a} \in \partial D$ such that $\|a - \widehat{a}\| = \mathrm{dist}(a;\partial D)$ and a lies on the inward normal to ∂D at $\widehat{a}$. Let r be a $\mathcal{C}^1$-smooth defining function for D near 0, and let $\Phi_a(z) := \frac{\partial r}{\partial z}(\widehat{a})(\widehat{a} - z)$. Put

$$E_\varepsilon := \{z \in \mathbb{C} : \mathrm{Re}\,z > -\varepsilon|z|\}, \quad F_\varepsilon := \{z \in \mathbb{C} : |z| > \varepsilon\}.$$

Then, for any $\varepsilon > 0$ small enough, we have $\Phi_a(D) \subset E_\varepsilon \cup F_\varepsilon$ if $|a| < \varepsilon$. Since $\widetilde{a} := \Phi_a(a) > 0$, it follows that

$$(10) \qquad C_D(a;1) \geq C_{E_\varepsilon \cup F_\varepsilon}(\widetilde{a}; X(a)) = C_{G_{\varepsilon,a}}(1;1)\frac{|X(a)|}{\widetilde{a}} = \frac{C_{G_{\varepsilon,a}}(1;1)}{\operatorname{dist}(a;\partial D)},$$

where $X(a) := -\frac{\partial r}{\partial z}(\widehat{a})$ and $G_{\varepsilon,a} := E_\varepsilon \cup F_{\frac{\varepsilon}{\widetilde{a}}}$. Note that

$$(11) \qquad \lim_{a \to a_0} C_{G_{\varepsilon,a}}(1;1) = C_{E_\varepsilon}(1;1)$$

and

$$(12) \qquad \lim_{\varepsilon \to 0+} C_{E_\varepsilon}(1;1) = C_{E_0}(1;1) = \frac{1}{2}.$$

Indeed, to prove (12), let H_ε and $H_{\varepsilon,a}$ be the images of E_ε and $G_{\varepsilon,a}$, respectively, under the transformation $z \longrightarrow \frac{2}{z+1}$ if $\widetilde{a} < \varepsilon < 1$. Then H_ε and $\widetilde{H}_{\varepsilon,a} = H_{\varepsilon,a} \cup \{0\}$ are bounded simply connected domains, and hence $C_{H_\varepsilon} = K_{H_\varepsilon}$ and $C_{H_{\varepsilon,a}} = C_{\widetilde{H}_{\varepsilon,a}} = K_{\widetilde{H}_{\varepsilon,a}}$. By a normal family argument, it easy to see that $\lim_{a \to a_0} K_{\widetilde{H}_{\varepsilon,a}}(1;1) = K_{H_\varepsilon}(1;1)$ which implies (11). Equality (12) can be proved in the same way (or, using the fact that E_ε and E_0 are biholomorphically equivalent). Now, (9) follows from (10), (11), and (12). □

Remark. In a similar way as above, it can be proved that if a_0 is a $\mathcal{C}^1$-smooth boundary point of a plane domain D, then

$$\lim_{a \to a_0} \widetilde{K}_D(a,a)\operatorname{dist}^2(a;\partial D) = \frac{1}{4\pi} \text{ and } \lim_{a \to a_0} B_D(a;1)\operatorname{dist}(a;\partial D) = \frac{\sqrt{2}}{2},$$

where $\widetilde{K}_D$ and B_D denote the Bergman kernel and metric of D, respectively.

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