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INTEGRAL CLOSURE OF $\mathbb{C}\{X\}$ IN $\mathbb{C}[[X]]$ VIA THE PUISEUX THEOREM

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Abstract. The aim of this paper is to prove the integral closedness of the ring of convergent power series in the ring of formal power series basing on the Puiseux Theorem.

Let $X = (X_1, \dots, X_n)$ be n variables. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ let X^α denote the monomial $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$. As usual we will denote by $\mathbb{C}\,[[X]]$ the ring of formal power series of the variables $X_1, \dots, X_n$ over $\mathbb{C}$ and by $\mathbb{C}\,\{X\}$ its subring of convergent power series. (Recall that a power series $\sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha$ is called convergent if and only if there are positive constants C and M such that for each $\alpha \in \mathbb{N}^n$, $|c_\alpha| < CM^{|\alpha|}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.) Note that a convergent power series can be identified with a germ of a holomorphic function.

The aim of this article is to give a proof of the following

Theorem 1. *$\mathbb{C}\,\{X\}$ is integrally closed in $\mathbb{C}\,[[X]]$.*

Our proof will be based on the following version of the Puiseux Theorem (cf. [**2**])

Theorem 2. *Let W be a polynomial of the form:*

$$W\,(x; z) = z^d + a_1\,(x)\,z^{d-1} + \cdots + a_{d-1}\,(x)\,z + a_d\,(x)$$

whose coefficients $a_j\,(x) = a_j\,(x_1, \dots, x_n)$ are functions holomorphic on $U \times \mathbb{C}$, where U is a neighbourhood of 0 in $\mathbb{C}^n$. Assume that the discriminant $\Delta_W\,(x)$ of W is such that $\Delta_W^{-1}\,(0) \subset \{x \in U : x_n = 0\}$. Then there exists positive integer r such that

$$W\left(x', y_n^r; z\right) = \prod_{j=1}^{d} \left(z - b_j\left(x', y_n\right)\right)$$

for $(x', y_n) \in \widetilde{U}, z \in \mathbb{C}$*, where* $\widetilde{U}$ *is a neighbourhood of* 0 *in* $\mathbb{C}^n$ *and* $b_j(x', y_n)$ *are functions holomorphic on* $\widetilde{U}$*.*

Before starting the proof, observe some easy properties of convergent power series to be used later. Let $h(X) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha$ be a power series in $\mathbb{C}[[X]]$. For $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$ where all $r_j > 0$, let $\phi_r : \mathbb{C}^n \ni (X_1, \ldots, X_n) \to (X_1^{r_1}, \ldots, X_n^{r_n}) \in \mathbb{C}^n$. If a power series $(h \circ \phi_r)(X) = \sum_{\nu \in \mathbb{N}^n} b_\nu X^\nu$ is convergent then $h(X)$ is too. Similarly, for a mapping $\phi : \mathbb{C}^n \ni (X_1, \ldots, X_n) \to (X_1 X_n, \ldots, X_{n-1} X_n, X_n) \in \mathbb{C}^n$, if a power series $(h \circ \phi)(X) = \sum_{\nu \in \mathbb{N}^n} b_\nu X^\nu$ is convergent then $h(X)$ is too. Finally note that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a linear automorphism then the convergence of $h(F(X))$ implies that of h.

Proof. Let $h(X) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{C}[[X]]$ be such that $W(X, h(X)) = 0$ for $W(X; Z) = Z^d + a_1(X) Z^{d-1} + \cdots + a_{d-1}(X) Z + a_d(X) \in \mathbb{C}\{X\}[Z]$. Since $\mathbb{C}\{X\}$ is factorial (cf. [**1**], Chap. I, § 2, Prop. 3) we can assume that W is irreducible, so its discriminant $\Delta_W(X)$ is nonzero.
Case I. $\Delta_W(X)$ is as in the assumption of the Puiseux Theorem. By the theorem we get $W(Y_1, \ldots, Y_{n-1}, Y_n^r; Z) = \prod_{i=1}^d (Z - b_i(Y))$, where r is a positive integer and $b_i(Y)$ are convergent power series of variables $Y_1, \ldots, Y_n$. Hence there exists $i_0 \in \{1, \ldots, n\}$ such that $b_{i_0}(Y) = h(Y_1, \ldots, Y_{n-1}, Y_n^r)$. Hence $h \in \mathbb{C}\{X\}$.
Case II (general). reduces to Case I in the following way. Let $A = \Delta_W^{-1}(0)$, and $S_0(A) = \{x \in \mathbb{C}^n : \delta(x) = 0\}$ where δ is the initial form of Δ_W. There exists $v \in \mathbb{C}^n \setminus \{0\}$ such that $\ell = \mathbb{C}v \in \mathbb{C}^n \setminus S_0(A)$. Hence, there exists U_ℓ, an open neighbourhood of ℓ in $\mathbb{P}(\mathbb{C}^n)$, such that $U_\ell \cap S_0(A) = \{0\}$. Taking an appropriate linear automorphism we can assume that $\ell = \left(0, \ldots, 0, \underset{(n)}{1}\right)$. Thus, without loss of generality we may assume that a neighbourhood U_ℓ is of the form:

$$\begin{aligned} U_\ell &= \{y_n(y_1, y_2, \ldots, y_{n-1}, 1) \in \mathbb{C}^n : \|y'\| < \varepsilon, y_n \in \mathbb{C}\} \\ &= \{(y_n y_1, y_n y_2, \ldots, y_n y_{n-1}, y_n) \in \mathbb{C}^n : \|y'\| < \varepsilon\} \end{aligned}$$

for sufficiently small $\varepsilon > 0$ and $y' = (y_1, \ldots, y_{n-1}) \in \mathbb{C}^{n-1}$. After replacing ε again by a smaller positive constant we can also assume that $A \cap B = \{0\}$, where $B = U_\ell \cap \left(\mathbb{C}^{n-1} \times \{t \in \mathbb{C} : |t| < \varepsilon\}\right)$ (cf. [**3**], Chap. 7, Theorem 4A and Lemma 2G). Hence, composing h with the mapping $\phi(y_1, \ldots, y_{n-1}, y_n) = (y_1 y_n, \ldots, y_{n-1} y_n, y_n)$ ends the reduction to Case I. □

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