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THE GLEASON PROBLEM FOR $\mathbb{A}^k(\Omega)$, $\mathbb{H}^k(\Omega)$, $Lip_{k+\epsilon}(\Omega)$

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Abstract. Let $T(\Omega)$ be the algebra of functions holomorphic in Ω. We consider the following problem (known as the Gleason problem):

Let p be a point in Ω. Does the maximal ideal in $T(\Omega)$ consist of functions vanishing at p generated by $(z_1 - p_1), \dots, (z_n - p_n)$? We give the answer for star-shaped domains with more general situation than in **[2]**. Moreover we give some results for the Gleason problem in products of sets.

1. Introduction. For a given domain Ω in $\mathbb{C}^n$, let $\mathbb{A}(\Omega)$ be the algebra of functions holomorphic in Ω and continuous on the closure of Ω. We consider the following problem (known as the Gleason Problem):

Let p be a point in Ω. Does the maximal ideal in $\mathbb{A}(\Omega)$ consist of functions vanishing at p generated by $(z_1 - p_1), \dots, (z_n - p_n)$?

This question was considered by A.M. Gleason **[6]** in the special case when Ω is the unit ball and p is the origin. The problem is interesting since the ideals in a neighbourhood U of an algebraically finitely generated ideal in the maximal ideal space of $\mathbb{A}(\Omega)$ are also finitely generated and consequently U contains an analytic variety.

The Gleason problem, as well as other related division problems for function algebras ($\mathbb{A}^k(\Omega)$, $\mathbb{H}^k(\Omega)$, $Lip_{k+\epsilon}(\Omega)$) were considered earlier by several authors, e.g. **[14]**, **[9]**, **[3]**, **[4]**, **[11]**, **[5]**, **[12]**, **[8]**. The results, which appeared in the papers cited above, were usually the extensions of the results on the division problems for algebras $\mathbb{A}(\Omega)$, $\mathbb{H}^\infty(\Omega)$ in bounded stricly pseudoconvex domanis with sufficiently smooth boundary. These results were derived with use of theorems on the solution of the $\overline{\partial}$-problem in stricly pseudoconvex domains with boundary regularity, together with the use of some integral formulas, invented originally for the solution of the $\overline{\partial}$-problem in stricly pseudoconvex domains by Lieb **[10]**, Henkin **[7]** and Ramirez de Arellano **[13]**. In our paper we would like, similarly to the case of stricly pseudoconvex domains, to extend the results on the solution of the Gleason problem in convex domains with a $Lip_{1+\epsilon}$-boundary, obtained by Backlund and Fälström in **[2]**, into the case of more general function algebras.

We introduce a definition for the new $Lip_{k,\varphi}$ algebra (for a definition see below), which allows us to extend a result from **[2]** into convex domains with a $\bigcup_{\epsilon>0} Lip_{1+\epsilon}$ -boundary. Moreover we prove our results for algebras $\mathbb{A}^k(\Omega)$, $\mathbb{H}^k(\Omega)$ and $Lip_{k+\epsilon}(\Omega)$ (for a definition see below). We consider extending the result of **[2]** into $Lip_{k+\epsilon}(\Omega)$ algebra to be the most important. We also give the solution for the Gleason problem in some products of sets. It enables us to give a natural example of sets which have a merely continuous boundary and for which the Gleason problem has a solution.

2. Notation and Geometric Preliminaries. In this chapter we would like to explain the geometric background for the new $Lip_{k,\varphi}$ algebra used throughout the paper.

DEFINITION 2.1. Let $\varphi : [0,1] \to \mathbb{R}_+$ be a continuous function such that:

1. $\varphi(0) = 0$,
2. $\varphi(x) < \varphi(y)$ for all $x < y < 1$,
3. $\int_0^1 \frac{\varphi(\eta)}{\eta} d\eta < +\infty$,
4. there exists $M > 0$ such that if $0 < \eta < 1$ and $\eta \ln^2(\eta/2) < 1$ then: $\varphi(\eta \ln^2(\eta/2)) \leq M\varphi(\eta)$.

DEFINITION 2.2. By $\mathbb{O}(\Omega)$ we denote a set of all functions holomorphic on Ω. For a given multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ we set $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} ... \partial z_n^{\alpha_n}}$. Moreover for a non-negative integer k and $\epsilon > 0$ we define:

$$
\begin{aligned}
\mathbb{A}^k(\Omega) &:= \{h \in \mathbb{O}(\Omega) : D^\alpha h \text{ is continuous on } \overline{\Omega} \text{ for all } |\alpha| \leq k\}, \\
\mathbb{H}^k(\Omega) &:= \{h \in \mathbb{O}(\Omega) : D^\alpha h \text{ is bounded on } \Omega \text{ for all } |\alpha| \leq k\}, \\
Lip_{k+\epsilon}(U) &:= \{h \in C^k(\overline{U}) : \sup_{w,z \in \overline{U}} \frac{|D^\alpha h(z) - D^\alpha h(w)|}{|z-w|^\epsilon} < \infty \text{ for all } |\alpha| \leq k\}.
\end{aligned}
$$

Now for a given continuous function φ satisfying the conditions 1 and 2 from Definition 2.1 we set:

$$
Lip_{k,\varphi}(U) := \{h \in C^k(\overline{U}) : \lim_{(w,z)\to(x,x)} \frac{|D^\alpha h(z) - D^\alpha h(w)|}{\varphi(|z-w|)} = 0 \text{ for } x \in \overline{U}, |\alpha| \leq k\}.
$$

REMARK. If $\varphi(\eta) = \frac{1}{\ln^2(\eta/2)}$ then φ satisfies all the conditions of Definition 2.1 and moreover $Lip_{k+\epsilon}(U) \subset Lip_{k,\varphi}(U)$ for all $\epsilon > 0$.

Definition 2.3. For $x, y \in \mathbb{R}^n$ we set
$$[x, y] := \{x + t(y - x) : \ 0 \le t \le 1\}$$

Lemma 2.4. *Let $r \in Lip_{1,\varphi}(U)$ where U is a convex subdomain of $\mathbb{R}^n$. If we define ϑ_r so that*

$$r(w) - r(z) = \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z)(w - z)_j + \vartheta_r(w, z)|w - z|\varphi(|w - z|) \tag{2.1}$$

and $\vartheta_r|_{\{(z,z):\, z \in U\}} = 0$, then ϑ_r is a continuous function.

Proof. Let $h = w - z$. Without the loss of generality we can assume that $|h| < 1$. We get $r(z + h) - r(z) = \sum_j \int_0^1 \frac{\partial r}{\partial z_j}(z + th)h_j dt$. We can calculate

$$r(z + h) - r(z) - \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z)h_j = \sum_{j=1}^{n} \int_0^1 \left(\frac{\partial r}{\partial z_j}(z + th)h_j - \frac{\partial r}{\partial z_j}(z)h_j \right) dt.$$

Let

$$\psi(x, y) := \sum_{j=1}^{n} \frac{\left| \frac{\partial r}{\partial z_j}(x) - \frac{\partial r}{\partial z_j}(y) \right|}{\varphi(|x - y|)}.$$

From the definition of $Lip_{1,\varphi}(U)$ it follows that ψ is a continuous function and $\psi(x, x) = 0$ for $x \in U$. Now we can estimate

$$\frac{\left| \frac{\partial r}{\partial z_j}(z + th)h_j - \frac{\partial r}{\partial z_j}(z)h_j \right|}{|h|\varphi(|h|)} \le \frac{\left| \frac{\partial r}{\partial z_j}(z + th) - \frac{\partial r}{\partial z_j}(z) \right|}{\varphi(|th|)}$$

for $t \in (0, 1)$. From this there follows that

$$|\vartheta_r(z + h, z)| \le \sum_{j=1}^{n} \int_0^1 \frac{\left| \frac{\partial r}{\partial z_j}(z + th)h_j - \frac{\partial r}{\partial z_j}(z)h_j \right|}{|h|\varphi(|h|)} dt \le \int_0^1 \psi(z + th, z)dt.$$

We conclude that ϑ_r is continuous and $\vartheta_r|_{\{(z,z):\, z \in U\}} = 0$. □

Definition 2.5. Let φ be a function satisfying Definition 2.1. We define
$$\xi : (0, 1) \ni \eta \to \xi(\eta) = \min \left\{ \ln^2(\eta/2), \frac{1}{\varphi(\eta)} \right\} \in \mathbb{R}_+$$

Lemma 2.6. *Function ξ has the following properties:*
1. $\lim_{\eta \to 0^+} \eta\xi(\eta) = 0$;
2. $\lim_{\eta \to 0^+} \xi(\eta) = \infty$;
3. $\int_0^1 \frac{1}{\eta\xi(\eta)} d\eta < +\infty$;
4. *if* $0 < \eta < 1$ *and* $\eta \ln^2(\eta/2) < 1$ *then* $\xi(\eta)\varphi(\eta\xi(\eta)) \le M$.

Proof. We have $\lim_{\eta\to 0^+}\xi(\eta)=\infty$ because $\varphi(0)=0$. We can calculate

$$\lim_{\eta\to 0^+}\eta\xi(\eta)\le \lim_{\eta\to 0^+}\eta\ln^2(\eta/2)=0.$$

Now we can estimate:

$$\begin{array}{llll} \xi(\eta)\varphi(\eta\xi(\eta))\le & \ln^2(\eta/2)M\varphi(\eta) & \le M & \text{if } \xi(\eta)=\ln^2(\eta/2),\\ \xi(\eta)\varphi(\eta\xi(\eta))\le & \frac{1}{\varphi(\eta)}\varphi(\eta\ln^2(\eta/2)) & \le M & \text{if } \xi(\eta)=\dfrac{1}{\varphi(\eta)}. \end{array}$$

Moreover:

$$\int_0^1\frac{1}{\eta\xi(\eta)}d\eta\le\int_0^1\frac{1}{\eta\ln^2(\eta/2)}d\eta+\int_0^1\frac{\varphi(\eta)}{\eta}d\eta<+\infty.$$

□

Definition 2.7. By $K(z,r)$ we denote the ball in the $\mathbb{C}^n$ with the centre at z and radius r.

We need a geometric lemma describing precisely the properties of star-shaped domains with a $Lip_{1,\varphi}$ boundary.

Lemma 2.8. *Let Ω be a domain in $\mathbb{C}^n$ with a $Lip_{1,\varphi}$ boundary such that $[0,z]$ is not tangent to $\partial\Omega$ for every $z\in\partial\Omega$. Let $z^*\in\partial\Omega$ and let $e(z)$ be a unit tangent vector to $\partial\Omega$ at a point $z\in\partial\Omega$. There exists $\varepsilon_0\in(0,1)$ so that for all $\eta\in(0,1)$, $t\in\mathbb{C}$, $z\in\mathbb{C}^n$: if*

- $0<\eta<\varepsilon_0$,
- $2|t|<\eta\xi(\eta)$,
- $z\in K(z^*,\varepsilon_0)\cap\partial\Omega$

then we have the following property: $(1-\eta)z+te(z)\in\Omega$.

Proof. Since $\xi(\eta)\to\infty$ and $\eta\xi(\eta)\to 0$ as $\eta\to 0$ we can choose ε_0 so small that $2|z|<\xi(\eta)$ and $\eta\xi(\eta)<1$ for $0<\eta<\varepsilon_0$ and $z\in K(z^*,\varepsilon_0)\cap\partial\Omega$. Let $z_j=x_{2j}+ix_{2j-1}$. We can assume that there exists a $Lip_{1,\varphi}(K(z^*,\varepsilon_0))$–defining function r, so that $\Omega\cap K(z^*,\varepsilon_0)=\{z\in K(z^*,\varepsilon_0):\ r(z)<0\}$. We can assume that $0\notin K(z^*,\varepsilon_0)$.
Let $h=h(z,t,\eta)=te(z)-\eta z$. Firstly we can easily see that $\lim_{\varepsilon_0\to 0}h(z,t,\eta)=0$.

It is enough to prove that $r(z+h)<0$. By Lemma 2.4 we have

$$r(z+h)-r(z)=\sum_{j=1}^n\frac{\partial r}{\partial x_{2j}}(z)Re(h_j)+\sum_{j=1}^n\frac{\partial r}{\partial x_{2j-1}}(z)Im(h_j)+\vartheta_r(z+h,z)|h|\varphi(|h|).$$

From this, because $r(z)=0$, it follows that

$$\frac{r(z+h)}{\eta}=-\sum_{j=1}^n\frac{\partial r}{\partial x_{2j}}(z)Re(z_j)-\sum_{j=1}^n\frac{\partial r}{\partial x_{2j-1}}(z)Im(z_j)+\vartheta_r(z+h,z)\Big(\frac{|h|}{\eta}\varphi(|h|)\Big).$$

Moreover $|h| = |te(z) - \eta z| \leq \eta\xi(\eta)$ and:

$$\frac{|h|}{\eta}\varphi(|h|) \leq \xi(\eta)\varphi(|h|) \leq \xi(\eta)\varphi(\eta\xi(\eta)) \leq M. \tag{2.2}$$

Because $[0, z]$ is not tangent to $\partial\Omega$ we can conclude that

$$-\sum_{j=1}^{n} \frac{\partial r}{\partial x_{2j}}(z)Re(z_j) - \sum_{j=1}^{n} \frac{\partial r}{\partial x_{2j-1}}(z)Im(z_j) \neq 0. \tag{2.3}$$

Moreover from this it follows that we can choose ε_0 so small that $z - \eta z \in \Omega$ for $0 < \eta < \varepsilon_0$ and $z \in K(z^*, \varepsilon_0) \cap \partial\Omega$. Therefore

$$\frac{r(z - \eta z)}{\eta} < 0. \tag{2.4}$$

This argument is valid for all $z \in K(z^*, \varepsilon_0) \cap \partial\Omega$. Because ϑ_r is continuous and $\vartheta_r(z, z) = 0$, by (2.2) and (2.4) it follows that

$$\lim_{\varepsilon_0 \to 0} \frac{r(z+h)}{\eta} - \vartheta_r(z+h, z)\left(\frac{|h|}{\eta}\varphi(|h|)\right) \leq 0.$$

Now from property (2.3) it can be concluded that

$$-\sum_{j=1}^{n} \frac{\partial r}{\partial x_{2j}}(z)Re(z_j) - \sum_{j=1}^{n} \frac{\partial r}{\partial x_{2j-1}}(z)Im(z_j) < 0$$

for all $z \in K(z^*, \varepsilon_0) \cap \partial\Omega$. Now we can easily see that we can choose ε_0 so small that:

$$-\sum_{j=1}^{n} \frac{\partial r}{\partial x_{2j}}(z)Re(z_j) - \sum_{j=1}^{n} \frac{\partial r}{\partial x_{2j-1}}(z)Im(z_j) + \vartheta_r(z+h, z)\left(\frac{|h|}{\eta}\varphi(|h|)\right) < 0$$

for all $z \in K(z^*, \varepsilon_0) \cap \partial\Omega$. From this it follows that $r(z+h) < 0$ for all desired $z \in K(z^*, \varepsilon_0) \cap \partial\Omega$, which finishes the proof. □

3. Results for the Banach algebras $\mathbb{A}^k(\Omega)$ and $\mathbb{H}^k(\Omega)$. In this chapter we would like to solve the Gleason problem for a star-shaped domain with respect to zero and with $Lip_{1,\varphi}$ regularity on the boundary. We note that a star-shaped domain with respect to each point in its interior is convex.

We use the technique of proof which is very close to that of the Backlund and Fälström paper (compare **[2]**). This technique of the proof will be used in the next sections.

THEOREM 3.1. *If Ω is a bounded domain with a $Lip_{1,\varphi}$ boundary such that $[0, z) \subset \Omega$ and $[0, z]$ is not tangent to $\partial\Omega$ for every $z \in \partial\Omega$, then the Gleason problem has a solution for the Banach algebra $\mathbb{A}^k(\Omega)$ (resp. $\mathbb{H}^k(\Omega)$) with respect to the point* 0.

Proof. Let $z^* \in \partial\Omega$, $f \in \mathbb{A}^k(\Omega)$ (resp. $f \in \mathbb{H}^k(\Omega)$). Assume that $f(0) = 0$. We have $f(z) = \sum_{i=1}^n z_i f_i(z)$ for $f_i(z) = \int_0^1 \frac{\partial}{\partial z_i} f(\lambda z) d\lambda$. We will prove that $f_i \in \mathbb{A}^k(\Omega)$(resp. $f_i \in \mathbb{H}^k(\Omega)$).

Because Ω has a C^1-boundary, there exist an open set U such that $z^* \in U$ and the continuous functions $e^1, \dots, e^{n-1}$ on U such that:

- $e^i(z)$ is a unit tangent vector to $\partial\Omega$ for all $z \in \partial\Omega \cap U$,
- $tz \in U$ and $e^i(tz) = e^i(z)$ for all $t > 0$, $z \in \partial\Omega \cap U$,
- $e^1(z), \dots, e^{n-1}(z)$ are linearly independent vectors for every $z \in U$.

Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a multiindex so that $|\alpha| \le k$. We define $\widehat{\alpha}(i) := (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_n)$. We consider the system of n equations:

$$\begin{aligned}
\sum_{i=1}^n z_i D^\alpha f_i(z) &= D^\alpha f(z) - \sum_{\alpha_i > 0} D^{\widehat{\alpha}(i)} f_i(z), \\
\sum_{i=1}^n e_i^1(z) D^\alpha f_i(z) &= \int_0^1 \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^\alpha f(\lambda z + t e^1(z)) \right)\Big|_{t=0} d\lambda, \\
&\cdots \\
\sum_{i=1}^n e_i^{n-1}(z) D^\alpha f_i(z) &= \int_0^1 \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^\alpha f(\lambda z + t e^{n-1}(z)) \right)\Big|_{t=0} d\lambda.
\end{aligned}$$

Since the vectors $e^1(z), \dots, e^{n-1}(z), z$ are linearly independent for every $z \in U$, the above system of n equations has the unique solution $D^\alpha f_1, ..., D^\alpha f_n$. Let $\lambda \in (0, 1)$. It follows from Lemma 2.8 that there exists $\varepsilon_0 > 0$ such that if

- $1 - \lambda < \varepsilon_0$,
- $2|t| < (1 - \lambda)\xi(1 - \lambda)$,
- $z \in K(z^*, \varepsilon) \cap \partial\Omega$,
- $0 < \varepsilon \le \varepsilon_0$

then $\lambda z + t e^j(z) \in \Omega$. If for a given pair $(\varepsilon_0, \varepsilon)$: λ, z and t satisfy the above conditions we say that (λ, z, t) is admissible for $(\varepsilon_0, \varepsilon)$. Now let

$$R(\varepsilon_0, \varepsilon) := \{\lambda z + t e^j(z) \in \Omega : (\lambda, z, t) \text{ is admissible for } (\varepsilon_0, \varepsilon)\}.$$

We can easily see that $\overline{R(\varepsilon_0, \varepsilon)}$ is compact.

Using Cauchy estimates we get:

$$\left| \frac{\partial}{\partial t} \left(D^\alpha f(\lambda z + t e^j(z)) \right) \Big|_{t=0} \right| \le \frac{2 \sup_{w \in R(\varepsilon_0, \varepsilon_0)} |D^\alpha f(w)|}{(1 - \lambda)\xi(1 - \lambda)} \tag{3.1}$$

for $1 - \lambda < \varepsilon_0$, and $z \in K(z^*, \varepsilon) \cap \partial\Omega$. Moreover if we denote

$$D^\alpha \widetilde{f}(\lambda z + t e^j(z)) := D^\alpha f(\lambda z + t e^j(z)) - D^\alpha f(\lambda z^* + t e^j(z^*))$$

then

$$\left| \frac{\partial}{\partial t} \left(D^\alpha \widetilde{f}(\lambda z + t e^j(z)) \right) \Big|_{t=0} \right| \leq \frac{2 \sup_{w \in R(\varepsilon_0, \varepsilon)} \left| D^\alpha \widetilde{f}(w) \right|}{(1-\lambda)\xi(1-\lambda)} \tag{3.2}$$

for $1-\lambda < \varepsilon_0$ and $z \in K(z^*, \varepsilon) \cap \partial\Omega$. Now let $G := \overline{\Omega \cap K(z^*, \varepsilon_0/2)}$ and

$$\psi : G \times [0,1] \ni (z, \varepsilon) \to \psi(z, \varepsilon) := \int_{1-\varepsilon}^{1} \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^\alpha f(\lambda z + t e^j(z)) \right) \big|_{t=0} \, d\lambda.$$

If $f \in \mathbb{H}^k(\Omega)$ then because $\int_0^1 \frac{1}{\eta \xi(\eta)} d\eta < +\infty$ we can conclude from (3.1) that

$$\limsup_{z \in G,\, \varepsilon \to 0} |\psi(z, \varepsilon)| = 0.$$

Therefore ψ is bounded (we need the asumption that Ω is bounded) and from this it follows that $f_i \in \mathbb{H}^k(\Omega)$.

If $f \in \mathbb{A}^k(\Omega)$ then

$$\limsup_{\varepsilon \to 0,\, w \in R(\varepsilon_0, \varepsilon)} \left| D^\alpha \widetilde{f}(w) \right| = 0$$

moreover because $\int_0^1 \frac{1}{\eta \xi(\eta)} d\eta < +\infty$ we can conclude from (3.2) that ψ is continuous at the point $(z^*, 1)$ and from this follows that $D^\alpha f_i$ is continuous at z^*. Therefore $f_i \in \mathbb{A}^k(\Omega)$ and the proof is finished. $\square$

4. Results for the Banach algebra $Lip_{k+\varepsilon}(\Omega)$.

Definition 4.1. For the sets S, T, $\Omega \subset \mathbb{C}^n$ we define

$$\begin{aligned} d_S(T) &:= \inf_{w \in S, z \in T} |w - z|, \\ d_S(z) &:= \inf_{w \in S} |w - z|, \\ \Omega_\delta &:= (1-\delta)\Omega, \\ \widehat{\partial}_\alpha &:= \operatorname{grad} \circ D^\alpha. \end{aligned}$$

Definition 4.2. We denote

$$\Lambda_\varepsilon^k(\Omega) := \{ f \in C^k(\overline{\Omega}) \cap \mathbb{O}(\Omega) : \sup_{z \in \overline{\Omega},\, |\alpha| \leq k} |\widehat{\partial}_\alpha f(z)| d_{\partial\Omega}(z)^{1-\varepsilon} < \infty \}$$

for $k \in \mathbb{N}$, $\varepsilon \in (0,1)$.

Remark 4.3. If $D \subset \mathbb{R}^n$, $f \in C^1(D)$, $s \in (0,1)$ then the following conditions are equivalent

1. There exists $M > 0$ such that $|\operatorname{grad} f(z)| d_{\partial D}(z)^{1-s} \leq M$ for all $z \in D$.
2. There exists $\widetilde{M} > 0$ such that $|f(z) - f(w)| \leq \widetilde{M} |z - w|^s$ for all z, $w \in D$.

PROOF. This is a standard "Hardy-Littlewood" result; see [**1**]. □

REMARK 4.4. From the Remark 4.3 it follows that

$$\Lambda^k_\varepsilon(\Omega) = Lip_{k+\varepsilon}(\Omega) \cap \mathbb{O}(\Omega)$$

for $\varepsilon \in (0,1)$, $k \in \mathbb{N}$.

THEOREM 4.5. *If Ω is a bounded domain with a $Lip_{1,\varphi}$ boundary such that $[0,z) \subset \Omega$ and $[0,z]$ is not tangent to $\partial\Omega$ for every $z \in \partial\Omega$ then the Gleason problem has a solution for the Banach algebra $\Lambda^k_\epsilon(\Omega)$ ($k \in N$, $0 < \epsilon < 1$) at the point* 0.

First we need the following lemmas:

LEMMA 4.6. *If Ω is a bounded domain with a C^1 boundary such that $[0,z]$ is not tangent to $\partial\Omega$ for every $z \in \partial\Omega$, then there exists $M > 0$ such that:*

$$\frac{\varepsilon}{d_{\partial\Omega}(\partial\Omega_\varepsilon)} \leq M$$

for all $\varepsilon \in (0,1)$.

PROOF. Let r be a defining function for Ω.

Assume that there exists a sequence ε_n such that $\frac{\varepsilon_n}{d_{\partial\Omega}(\partial\Omega_{\varepsilon_n})} \to \infty$. From this it follows that $d_{\partial\Omega}((1-\varepsilon_n)\partial\Omega) \to 0$ and $\varepsilon_n \to 0$. We denote $\delta_n := 1 - \varepsilon_n$. There exist a_n and $b_n \in \partial\Omega$ such that $d_{\partial\Omega}(\delta_n \partial\Omega) = |\delta_n a_n - b_n|$. Because $\partial\Omega$ is a compact set we can asssume that $b_n \to b \in \partial\Omega$. This implies that $a_n \to b$.

Because $[0,b]$ is not tangent to $\partial\Omega$ at the point b we have the following inequality:

$$\sum \frac{\partial r}{\partial x_i}(b) b_i \neq 0.$$

Now because

$$\frac{d_{\partial\Omega}(\delta_n \partial\Omega)}{1-\delta_n} = \frac{|\delta_n a_n - b_n|}{1-\delta_n} \to 0$$

we can conclude that $a_n + \frac{\delta_n a_n - b_n}{1-\delta_n} = \frac{a_n - b_n}{1-\delta_n} \to b$.

There exists a sequence $\xi_n \in [a_n, b_n]$ ($\lim_{n\to\infty} \xi_n = b$) such that

$$0 = r(a_n) - r(b_n) = \sum \frac{\partial r}{\partial x_i}(\xi_n)(a_n - b_n)_i.$$

Now we can calculate

$$0 = \lim_{n\to\infty} \sum \frac{\partial r}{\partial x_i}(\xi_n)(a_n - b_n)_i \frac{1}{1-\delta_n} \to \sum \frac{\partial r}{\partial x_i}(b) b_i \neq 0$$

and we get a contradiction. □

Definition 4.7. Let $\Omega \subset \mathbb{C}^n$ be a domain such that $[0,z) \subset \Omega$ for every $z \in \partial\Omega$. We denote by ω and δ the functions defined on $\overline{\Omega} \setminus \{0\}$ so that for every $z \in \overline{\Omega} \setminus \{0\}$ we have the following properties:

1. $\omega(z) \in \partial\Omega$;
2. $\delta(z) \in [0,1)$;
3. $z = (1-\delta(z))\omega(z)$.

Lemma 4.8. *Let $\Omega \subset \mathbb{C}^n$ be a domain with $Lip_{1,\varphi}$ boundary such that $[0,z) \subset \Omega$ and $[0,z]$ is not tangent to $\partial\Omega$ for every $z \in \partial\Omega$. Let $z^* \in \partial\Omega$ and let $e(z)$ be a unit tangent vector to $\partial\Omega$ at the point $z \in \partial\Omega$. There exists $\varepsilon_0 \in (0,1)$ so that for all $\eta \in (0,1)$, $t \in \mathbb{C}$, $z \in \mathbb{C}^n$: if*

- $\eta < \varepsilon_0$,
- $z \in \omega^{-1}(K(z^*, \varepsilon_0) \cap \partial\Omega)$,
- $2|t| < (1-\delta(z))\eta\xi(\eta)$

then $(1-\eta)z + te(\omega(z)) \in \Omega_{\delta(z)}$.

Proof. It follows from Lemma 2.8 that there exists $\varepsilon_0 > 0$ such that for all $\eta \in (0,1)$, $s \in \mathbb{C}$, $z \in \mathbb{C}^n$ if

- $\eta < \varepsilon_0$,
- $2|s| < \eta\xi(\eta)$,
- $z \in K(z^*, \varepsilon_0) \cap \partial\Omega$

then $(1-\eta)z + se(z) \in \Omega$.

If $z \in \omega^{-1}(K(z^*, \varepsilon_0) \cap \partial\Omega)$ then $(1-\eta)\omega(z) + se(\omega(z)) \in \Omega$. From this we can conclude that $(1-\eta)(1-\delta(z))\omega(z) + (1-\delta(z))se(\omega(z)) \in (1-\delta(z))\Omega$. Therefore $(1-\eta)z + (1-\delta(z))se(\omega(z)) \in \Omega_{\delta(z)}$. Now it is enough to denote $t := s(1-\delta(z))$. $\square$

Now we can prove Theorem 4.5

Proof. Let $z^* \in \partial\Omega$, $f \in \Lambda^k_\epsilon(\Omega)$. Let us assume that $f(0) = 0$. We have $f(z) = \sum_{i=1}^n z_i f_i(z)$ for $f_i(z) = \int_0^1 \frac{\partial}{\partial z_i} f(\lambda z) d\lambda$. We will prove that $f_i \in \Lambda^k_\epsilon(\Omega)$.

Because Ω has a C^1-boundary, there exists an open set U such that $z^* \in U$ and there exist continuous functions $e^1, \dots, e^{n-1}$ on U such that:

- $e^i(z)$ is a unit vector tangent to $\partial\Omega$ for all $z \in \partial\Omega \cap U$,
- $tz \in U$ and $e^i(tz) = e^i(z)$ for all $t > 0$, $z \in \partial\Omega \cap U$,
- $e^1(z), \dots, e^{n-1}(z)$ are linearly independent vectors for every $z \in U$.

Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a multiindex such that $|\alpha| \leq k$. We define $\widetilde{\alpha}(i) := (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n)$. Now we can consider the system of n equations:

$$\begin{aligned}
\sum_{i=1}^{n} z_i \widehat{\partial}_\alpha f_i(z) &= \widehat{\partial}_\alpha f(z) - \sum_{\alpha_i > 0} \widehat{\partial}_{\widetilde{\alpha}(i)} f_i(z) - (D^\alpha f_1(z), \ldots, D^\alpha f_n(z)), \\
\sum_{i=1}^{n} e_i^1(z) \widehat{\partial}_\alpha f_i(z) &= \int_0^1 \lambda^{|\alpha|+1} \frac{\partial}{\partial t} \left(\widehat{\partial}_\alpha f(\lambda z + t e^1(z)) \right) \Big|_{t=0} d\lambda, \\
&\cdots \\
\sum_{i=1}^{n} e_i^{n-1}(z) \widehat{\partial}_\alpha f_i(z) &= \int_0^1 \lambda^{|\alpha|+1} \frac{\partial}{\partial t} \left(\widehat{\partial}_\alpha f(\lambda z + t e^{n-1}(z)) \right) \Big|_{t=0} d\lambda.
\end{aligned}$$

Since the vectors $e^1(z), \ldots, e^{n-1}(z), z$ are linearly independent for every $z \in U$, the above system of n equations has the unique solution $\widehat{\partial}_\alpha f_1, ..., \widehat{\partial}_\alpha f_n$. Let $\lambda \in (0, 1)$. From Lemma 4.8 it follows that there exists $\varepsilon_0 \in (0, 1)$ so that for all $\lambda \in (0; 1)$, $t \in \mathbb{C}$, $z \in \mathbb{C}^n$: if

- $1 - \lambda < \varepsilon_0$,
- $z \in \omega^{-1}(K(z^*, \varepsilon_0) \cap \partial\Omega)$,
- $2|t| < (1 - \delta(z))(1 - \lambda)\xi(1 - \lambda)$

then $\lambda z + te(\omega(z)) \in \Omega_{\delta(z)}$. Moreover we can assume that $K(z^*, \varepsilon_0) \subset U$.

Because $f \in \Lambda^k_\epsilon(\Omega)$, from Lemma 4.6 it follows that there exists T such that

- $|\widehat{\partial_\alpha} f(z)| d_{\partial\Omega}(z)^{1-\epsilon} \leq T$ for all $z \in \Omega$,
- $\frac{\delta}{d_{\partial\Omega}(\partial\Omega_\delta)} \leq T$ for all $\delta \in (0, 1)$,
- $\sup_{z \in \Omega} |z| \leq T$.

Let $z \in \Omega \setminus \{0\}$. If $w \in \Omega_{\delta(z)}$ then

$$d_{\partial\Omega}(\partial\Omega_{\delta(z)}) \leq d_{\partial\Omega}(w).$$

Moreover $d_{\partial\Omega}(z) \leq |\omega(z) - z| = |\delta(z)\omega(z)|$ and $|\omega(z)| \leq T$. We can estimate:

(4.1)
$$\frac{1}{d_{\partial\Omega}(w)^{1-\epsilon}} \leq \frac{1}{d_{\partial\Omega}(\partial\Omega_{\delta(z)})^{1-\epsilon}} \leq \frac{T^{1-\epsilon}}{\delta(z)^{1-\epsilon}} = \frac{T^{1-\epsilon}|\omega(z)|^{1-\epsilon}}{\delta(z)^{1-\epsilon}|\omega(z)|^{1-\epsilon}} \leq \frac{T^{1-\epsilon}T^{1-\epsilon}}{d_{\partial\Omega}(z)^{1-\epsilon}}.$$

Now let $z \in \omega^{-1}(K(z^*, \varepsilon_0) \cap \partial\Omega) \cap \Omega$ be such that $\delta(z) < 1/2$. We can calculate

$$\sup_{w \in \Omega_{\delta(z)}} \left| \widehat{\partial_\alpha} f(w) \right| \leq T \sup_{w \in \Omega_{\delta(z)}} \frac{1}{d_{\partial\Omega}(w)^{1-\epsilon}} \leq \frac{T^{3-2\epsilon}}{d_{\partial\Omega}(z)^{1-\epsilon}}.$$

Now from Cauchy estimate for $z \in \Omega$ with $\delta(z) < 1/2$ we get:

$$\begin{aligned}\left|\frac{\partial}{\partial t}\left(\widehat{\partial_\alpha} f(\lambda z + te^j(z))\right)\Big|_{t=0}\right| d\eta &\leq \frac{2\sup_{w\in\Omega_{\delta(z)}}\left|\widehat{\partial_\alpha} f(w)\right|}{(1-\delta(z))(1-\lambda)\xi(1-\lambda)} \\ &\leq \frac{4T^{3-2\epsilon}}{(1-\lambda)\xi(1-\lambda)d_{\partial\Omega}(z)^{1-\epsilon}}\end{aligned}$$

for $1-\varepsilon_0 < \lambda < 1$, $z \in \omega^{-1}(K(z^*,\varepsilon_0)\cap\partial\Omega)$ and $2|t| < (1-\delta(z))(1-\lambda)\xi(1-\lambda)$

Because $\int_0^1 \frac{1}{\eta\xi(\eta)}d\eta < +\infty$ then it is clear that there exists $M > 0$ such that

$$\left|\int_0^1 \lambda^{|\alpha|+1}\frac{\partial}{\partial t}\left(\widehat{\partial}_\alpha f(\lambda z + te^j(z))\right)\Big|_{t=0} d\lambda\right| \leq \frac{M}{d_{\partial\Omega}(z)^{1-\epsilon}}. \tag{4.2}$$

Moreover because

$$\begin{aligned}|D^\alpha f_i(z)|^2 &\leq \int_0^1 \sum_i \left|t^{|\alpha|}D^\alpha \frac{\partial}{\partial z_i}f(tz)\right|^2 dt \leq \int_0^1 \left|\widehat{\partial_\alpha} f(tz)\right|^2 dt \\ &\leq \int_0^1 \left(\frac{T}{d_{\partial\Omega}(tz)^{1-\epsilon}}\right)^2 dt\end{aligned}$$

therefore by (4.1) it follows that:

$$|(D^\alpha f_1(z),\dots,D^\alpha f_n(z))| \leq \sqrt{n\int_0^1\left(\frac{T}{d_{\partial\Omega}(tz)^{1-\epsilon}}\right)^2 dt} \leq \frac{\sqrt{n}T^{3-2\epsilon}}{d_{\partial\Omega}(z)^{1-\epsilon}} \tag{4.3}$$

for all $z \in \Omega$. We can conclude from (4.2) and (4.3) that $f_i \in \Lambda^k_\epsilon(\Omega)$. $\square$

5. Gleason Problem in products of sets. For a given multiindex $\alpha = (\alpha_1,...,\alpha_n)$, $\beta = (\beta_1,...,\beta_m)$ we denote $|\alpha| = \alpha_1 + ... + \alpha_n$, $|\beta| = \beta_1 + ... + \beta_m$ and $D^{\alpha,\beta} = \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1}...\partial z_n^{\alpha_n}\partial w_1^{\beta_1}...\partial w_m^{\beta_m}}$. Moreover let $\widehat{\partial}_{\alpha,\beta} :=$ grad $\circ D^{\alpha,\beta}$. If $\alpha_i > 0$ then $\widetilde{\alpha}(i) := (\alpha_1,\dots,\alpha_i - 1,\dots,\alpha_n)$.

Corollary 5.1. *Let $\Omega_2 \subset \mathbb{C}^m$ be a bounded domain. Let us assume that the Gleason problem has a solution for the Banach algebra $\mathbb{A}^k(\Omega_2)$ (resp. $\mathbb{H}^k(\Omega_2)$) at the point* 0*. If $\Omega_1 \subset \mathbb{C}^n$ is a bounded domain with a $Lip_{1,\varphi}$ boundary such that $[0,z) \subset \Omega_1$ and $[0,z]$ is not tangent to $\partial\Omega_1$ for every $z \in \partial\Omega_1$ then the Gleason problem has a solution for the Banach algebra $\mathbb{A}^k(\Omega_1\times\Omega_2)$ (resp. $\mathbb{H}^k(\Omega_1\times\Omega_2)$) at the point* 0.

Proof. For $x \in \Omega_1\times\Omega_2$ we denote $x = (x_!, x_2)$ for $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. Let $f \in \mathbb{A}^k(\Omega_1\times\Omega_2)$ (resp. $f \in \mathbb{H}^k(\Omega_1\times\Omega_2)$). We have $f(z,w) - f(0,w) = \sum_{i=1}^n z_i f_i(z,w)$ for $f_i(z,w) = \int_0^1 \frac{\partial}{\partial z_i}f(\lambda z, w)d\lambda$. Because $f(z,w) - f(0,0) = f(z,w) - f(0,w) + f(0,w) - f(0,0)$ it is enough to prove that $f_i \in \mathbb{A}^k(\Omega_1\times\Omega_2)$

(resp. $f_i \in \mathbb{H}^k(\Omega_1 \times \Omega_2)$). Without the loss of generality we can assume that $f_i(0, w) = 0$ for all $w \in \Omega_2$.

Let $(z^*, w^*) \in \partial(\Omega_1 \times \Omega_2)$. We consider the system of n equations:

$$\begin{aligned}
\sum_{i=1}^{n} z_i D^{\alpha,\beta} f_i(z, w) &= D^{\alpha,\beta} f(z, w) - \sum_{i=1,\alpha_i>0}^{n} D^{\widetilde{\alpha}(i),\beta} f_i(z, w), \\
\sum_{i=1}^{n} e_i^1(z) D^{\alpha,\beta} f_i(z, w) &= \int_0^1 \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^{\alpha,\beta} f(\lambda z + t e^1(z), w) \right)\Big|_{t=0} d\lambda, \\
&\ \ \ldots \\
\sum_{i=1}^{n} e_i^{n-1}(z) D^{\alpha,\beta} f_i(z, w) &= \int_0^1 \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^{\alpha,\beta} f(\lambda z + t e^{n-1}(z), w) \right)\Big|_{t=0} d\lambda,
\end{aligned}$$

where $e^1, ..., e^{n-1}$ are linearly independent vectors chosen for the set Ω_1 as in the proof of Theorem 3.1.

We denote

$$D^{\alpha,\beta} \widetilde{f}(\lambda z + t e^j(z), w) := D^{\alpha,\beta} f(\lambda z + t e^j(z), w) - D^{\alpha,\beta} f(\lambda z^* + t e^j(z^*), w).$$

Let ε_0, $R(\varepsilon_0, \varepsilon)$ be the same as in the proof in the Theorem 3.1. Using the same arguments (as in the proof in the Theorem 3.1) we can obtain main estimates:

$$\left| \frac{\partial}{\partial t} \left(D^{\alpha,\beta} f(\lambda z + t e^j(z), w) \right)\Big|_{t=0} \right| \leq \frac{2 \sup_{v \in R(\varepsilon_0, \varepsilon_0), w \in \Omega_2} \left| D^{\alpha,\beta} f(v, w) \right|}{(1 - \lambda)\xi(1 - \lambda)}$$

and

$$\left| \frac{\partial}{\partial t} \left(D^{\alpha,\beta} \widetilde{f}(\lambda z + t e^j(z), w) \right)\Big|_{t=0} \right| \leq \frac{2 \sup_{v \in R(\varepsilon_0, \varepsilon), w \in \Omega_2} \left| D^{\alpha,\beta} \widetilde{f}(v, w) \right|}{(1 - \lambda)\xi(1 - \lambda)}$$

for $1 - \lambda < \varepsilon_0$ and $z \in K(z^*, \varepsilon) \cap \partial\Omega$.

From this it follows that $\int_0^1 \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^{\alpha,\beta} f(\lambda z + t e^i(z), w) \right)\big|_{t=0} d\lambda \in \mathbb{A}^k(\Omega_1 \times \Omega_2)$ and $D^{\widetilde{\alpha}(i),\beta} f_i(z, w) \in \mathbb{A}^k(\Omega_1 \times \Omega_2)$ (resp. $D^{\widetilde{\alpha}(i),\beta} f_i(z, w) \in \mathbb{H}^k(\Omega_1 \times \Omega_2)$ and $\int_0^1 \lambda^{|\alpha|} \frac{\partial}{\partial t} \left(D^{\alpha,\beta} f(\lambda z + t e^i(z), w) \right)\big|_{t=0} d\lambda \in \mathbb{H}^k(\Omega_1 \times \Omega_2)$). Therefore we can conclude that $f_i \in \mathbb{A}^k(\Omega_1 \times \Omega_2)$ (resp. $f_i \in \mathbb{H}^k(\Omega_1 \times \Omega_2)$). $\square$

Corollary 5.2. *Let $\Omega_2 \subset \mathbb{C}^m$ be a bounded domain. Let us assume that the Gleason problem has a solution for the Banach algebra $\mathbb{A}^k(\Omega_2)$ (respectively $\mathbb{H}^k(\Omega_2)$) at the point 0. If $\Omega_1 \subset \mathbb{C}$ is a bounded domain, $0 \in \Omega_1$ then the Gleason problem has a solution for the Banach algebra $\mathbb{A}^k(\Omega_1 \times \Omega_2)$ (resp. $\mathbb{H}^k(\Omega_1 \times \Omega_2)$) at the point 0.*

Proof. Let $f \in \mathbb{A}^k(\Omega_1 \times \Omega_2)$ (resp. $f \in \mathbb{H}^k(\Omega_1 \times \Omega_2)$). For $(z, w) \in \Omega_1 \times \Omega_2$ we denote $h(z, w) := \frac{1}{z}(f(z, w) - f(0, w))$. Let ε_0 be so small that $K(0, 2\varepsilon_0) \subset \Omega_1$. It is enough to show that $h \in \mathbb{A}^k(K(0, \varepsilon_0) \times \Omega_2)$ (resp.

$h \in \mathbb{H}^k(K(0,\varepsilon_0) \times \Omega_2))$. We can write $f(z,w) - f(0,w) = z\int_0^1 \frac{\partial}{\partial z} f(\lambda z, w)d\lambda$. From this it follows that $h(z,w) = \int_0^1 \frac{\partial}{\partial z} f(\lambda z, w)d\lambda$. Now from the Cauchy estimate we obtain

$$\left| \frac{\partial}{\partial z}\left(D^{\alpha,\beta} f(\lambda z, w)\right)\Big|_{z=z_0}\right| \leq \frac{2\sup_{v\in K(0,2\varepsilon_0), w\in\Omega_2}\left|D^{\alpha,\beta} f(v,w)\right|}{\varepsilon_0}$$

for $z_0 \in K(0,\varepsilon_0)$, $\lambda \in (0,1)$. Therefore, like in the proof in Theorem 3.1, we can conclude the required properties. □

Now we would like to prove some results for $\Lambda^k_\epsilon(\Omega)$ algebra.

Definition 5.3. Let $\Omega \subset \mathbb{C}^k$ be a domain such that $[0,z) \subset \Omega$ for every $z \in \partial\Omega$. We denote by ω_Ω and δ_Ω the functions defined on $\overline{\Omega} \setminus \{0\}$ so that for every $z \in \overline{\Omega} \setminus \{0\}$ we have the following properties:

1. $\omega_\Omega(z) \in \partial\Omega$,
2. $\delta_\Omega(z) \in [0,1)$,
3. $z = (1-\delta_\Omega(z))\omega_\Omega(z)$.

Moreover we denote $\delta_\Omega(0) = 1$.

Definition 5.4. For $x, y \in \mathbb{C}^k$ we set

$$(x,y) := \{x + t(y-x) : \ 0 < t < 1\}.$$

Definition 5.5. For sets S, T, $\Omega \subset \mathbb{C}^n$we define

$$\begin{aligned} d_S(T) &:= \inf_{w\in S, z\in T} |w-z|, \\ d_S(z) &:= \inf_{w\in S} |w-z|, \\ \Omega_\delta &:= (1-\delta)\Omega, \\ \widehat{\partial}_{\alpha,\beta} &:= \operatorname{grad}\circ D^{\alpha,\beta}. \end{aligned}$$

Lemma 5.6. *If $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ are open, bounded domains so that $[0,x_i) \subset \Omega_i$ for all $x_i \in \Omega_i$ $(i = 1,2)$ then*

$$\min_{i=1,2} d_{\partial\Omega_i}(\delta\partial\Omega_i) \leq d_{\partial(\Omega_1\times\Omega_2)}(\delta\partial(\Omega_1 \times \Omega_2)) \tag{5.1}$$

for all $\delta \in (0,1)$.

Proof. Let $\Omega := \Omega_1 \times \Omega_2$. Let $\delta \in (0,1)$. There exist $w, z \in \partial\Omega$ such that $d_{\partial\Omega}(\delta\partial\Omega) = |z - \delta w|$. We can write $z = (z_1, z_2)$ and $w = (w_1, w_2)$ for $z_i, w_i \in \overline{\Omega}_i$. Without the loss of generality we can assume that $w_1 \in \partial\Omega_1$.

If $w_2 \in \partial\Omega_2$ then because $z_1 \in \partial\Omega_1$ or $z_2 \in \partial\Omega_2$, we may estimate

$$\min_{i=1,2} d_{\partial\Omega_i}(\delta\partial\Omega_i) \leq \sqrt{|z_1 - \delta w_1|^2 + |z_2 - \delta w_2|^2} = |z - \delta w| = d_{\partial\Omega}(\delta\partial\Omega),$$

which completes in this case. If $z_1 \in \partial\Omega_1$ then we may estimate

$$d_{\partial\Omega_1}(\delta\partial\Omega_1) \leq |z_1 - \delta w_1| \leq \sqrt{|z_1 - \delta w_1|^2 + |z_2 - \delta w_2|^2} = |z - \delta w| = d_{\partial\Omega}(\delta\partial\Omega)$$

which completes in this case.

Now we assume that $w_2 \in \Omega_2$ and $z_1 \in \Omega_1$. We conclude that $z_2 \in \partial\Omega_2$.

If $(\frac{1}{\delta}z_2, w_2) \cap \partial\Omega_2 = \emptyset$ then $(\frac{1}{\delta}z_2, w_2) \subset \overline{\Omega_2}$ but from this it follows that $\frac{1}{\delta}z_2 \in \overline{\Omega}_2$ and we have a contradiction: $z_2 \in [0, \frac{1}{\delta}z_2) \subset \Omega_2$.

If there exists $w_3 \in (\frac{1}{\delta}z_2, w_2) \cap \partial\Omega_2$ then $\delta w_3 \in (z_2, \delta w_2)$ and we have the following estimate:

$$\begin{aligned} d_{\partial\Omega_2}(\delta\partial\Omega_2) &\leq |z_2 - \delta w_3| \leq |z_2 - \delta w_2| \leq \sqrt{|z_1 - \delta w_1|^2 + |z_2 - \delta w_2|^2} \\ &= |z - \delta w| = d_{\partial\Omega}(\delta\partial\Omega) \end{aligned}$$

which completes the proof. □

LEMMA 5.7. *Let $\Omega := \Omega_1 \times ... \times \Omega_k$ where $\Omega_i \subset \mathbb{C}^{n_i}$ are open bounded domains with C^1 boundaries. If $[0, x_i) \subset \Omega_i$ for all $x_i \in \Omega_i$ and $[0, x_i]$ is not tangent to $\partial\Omega_i$ for every $x_i \in \partial\Omega_i$ $(i = 1, 2, ..., k)$ then there exists $T > 0$ such that:*

$$\frac{\delta}{d_{\partial\Omega}(\delta\partial\Omega)} \leq T$$

for all $\delta \in (0, 1)$.

PROOF. From Lemma 4.6 it follows that there exists $T > 0$ such that

$$\frac{\delta}{d_{\partial\Omega_i}(\delta\partial\Omega_i)} \leq T$$

for all $\delta \in (0, 1)$. Now by Lemma 5.6 we may estimate

$$\frac{\delta}{d_{\partial\Omega}(\delta\partial\Omega)} \leq \frac{\delta}{\min_{i=1,..,k} d_{\partial\Omega_i}(\delta\partial\Omega_i)} \leq T$$

for all $\delta \in (0, 1)$. □

LEMMA 5.8. *Let $\Omega_2 \subset \mathbb{C}^m$ be a bounded domain such that $[0, w) \subset \Omega_2$ for every $w \in \partial\Omega_2$. Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with a $Lip_{1,\varphi}$ boundary such that $[0, z) \subset \Omega_1$ and $[0, z]$ is not tangent to $\partial\Omega_1$ for every $z \in \partial\Omega_1$. By $e(z)$ we denote a unit tangent vector to $\partial\Omega_1$ at the point $z \in \partial\Omega_1$. Moreover let $\Omega := \Omega_1 \times \Omega_2$. There exists $\varepsilon_0 \in (0, 1)$ such that if*

- *$\eta \in (0, \varepsilon_0) \subset \mathbb{R}$,*
- *$z \in \overline{\Omega}_1$, $w \in \overline{\Omega}_2$, $2\delta_\Omega(z, w) < 1$,*
- *$t \in \mathbb{C}$, $8|t| < \eta\xi(\eta)$.*

then $(1 - \eta)z + te(\omega_{\Omega_1}(z)) \in (1 - \delta_\Omega(z, w))\overline{\Omega}_1$ and $w \in (1 - \delta_\Omega(z, w))\overline{\Omega}_2$, Proof. If $x \in \overline{\Omega}$ then there exists $x_1 \in \overline{\Omega}_1$ and $x_2 \in \overline{\Omega}_2$ so that $x = (x_1, x_2)$. Now we denote $(x)_i = x_i$ for $i = 1, 2$. We have the following equality: $(1 - \delta_\Omega(x))(\omega_\Omega(x))_i = x_i = \varpi(1 - \delta_{\Omega_i}(x_i))\omega_{\Omega_i}(x_i)$. From this, because $(\omega_\Omega(x))_i \in \overline{\Omega}_i$, it follows that $1 - \delta_{\Omega_i}(x_i) \leq 1 - \delta_\Omega(x)$ and therefore $\delta_\Omega(x) \leq \delta_{\Omega_i}(x_i)$ for $x \in \overline{\Omega}$, $i = 1, 2$. Now we may calculate $w \in (1 - \delta_{\Omega_2}(w))\overline{\Omega}_2 \subset (1 - \delta_\Omega(z, w))\overline{\Omega}_2$ for all $z \in \overline{\Omega}_1$, $w \in \overline{\Omega}_2$.

Because $\lim_{\eta \to 0} \eta\xi(\eta) = 0$ there exists $\varepsilon_1 > 0$ so that if

- $\eta \in (0, \varepsilon_1)$,
- $z \in \overline{\Omega}_1$, $4\delta_{\Omega_1}(z) > 3$,
- $t \in \mathbb{C}$, $8|t| < \eta\xi(\eta)$,
- $w \in \overline{\Omega}_2$, $2\delta_\Omega(z, w) < 1$

then

$$1 - \delta_{\Omega_1}\left((1 - \eta)z + te(\omega_{\Omega_1}(z))\right) < \frac{1}{2} < 1 - \delta_\Omega(z, w)$$

and therefore $(1 - \eta)z + te(\omega_{\Omega_1}(z)) \in (1 - \delta_\Omega(z, w))\overline{\Omega}_1$.

Now let us assume that $4\delta_{\Omega_1}(z) \leq 3$. If $8|t| < \eta\xi(\eta)$ then $2|t| < (1 - \delta_{\Omega_1}(z))\eta\xi(\eta)$ and therefore by Lemma 4.8 we may conclude that there exists ε_2 so that if

- $\eta \in (0, \varepsilon_2)$,
- $z \in \overline{\Omega}_1$, $4\delta_{\Omega_1}(z) \leq 3$,
- $t \in \mathbb{C}$, $8|t| < \eta\xi(\eta)$

then $(1 - \eta)z + te(\omega_{\Omega_1}(z)) \in (1 - \delta_{\Omega_1}(z))\Omega_1 \subset (1 - \delta_\Omega(z, w))\overline{\Omega}_1$.

Now it is enough to define $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$. $\square$

Theorem 5.9. *Let $\Omega_1 \subset \mathbb{C}^n$ be a bounded domain with a Lip^1_φ boundary such that $[0, z) \subset \Omega_1$ and $[0, z]$ is not tangent to $\partial\Omega_1$ for every $z \in \partial\Omega_1$. Moreover let $\Omega_2 := \widetilde{\Omega}_1 \times ... \times \widetilde{\Omega}_k \subset \mathbb{C}^m$ for bounded domains $\widetilde{\Omega}_i$ with $\mathbb{C}^1$ boundaries such that $[0, w) \subset \widetilde{\Omega}_i$ and $[0, w]$ is not tangent to $\partial\widetilde{\Omega}_i$ for every $w \in \partial\widetilde{\Omega}_i$. If $f \in \Lambda^k_\epsilon(\Omega_1 \times \Omega_2)$ then there exists $f_i \in \Lambda^k_\epsilon(\Omega_1 \times \Omega_2)$ such that $f(z, w) - f(0, w) = \sum_{i=1}^n z_i f_i(z, w)$.*

Proof. Let $\Omega := \Omega_1 \times \Omega_2$. From Lemma 5.7 it follows that we can choose $T > 0$ so that $\frac{\delta}{d_{\partial\Omega}(\partial\Omega_\delta)} \leq T$ for all $\delta \in (0, 1)$.

Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_m)$ be a multiindex such that $|\alpha| + |\beta| \leq k$. We denote $(z, w) \in \overline{\Omega}$ iff $z \in \overline{\Omega}_1$ and $w \in \overline{\Omega}_2$. Let $(z^*, w^*) \in \partial\Omega$. Moreover let $f \in \Lambda^k_\epsilon(\Omega)$. We have $f(z, w) - f(0, w) = \sum_{i=1}^n z_i f_i(z, w)$ for $f_i(z, w) = \int_0^1 \frac{\partial}{\partial z_i} f(\lambda z, w) d\lambda$. It is enough to show that $f_i \in \Lambda^k_\epsilon(\Omega)$. Now without the loss of generality we may assume that $f(0, w) = 0$ for all $w \in \Omega_2$. Because Ω_1 has a C^1-boundary, there exist an open set $U \subset \mathbb{C}^n$ such that $z^* \in U$ and continuous functions $e^1, \ldots, e^{n-1}$ on U such that:

- $e^i(z)$ is a unit tangent vector to $\partial\Omega_1$ for all $z \in \partial\Omega_1 \cap U$,
- $tz \in U$ and $e^i(tz) = e^i(z)$ for all $t > 0$, $z \in \partial\Omega_1 \cap U$,
- $e^1(z), \dots, e^{n-1}(z)$ are linearly independent vectors for every $z \in U$.

We denote $F_{n,\lambda}(z_1, ..., z_n, z_{n+1}, ..., z_{n+m}) := (\lambda z_1, ..., \lambda z_n, z_{n+1}, ..., z_{n+m})$ and $\pi(f) := (f_1, ..., f_n, 0, ..., 0)$. Now we can consider the system of n equations:

$$\begin{aligned}
\sum_{i=1}^{n} z_i \widehat{\partial}_{\alpha,\beta} f_i(z,w) &= \widehat{\partial}_{\alpha,\beta} f(z,w) - \sum_{\alpha_i > 0} \widehat{\partial}_{\widetilde{\alpha}(i),\beta} f_i(z,w) - D^{\alpha,\beta}\pi(f)(z,w), \\
\sum_{i=1}^{n} e_i^1(z) \widehat{\partial}_{\alpha,\beta} f_i(z,w) &= \int_0^1 \lambda^{|\alpha|} F_{n,\lambda}\left(\frac{\partial}{\partial t} \left(\widehat{\partial}_{\alpha,\beta} f(\lambda z + te^1(z), w) \right)\Big|_{t=0} \right) d\lambda, \\
&\dots \\
\sum_{i=1}^{n} e_i^{n-1}(z) \widehat{\partial}_{\alpha,\beta} f_i(z,w) &= \int_0^1 \lambda^{|\alpha|} F_{n,\lambda}\left(\frac{\partial}{\partial t} \left(\widehat{\partial}_{\alpha,\beta} f(\lambda z + te^{n-1}(z), w) \right)\Big|_{t=0} \right) d\lambda.
\end{aligned}$$

Since the vectors $e^1(z), \dots, e^{n-1}(z), z$ are linearly independent for every $z \in U$ the above system of n equations has the unique solution $\widehat{\partial}_{\alpha,\beta} f_1, ..., \widehat{\partial}_{\alpha,\beta} f_n$.

There exists T such that

- $|\widehat{\partial}_{\alpha,\beta} f(\tau)| d_{\partial\Omega}(\tau)^{1-\epsilon} \le T$ for all $\tau \in \Omega$,
- $\frac{\delta}{d_{\partial\Omega}(\partial\Omega_\delta)} \le T$ for all $\delta \in (0,1)$,
- $\sup_{\tau\in\Omega} |\tau| \le T$.

Let $\mu \in \Omega \setminus \{0\}$. We denote $\omega := \omega_\Omega$ and $\delta := \delta_\Omega$. If $\tau \in \Omega_{\delta(\mu)}$ then

$$d_{\partial\Omega}(\partial\Omega_{\delta(\mu)}) \le d_{\partial\Omega}(\tau).$$

Moreover $d_{\partial\Omega}(\mu) \le |\omega(\mu) - \mu| = |\delta(\mu)\omega(\mu)|$ and $|\omega(\mu)| \le T$. We may estimate:

(5.2)
$$\frac{1}{d_{\partial\Omega}(\tau)^{1-\epsilon}} \le \frac{1}{d_{\partial\Omega}(\partial\Omega_{\delta(\mu)})^{1-\epsilon}} \le \frac{T^{1-\epsilon}}{\delta(\mu)^{1-\epsilon}} = \frac{T^{1-\epsilon}|\omega(\mu)|^{1-\epsilon}}{\delta(\mu)^{1-\epsilon}|\omega(\mu)|^{1-\epsilon}} \le \frac{T^{1-\epsilon}T^{1-\epsilon}}{d_{\partial\Omega}(\mu)^{1-\epsilon}}.$$

We may calculate

$$\sup_{\tau\in\Omega_{\delta(\mu)}} \left| \widehat{\partial}_{\alpha,\beta} f(\tau) \right| \le T \sup_{\tau\in\Omega_{\delta(\mu)}} \frac{1}{d_{\partial\Omega}(\tau)^{1-\epsilon}} \le \frac{T^{3-2\epsilon}}{d_{\partial\Omega}(\mu)^{1-\epsilon}}.$$

By Lemma 5.8 there exists $\varepsilon_0 \in (0,1)$ such that if

- $\eta \in (0, \varepsilon_0) \subset \mathbb{R}$,
- $z \in \overline{\Omega}_1$, $w \in \overline{\Omega}_2$, $2\delta_\Omega(z,w) < 1$,
- $t \in \mathbb{C}$, $8|t| < \eta\xi(\eta)$

then $(1-\eta)z + te(\omega_{\Omega_1}(z)) \in (1 - \delta_\Omega(z,w))\overline{\Omega}_1$ and $w \in (1 - \delta_\Omega(z,w))\overline{\Omega}_2$.

Now from the Cauchy estimate for $\varepsilon_0 < \lambda < 1$ and $\mu = (\mu_1, \mu_2) \in \omega^{-1}(U \cap \Omega)$ with $2\delta(\mu) < 1$ we get:

$$\begin{aligned}
\left| \frac{\partial}{\partial t} \left(\widehat{\partial}_{\alpha,\beta} f(\lambda\mu_1 + te^j(\mu_1), \mu_2) \right) \Big|_{t=0} \right| &\leq \frac{8 \sup_{\tau \in \Omega_{\delta(\mu)}} \left| \widehat{\partial}_{\alpha,\beta} f(\tau) \right|}{(1-\lambda)\xi(1-\lambda)} \\
&\leq \frac{8T^{3-2\epsilon}}{(1-\lambda)\xi(1-\lambda) d_{\partial\Omega}(\mu)^{1-\epsilon}}.
\end{aligned}$$

Because $\int_0^1 \frac{1}{\eta\xi(\eta)} d\eta < +\infty$, it is clear that there exists $M > 0$ such that

$$\left| \int_0^1 \lambda^{|\alpha|} F_{n,\lambda} \left(\frac{\partial}{\partial t} \left(\widehat{\partial}_{\alpha,\beta} f(\lambda\mu_1 + te^j(\mu_1), \mu_2) \right) \Big|_{t=0} \right) d\lambda \right| \leq \frac{M}{d_{\partial\Omega}(\mu)^{1-\epsilon}} \tag{5.3}$$

for all $\mu \in \omega^{-1}(U \cap \Omega)$ with $2\delta(\mu) < 1$.

Moreover because

$$\left| D^{\alpha,\beta} f_j(z,w) \right| \leq \sqrt{\int_0^1 \sum_{i=1}^n \left| D^{\alpha,\beta} \frac{\partial}{\partial z_i} f(tz,w) \right|^2 + \sum_{i=1}^m \left| D^{\alpha,\beta} \frac{\partial}{\partial w_i} f(tz,w) \right|^2 dt}$$

by (5.2) we may estimate:

$$\begin{aligned}
\left| \left(D^{\alpha,\beta} f_1(z,w), \ldots, D^{\alpha,\beta} f_{n+m}(z,w) \right) \right| &\leq \sqrt{(n+m) \int_0^1 \left| \widehat{\partial}_{\alpha,\beta} f(tz,w) \right|^2 dt} \\
&\leq \sqrt{(n+m) \int_0^1 \left(\frac{T}{d_{\partial\Omega}(tz,w)^{1-\epsilon}} \right)^2 dt} \\
&\leq \frac{\sqrt{n+m}\, T^{3-2\epsilon}}{d_{\partial\Omega}(z,w)^{1-\epsilon}}
\end{aligned}$$

for all $(z,w) \in \Omega$. We may conclude from (5.3) and (5.2) that $f_i \in \Lambda^k_\epsilon(\Omega)$. $\square$

References

1. Ahern P., Schneider R., *Holomorphic Lipschitz functions in pseudoconvex domains,* Amer. J. Math., **101** (1979), 543–565.
2. Backlund U., Fälström A., *The Gleason problem for $A(\Omega)$,* New Zealand J. Math., **24** (1995), 17–22.
3. de Bartolomeis P., Tomassini G., *Finitely generated ideals in $A^\infty(D)$,* C. R. Acad. Sci. Paris, **293** (1981), 133–134.
4. Bonneau P., Cumenge A., Zeriahi A., *Division dans les espaces de Lipschitz de fonctions holomorphes,* C. R. Acad. Sci. Paris, **273** (1983), 145–164.
5. Bruna J., Ma Ortega J., *Closed finitely generated ideals in algebras of holomorphic functions and smooth to the boundary in stricly pseudoconvex domains,* Math. Ann., **268** (1984), 137–157.

6. Gleason A.M., *Finitely generated ideals in Banach algebras,* J. Math. Mech., **13** (1964), 125–132.
7. Henkin G.M., *Integral representations of functions holomorphic in stricly pseudoconvex domains and some applications (in Russian),* Math. USRR Sb., **11** (1970), 273–282.
8. Jakóbczak P., *Extension and decomposition with local boundary regularity properties in stricly pseudoconvex domains,* Math. Z., **188** (1985), 513–533.
9. Kerzman N., Nagel A., *Finitely generated ideals in certain function algebras,* J. Funct. Anal., **7** (1971), 212–215.
10. Lieb I., *Die Cauchy-Riemmannsche Differentialgleichungen auf streng pseudo-konvexen Gebieten: Beschränkte Lösungen,* Math. Ann., **190** (1970), 6–44.
11. Lieb I., Range M., *Lösungsoperatoren für den Cauchy–Riemann Komplex mit C^k–Abschätzungen,* Math. Ann., **253** (1980), 145–164.
12. Øvrelid N., *Generators of the maximal ideals of $A(\overline{D})$,* Pacific J. Math., **39** (1971), 219–233.
13. Ramirez de Arellano E., *Ein Divisionproblem und Randintegraldarstellungen in der komplexen Analysis,* Math. Ann., **184** (1970), 172–187.
14. Yum-Tong Siu, *The $\overline{\partial}$-problem with uniform norms on derivatives,* Math. Ann., **207** (1974), 163–176.

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