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EXISTENCE OF SOLUTIONS OF THE CAUCHY PROBLEM FOR SEMILINEAR INFINITE SYSTEMS OF PARABOLIC DIFFERENTIAL–FUNCTIONAL EQUATIONS

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Abstract. We consider the Cauchy problem for a countable system of weakly coupled semilinear differential–functional equations of parabolic type. The right-hand sides of the system are functionals of unknowns. The object of this paper is to transform the system of parabolic differential equations into the associated system of integral equation in order to prove the existence of the solution of the latter problem with use of the Schauder fixed point theorem.

1. Introduction. We consider a countable system of weakly coupled semilinear differential–functional equations of the form

$$\mathcal{F}^i[u^i](t,x) = f^i(t,x,u), \quad i \in S, \tag{1}$$

with the initial condition

$$u(0,x) = \varphi(x) \text{ for } x \in \mathbb{R}^m. \tag{2}$$

Here

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i, \mathcal{A}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{m} b^i_j(t,x)\frac{\partial}{\partial x_j} + c^i(t,x)id,$$

where id is the identity operator, $(t,x) \in \Omega := \{(t,x) : t \in (0,T),\ x \in \mathbb{R}^m\}$, $T < \infty$, S is the set of positive integers, $f^i : \overline{\Omega} \times CB^\infty_S(\overline{\Omega}) \ni (t,x,s) \to f^i(t,x,s) \in$

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$\mathbb{R}$, $i \in S$, u stands for the mapping

$$u : S \times \Omega \ni (i, t, x) \to u^i(t, x) \in \mathbb{R},$$

composed of unknowns u^i and $\varphi = (\varphi^1, \varphi^2, \dots)$.

A function u is said to be a *C–solution* of differential problem (1), (2) in $\overline{\Omega}$ if $u \in CB_S^\infty(\overline{\Omega})$ and satisfies the system of integral equations associated with the differential problem in Ω.

This paper can be considered as a continuation of the author's study of the certain infinite systems of parabolic differential–functional equations. Now the author considers a more general form of operator with lower order of terms with respect to the derivatives in x–variables. There have appeared papers devoted to an analysis of the initial-boundary value problem, e.g. [**2**], [**3**]. The infinite systems of parabolic type and properties of their solutions are treated, e.g., in [**7**] and [**14**].

The goal of this paper is to prove the existence of a solution of Cauchy problem (1), (2) for a countable systems using the Schauder fixed point theorem under weaker assumptions than in [**11**]. In paper [**11**], to solve the above problem the Banach fixed point theorem was used. This approach allowed us to prove the existence and uniqueness of the solution under rather restrictive assumptions which were previously imposed on the initial data (namely equi-boundeness of all components) and the right-hand sides (among other things, equi-boundeness of all components and the Lipschitz condition with respect to functional argument).

Now, resigning from that and applying the Schauder fixed point theorem (see [**4**], [**10**] or [**15**]), we will get the existence of the considered problem in a layer $[0, \tau) \times \mathbb{R}^m$, where τ is a sufficiently small number. Unfortunately, this theorem does not help us to obtain the uniqueness of the problem (1), (2).
In order to apply the Schauder fixed point theorem, we have to check, among other things, that certain operator is compact. We obtained it by using the well-known Fréchet compactness theorem. It is in some way an extension of the idea we have first come across in [**10**].

Let us stress that the result is obtained without using the weighted-spaces. Instead, we assume that the truncated operator $\mathbf{F}_N$ of the operator $\mathbf{F}$ (generated by right-hand sides) uniformly smothers the functions at the infinity.

An infinite system of equations was first considered by M. Smoluchowski as a model for coagulation of colloids moving according to a Brownian motion. A system of infinite number of reaction–diffusion equations related to the system of ODE derived by Smoluchowski is investigated in [**1**]. A nonlocal discrete model of cluster coagulation and fragmentation expressed in terms of an infinite system of integro–differential bilinear equations was studied in [**8**]. The coagulation–fragmentation local model which add spatial diffusion to the classical coagulation equations is considered in [**12**]. For a treatment of infinite systems of parabolic differential–functional equations with the initial-boundary value problem, where Schauder's method of fixed point is also applied, we refer the reader to [**2**].

This paper is organized as follows. In the next section the necessary notations and definitions are introduced. In section 3 we impose the necessary assumptions and formulate the main result of this paper. In section 4 we state and prove the auxiliary lemmas. The last section contains a proof of the main theorem.

2. Notations and definitions. Throughout the paper, we use the following notation. By Ω^τ we denote the set $(0,\tau)\times\mathbb{R}^m$ for each $\tau\in(0,T]$. The set Ω^T is denoted for short by Ω. The norm in $\mathbb{R}^N$ we note by $|\cdot|^N$. $B(x,\delta)$ denotes an open ball with center at x and radius $\delta>0$.

Here, by $CB^\infty(\overline{\Omega})$ we unusually denote the space of functions $h\in C(\overline{\Omega})$, such that h vanishes uniformly at infinity, i.e.

$$\forall\epsilon>0\ \exists R_\epsilon>0\ \forall t\in[0,T]\ \forall x\in\mathbb{R}^m\setminus\overline{B}(0,R_\epsilon):\ |h(t,x)|<\epsilon.$$

The author hopes that it causes no misunderstanding.

The space $CB^\infty_S(\overline{\Omega})$ comprises all functions $h=(h^1,h^2,\dots)$ such that $h^i\in CB^\infty(\overline{\Omega}),\ i\in S$ with the finite norm

$$\|h\|^\Sigma_\Omega=\sum_{i=1}^\infty\frac{1}{Q^i}\|h^i\|_\Omega,\ \text{ where } \|h^i\|_\Omega=\sup_{(t,x)\in\overline{\Omega}}|h^i(t,x)| \text{ for } i\in S,$$

and Q is an arbitrary real number. Obviously, the space $CB^\infty_S(\overline{\Omega})$ endowed with the norm $\|\cdot\|^\Sigma_\Omega$ is a Banach space.

For fixed $N\in\mathcal{N}$ let $CB^\infty_N(\overline{\Omega})$ be the space of all functions $h=(h^1,\dots,h^N)$ such that $h^i\in CB^\infty(\overline{\Omega})$ for $i=1,\dots,N$. We endow this space with the following norm: $\|h\|^N_\Omega=\max\limits_{i=1,\dots,N}\sup\limits_{(t,x)\in\overline{\Omega}}|h^i(t,x)|$.

We understand the spaces $CB^\infty(\mathbb{R}^m),\ CB^\infty_S(\mathbb{R}^m),\ CB^\infty_N(\mathbb{R}^m)$ analogously.

Now we introduce the following **($\star$)**–*condition.*

The set $K\subset CB^\infty_N(\overline{\Omega})$ is said to satisfy **($\star$)**–*condition* if and only if

$$\forall\epsilon>0\ \exists R_{\epsilon,N}>0\ \forall h\in K\ \forall t\in[0,T]\ \forall x\in\mathbb{R}^m\setminus\overline{B}(0,R_{\epsilon,N})$$

$$|h^i(t,x)|<\epsilon \text{ for } i=1,\dots,N.$$

Let $\eta\in CB^\infty_S(\overline{\Omega})$. We define the following operator $\mathbf{F}=\ (\mathbf{F}^1,\mathbf{F}^2,...)$

$$\mathbf{F}:\eta\to\mathbf{F}[\eta],$$

setting

$$\mathbf{F}^i[\eta](t,x):=f^i(t,x,\eta),\ \ i\in S.$$

3. Assumptions and the main result. Now we formulate the assumptions necessary for obtaining the result which is given at the end of this section. We will assume that:

(**H**) : the coefficients $a^i_{jk}(t,x)$, $b^i_j(t,x)$, $c^i(t,x)$, $i \in S$, $j,k = 1,\dots,m$ are bounded continuous functions such that $a^i_{jk}(t,x) = a^i_{kj}(t,x)$ and satisfy the following *Hölder continuous condition with exponent* α $(0 < \alpha \leqslant 1)$ in $\overline{\Omega}$:

$$\exists\ H > 0\ \forall\ i \in S\ \forall\ j,k = 1,\dots,m\ \forall\ t \in [0,T]\ \forall\ x,x' \in \mathbb{R}^m :$$

$$|a^i_{jk}(t,x) - a^i_{jk}(t,x')| \leqslant H|x-x'|^\alpha$$

$$|b^i_j(t,x) - b^i_j(t,x')| \leqslant H|x-x'|^\alpha$$

$$|c^i(t,x) - c^i(t,x')| \leqslant H|x-x'|^\alpha.$$

We suppose as well that the operators $\mathcal{F}^i$, $i \in S$, are uniformly parabolic in $\overline{\Omega}$ (the operators $\mathcal{A}^i$ are uniformly elliptic in $\overline{\Omega}$), i.e.

$$\exists\ \mu_1,\mu_2 > 0\ \forall\ \xi = (\xi_1,\dots,\xi_m) \in \mathbb{R}^m\ \forall\ (t,x) \in \overline{\Omega}\ \forall\ i \in S :$$

$$\mu_1 \sum_{j=1}^m \xi_j^2 \sum_{j,k=1}^m a^i_{jk}(t,x)\xi_j\xi_k \leqslant \mu_2 \sum_{j=1}^m \xi_j^2.$$

The crucial assumptions related to the initial data $\varphi = (\varphi^1,\varphi^2,\dots)$ are:

(φ_1) : $\varphi^i \in CB^\infty(\mathbb{R}^m)$ for each $i \in S$ (i.e. $\varphi \in CB^\infty_S(\mathbb{R}^m)$)

(φ_2) : there exist $Q' \in (0,Q)$ and $\bar{M} > 0$ such that $\|\varphi^i\|_{\mathbb{R}^m} \leqslant \bar{M}(Q')^{i-1}$ for each $i \in S$.

Our main requirements concerning the right-hand sides are as follows: Let the function $f = (f^1,f^2,\dots)$ generating the operator $\mathbf{F}$ be such that for each $\tau \in (0,T]$

(F_1) : $\mathbf{F} :\ CB^\infty_S(\overline{\Omega}^\tau) \to CB^\infty_S(\overline{\Omega}^\tau)$ is continuous;

(F_2) : if K is a closed, bounded set in $CB^\infty_S(\overline{\Omega}^\tau)$ then there exist $Q' \in (0,Q)$ and $\bar{M} > 0$ such that $\sup\limits_{w \in K} \|\mathbf{F}^i[w]\|_{\Omega^\tau} \leqslant \bar{M}(Q')^{i-1}$ for each $i \in S$;

(F_3) : for each $N \in \mathcal{N}$, the mapping $\mathbf{F}_N := (\mathbf{F}^1,\dots,\mathbf{F}^N)$, $\mathbf{F}_N : CB^\infty_S(\overline{\Omega}^\tau) \to CB^\infty_N(\overline{\Omega}^\tau)$ transforms a bounded set in $CB^\infty_S(\overline{\Omega}^\tau)$ into a set which satisfies **(⋆)**–*condition*;

(V) : functions f^i satisfy the *Volterra condition*, i.e. for arbitrary $(t,x) \in \Omega$ and arbitrary $\eta,\tilde{\eta} \in CB^\infty_S(\overline{\Omega})$ such that $\eta^j(\bar{t},x) = \tilde{\eta}^j(\bar{t},x)$, for $0 \leqslant \bar{t} \leqslant t$, $j \in S$ there is $f^i(t,x,\eta) = f^i(t,x,\tilde{\eta})$, $i \in S$.

The examples of the **F** operator which satisfy the imposed conditions include:

$$\mathbf{F}^i[u](t,x) = \begin{cases} \sum_{n=1}^{i} b_n(t,x)u^n(t,x) & \text{for } (t,x) \in [0,T] \times B(0,1) \\ \frac{\sum_{n=1}^{i} b_n(t,x)u^n(t,x)}{|x|^{1/i}} & \text{for } (t,x) \in [0,T] \times (\mathbb{R}^m \setminus B(0,1)), \end{cases}$$

where $b_n \in C(\overline{\Omega})$ satisfy $\|b_n\|_\Omega < \overline{M}\left(\frac{Q'}{Q}\right)^{i-1}$ for some $\overline{M} > 0,\ Q' \in (1, Q)$ or

$$\mathbf{F}^i[u](t,x) = \int_0^t b_i(\tau,x)u^i(\tau,x)u^{i-1}(\tau,x)\ d\tau,$$

where $b_i \in C(\overline{\Omega})$ satisfy $\|b_i\|_\Omega < \overline{M}\left(\frac{Q'}{Q^2}\right)^{i-1}$ for some $\overline{M} > 0,\ Q' \in (1, Q)$.

Now, let us state the main result of the paper.

THEOREM. *Let all the above assumptions hold and* $\tau^* \in (0,T]$ *be a sufficiently small number. Then the Cauchy problem* (1), (2) *has at least one* C*–solution* $u \in CB_S^\infty(\overline{\Omega}^\tau)$, *where* $0 < \tau \leqslant \tau^* \leqslant T$.

4. Auxiliary lemmas. To get the proof of the foregoing theorem, we first formulate the following lemmas.

LEMMA 1. ([**5**], Th. 2.1, p. 71 [**6**], Th. 10, p. 23)
If the operators $\mathcal{F}^i$ $(i \in S)$ *are uniformly parabolic in* $\overline{\Omega}$ *with the constants* μ_1, μ_2 *and the coefficients* $a^i_{jk}(t,x),\ b^i_j(t,x),\ c^i(t,x),\ i{\in}S,\ j,k = 1,\dots,m$ *satisfy the condition* (**H**) *in* $\overline{\Omega}$ *then there exist the fundamental solutions* $\Gamma^i(t,x;\tau,\xi)$ *of the equations*

$$\mathcal{F}^i[u^i](t,x) = 0,\ \ i \in S$$

and the following inequalities hold

$$|\Gamma^i(t,x;\tau,\xi)| \leqslant C(t-\tau)^{-\frac{m}{2}} \exp\left(- \frac{\mu^*|x-\xi|^2}{4(t-\tau)} \right),\ \ i \in S$$

for any $\mu^* < \mu$, *where* μ *depends on* μ_1, μ_2, H *and* C *depends on* μ_1, μ_2, α, T *and the nature of continuity* $a^i_{jk}(t,x)$ *in* t.

Owing to the fact that $\mathcal{F}^i$ $(i \in S)$ are uniformly parabolic in $\overline{\Omega}$ and the coefficients $a^i_{jk}(t,x)$ $(i \in S,\ j,k = 1,\dots,m)$, satisfy condition (H_a) in $\overline{\Omega}$, we notice that $\int_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)|d\xi,\ i \in S$ are equi-bounded. By C we denote the infimum of their upper bounds.

Using the fundamental solutions $\Gamma^i(t,x;\tau,\xi)$, $i \in S$, for the equations $\mathcal{F}^i[u^i](t,x) = 0$, we consider the following system of integral equations associated with differential problem (1), (2)

$$u^i(t,x) = \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)\mathbf{F}^i[u](\tau,\xi)d\xi d\tau \tag{3}$$
$$i \in S \text{ for } t > 0,\ x \in \mathbb{R}^m.$$

Now, we define $\mathbf{T} : CB^\infty_S(\overline{\Omega}) \longrightarrow CB^\infty_S(\overline{\Omega})$ in the following way

$$\mathbf{T}[z] := (\mathbf{T}^1[z^1], \mathbf{T}^2[z^2], \dots),$$

where

$$\mathbf{T}^i[z^i] = \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)z^i(\tau,\xi)d\xi d\tau.$$

By $\mathbf{T}_N$ we denote the operator $\mathbf{T}_N = (\mathbf{T}^1, \dots, \mathbf{T}^N)$.

We will now prove the following lemmas.

Lemma 2. *Let $N \in \mathcal{N}$ be fixed, $\varphi^i \in CB^\infty(\mathbb{R}^m)$, $i = 1, \dots, N$. If $G \subset CB^\infty_N(\overline{\Omega})$ is a bounded set which satisfies $(\star)$–condition, then*

$$\mathbf{T}_N(G) \subset CB^\infty_N(\overline{\Omega})$$

is a family of equicontinuous functions.

Proof. To begin with, we notice that φ^i, $i = 1, \dots, N$, are uniformly continuous. We denote $M' := \sup\limits_{z \in G} \|z\|^N_\Omega$.
Fix $\eta > 0$. Let (t,x), $(t',x') \in \overline{\Omega}$. First of all, we consider the easiest case $t = t' = 0$. Since φ^i, $i = 1, \dots N$ are uniformly continuous, there exists $\delta_1 > 0$ such that

$$\Big|\mathbf{T}^i[z^i](0,x) - \mathbf{T}^i[z^i](0,x')\Big| = |\varphi^i(x) - \varphi^i(x')| < \eta \tag{2.1}$$

if only $|x - x'|^m < \delta_1$. Now let $t > t' = 0$. It is obvious that

$$\Big|\mathbf{T}^i[z^i](t,x) - \mathbf{T}^i[z^i](0,x')\Big|$$
$$= \Big| \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)z^i(\tau,\xi)d\xi d\tau - \varphi^i(x')\Big|$$
$$\leqslant \int\limits_0^t \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau + \Big| \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \varphi^i(x')\Big|.$$

We estimate the first term in the following way

(2.2)
$$\int_0^t \int_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau \leqslant M' \int_0^t \int_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)|d\xi d\tau = CM't < \frac{\eta}{2}$$

if only $t < \delta_2 := \frac{\eta}{2CM'}$. In order to estimate the second one, we select $R^1_\eta > 0$ such that for $x \in \mathbb{R}^m \setminus \overline{B}(0, R^1_\eta)$

$$|\varphi^i(x)| < \frac{\eta}{6C} \text{ for } i = 1, \ldots, N. \tag{2.3}$$

Now, we choose $\overline{R}^1_\eta > R^1_\eta$ such that for all $x \in \mathbb{R}^m \setminus \overline{B}(0, \overline{R}^1_\eta)$, $\xi \in \overline{B}(0, R^1_\eta)$, $t \in [0, T]$

$$|\Gamma^i(t,x;0,\xi)| < \frac{\eta}{6\|\varphi^i\|_{\mathbb{R}^m} m(\overline{B}(0, R^1_\eta))}\ , \tag{2.4}$$

where m stands for the Lebesgue measure.

Let us remark that it is enough to consider x, x' such that $|x-x'|^m < \delta_2$. Thus, if $x \in \overline{B}(0, \overline{R}^1_\eta + \delta_2)$, then $x,\ x' \in \overline{B}(0, \overline{R}^1_\eta + 2\delta_2)$. According to the definition of the fundamental solution,

$$\lim_{t \searrow 0} \int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi = \varphi^i(x)$$

follows for each $x \in \mathbb{R}^m$. Since $\overline{B}(0, \overline{R}^1_\eta + 2\delta_2)$ is a compact set and φ^i, $i = 1, \ldots, N$ are continuous functions, $\int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi$ converges to $\varphi^i(x)$ uniformly on $\overline{B}(0, \overline{R}^1_\eta + 2\delta_2)$, that is, there exists $\delta_3 > 0$ independent of x such that for $t \in (0, \delta_3),\ x \in \overline{B}(0, \overline{R}^1_\eta + 2\delta_2)$

$$\left| \int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \varphi^i(x) \right| < \frac{\eta}{4} \tag{2.5}$$

follows. Next, we choose $0 < \delta_4 < \delta_3$ such that for all $|x - x'|^m < \delta_4$

$$\left| \varphi^i(x) - \varphi^i(x') \right| < \frac{\eta}{4} \text{ for } i = 1, \ldots, N. \tag{2.6}$$

Then, (2.5) and (2.6) yield the estimate

$$\begin{aligned}&\Big|\int_{\mathbb{R}^m}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi\ -\varphi^i(x')\Big|\\ &\leqslant\Big|\int_{\mathbb{R}^m}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi-\varphi^i(x)\Big|\ +|\varphi^i(x)-\varphi^i(x')|<\frac{\eta}{2}\end{aligned}\tag{2.7}$$

for $t\in(0,\delta_4),\ x,x'\in\overline{B}(0,\overline{R}^1_\eta+2\delta_2),\ |x-x'|<\delta_4$.
If $x\notin\overline{B}(0,\overline{R}^1_\eta+\delta_2)$ then $x,x'\notin\overline{B}(0,\overline{R}^1_\eta)$. Certainly,

$$\Big|\int_{\mathbb{R}^m}\Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi-\varphi^i(x')\Big|\leqslant\int_{\mathbb{R}^m}|\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi+|\varphi^i(x')|.$$

We see from (2.3) that the second term is less than $\frac{\eta}{6}$, (as it is less than $\frac{\eta}{6C}$) and we present the first one as follows

$$\int_{\mathbb{R}^m}|\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi$$

$$=\int_{\mathbb{R}^m\setminus\overline{B}(0,R^1_\eta)}|\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi+\int_{\overline{B}(0,R^1_\eta)}|\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi.$$

To estimate the first integral, we employ (2.3):

$$\int_{\mathbb{R}^m\setminus\overline{B}(0,R^1_\eta)}|\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi<\frac{\eta}{6C}\int_{\mathbb{R}^m}|\Gamma^i(t,x;0,\xi)|d\xi<\frac{\eta}{6},\tag{2.8}$$

and the second one is a directly estimated by (2.4) as follows

$$\int_{\overline{B}(0,R^1_\eta)}|\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi<\int_{\overline{B}(0,R^1_\eta)}\frac{\eta}{6\|\varphi^i\|_{\mathbb{R}^m}m(B(0,R^1_\eta))}\|\varphi^i\|_{\mathbb{R}^m}d\xi=\frac{\eta}{6}.\tag{2.9}$$

Hence (2.2), (2.7), (2.8) and (2.9) lead to the assertion that
if $\max\{t,\ |x-x'|^m\}<\delta_5:=\min\{\delta_1,\ \delta_2,\ \delta_4\}$ then

$$\Big|\mathbf{T}^i[z^i](t,x)-\mathbf{T}^i[z^i](0,x')\Big|<\eta.\tag{2.10}$$

In the next step of our proof we consider the case $t, t' > 0$ the sake of simplicity let us now assume that $\min\{t,\ t'\} = t'$. We estimate

$$\left|\mathbf{T}^i[z^i](t,x) - \mathbf{T}^i[z^i](t',x')\right|$$

$$\leqslant \left| \int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \int_{\mathbb{R}^m} \Gamma^i(t',x';0,\xi)\varphi^i(\xi)d\xi \right|$$

$$+ \left| \int_0^t \int_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)z^i(\tau,\xi)d\xi d\tau - \int_0^{t'} \int_{\mathbb{R}^m} \Gamma^i(t',x';\tau,\xi)z^i(\tau,\xi)d\xi d\tau \right| =: I_1 + I_2.$$

To estimate I_1, we select $R^2_\eta > 0$ such that, for each $x \in \mathbb{R}^m \setminus \overline{B}(0, R^2_\eta)$

$$|\varphi^i(x)| < \frac{\eta}{8C} \text{ for } i = 1, \ldots, N \tag{2.11}$$

takes place. Next, we choose $\overline{R}^2_\eta > R^2_\eta$ such that, for all $x \in \mathbb{R}^m \setminus \overline{B}(0, \overline{R}^2_\eta)$, $\xi \in \overline{B}(0, R^2_\eta)$, $t \in (0, T]$

$$|\Gamma^i(t,x;0,\xi)| < \frac{\eta}{8\|\varphi^i\|_{\mathbb{R}^m} m(B(0, R^2_\eta))} \text{ for } i = 1, \ldots, N \tag{2.12}$$

follows. As previously, let us remark that it is enough to consider x, x' such that $|x - x'|^m < \delta_5$.
Obviously, if $x \in \overline{B}(0, \overline{R}^2_\eta + \delta_5)$, then $x,\ x' \in \overline{B}(0, \overline{R}^2_\eta + 2\delta_5)$. By the same argument as before, there exists $0 < \delta_6 < \delta_5$ independent of x such that for $t \in (0, \delta_6),\ x \in \overline{B}(0, \overline{R}^2_\eta + 2\delta_5)$, the following inequality holds

$$\left| \int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \varphi^i(x) \right| < \frac{\eta}{8} \text{ for } i = 1, \ldots, N. \tag{2.13}$$

If we select $0 < \delta_7 < \delta_6$ such that

$$|\varphi^i(x) - \varphi^i(x')| < \frac{\eta}{4} \text{ for } i = 1, \ldots, N, \tag{2.14}$$

takes place for all $|x - x'|^m < \delta_7$ then for $t < \delta_6,\ x,\ x' \in \overline{B}(0, \overline{R}^2_\eta)$ such that $|x - x'|^m < \delta_7$, using (2.13) and (2.14) we immediately obtain

$$I_1 \leqslant \left| \int_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \varphi^i(x) \right| \tag{2.15}$$

$$+ \left| \int_{\mathbb{R}^m} \Gamma^i(t',x';0,\xi)\varphi^i(\xi)d\xi - \varphi^i(x') \right| + |\varphi^i(x) - \varphi^i(x')| < \frac{\eta}{2}.$$

Now, let $t \in [\delta_6, T]$. We remark that it is enough to consider t' such that $|t-t'| < \delta_7$. Then certainly $t, t' \in [\delta_6 - \delta_7, T]$. Since $\Gamma^i(\cdot, \cdot, 0, \xi),\ i = 1, \dots, N$ are uniformly continuous on $[\delta_6 - \delta_7, T] \times \overline{B}(0, \overline{R}_\eta^2 + 2\delta_5)$ there exists $0 < \delta_8 < \delta_7$ such that, if $\max\{|t-t'|,\ |x-x'|^m\} < \delta_8$, then

$$|\Gamma^i(t,x;0,\xi) - \Gamma^i(t',x';0,\xi)| < \frac{\eta}{4\|\varphi^i\|_{\mathbb{R}^m} m(\overline{B}(0,R_\eta^2))}. \tag{2.16}$$

We estimate the term I_1 in the following way

$$\begin{aligned} I_1 &\leqslant \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;0,\xi) - \Gamma^i(t',x';0,\xi)||\varphi^i(\xi)|d\xi \\ &= \int\limits_{\overline{B}(0,R_\eta^2)} |\Gamma^i(t,x;0,\xi) - \Gamma^i(t',x';0,\xi)||\varphi^i(\xi)|d\xi \\ &+ \int\limits_{\mathbb{R}^m \setminus \overline{B}(0,R_\eta^2)} |\Gamma^i(t,x;0,\xi) - \Gamma^i(t',x';0,\xi)||\varphi^i(\xi)|d\xi =: I_{11} + I_{12}. \end{aligned} \tag{2.17}$$

Hence, applying (2.16) and (2.11), we easily see that for $(t,x),\ (t',x') \in [\delta_6 - \delta_7, T] \times \overline{B}(0, \overline{R}_\eta^2 + 2\delta_5)$ such that $\max\{|t-t'|,\ |x-x'|^m\} < \delta_8$ there is

$$I_{11} < \int\limits_{\overline{B}(0,R_\eta^2)} \frac{\eta}{4\|\varphi^i\|_{\mathbb{R}^m} m(\overline{B}(0,R_\eta^2))} \|\varphi^i\|_{\mathbb{R}^m} d\xi = \frac{\eta}{4} \tag{2.18}$$

as well as

$$I_{12} < \frac{\eta}{8C} \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;0,\xi)| + |\Gamma^i(t',x';0,\xi)| d\xi < \frac{\eta}{4}. \tag{2.19}$$

If, however, $x \notin \overline{B}(0, \overline{R}_\eta^2 + \delta_5)$, then $x, x' \notin \overline{B}(0, \overline{R}_\eta^2)$. Consequently, using (2.17), (2.11) and (2.12) we estimate I_1 as follows

$$\begin{aligned} I_1 &\leqslant I_{11} + I_{12} < \|\varphi^i\|_{\mathbb{R}^m} \int\limits_{B(0,R_\eta^2)} |\Gamma^i(t,x;0,\xi)| + |\Gamma^i(t',x';0,\xi)| d\xi \\ &+ \frac{\eta}{8C} \int\limits_{\mathbb{R}^m \setminus \overline{B}(0,R_\eta^2)} |\Gamma^i(t,x;0,\xi)| + |\Gamma^i(t',x';0,\xi)| d\xi \leqslant \frac{\eta}{2}. \end{aligned} \tag{2.20}$$

Let us now recall (2.15), (2.18), (2.19) and (2.20) to state that for all $(t,x),\ (t',x') \in \overline{\Omega}$ such that $\max\{|t-t'|,\ |x-x'|^m\} < \delta_8$ the following estimate follows

$$I_1 = \left| \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi - \int\limits_{\mathbb{R}^m} \Gamma^i(t',x';0,\xi)\varphi^i(\xi)d\xi \right| < \frac{\eta}{2}. \tag{2.21}$$

Next, let us demonstrate that $I_2 < \frac{\eta}{2}$ as well. To confirm this assertion we note that

$$\begin{aligned} I_2 \leqslant & \int\limits_0^{t'} \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau \\ & + \int\limits_{t'}^{t} \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau =: I_{21} + I_{22}. \end{aligned}$$

There exists $\delta_9 := \min\{\frac{\eta}{4CM'}, \delta_8\}$, where $M' := \sup\limits_{z\in G} \|z\|_{\Omega}^N$, such that for all $x \in \mathbb{R}^m$ and t, t' such that $t - t' < \delta_9$

$$I_{22} \leqslant CM'(t-t') < CM'\delta_9 < \frac{\eta}{4}. \tag{2.22}$$

Since G satisfies **(⋆)**–*condition*, there exists $R_\eta^3 > 0$ such that for all $t \in [0,T]$, $x \in \mathbb{R}^m \setminus \overline{B}(0, R_\eta^3)$ the inequality

$$|z^i(t,x)| < \frac{\eta}{16CT} \text{ for } i = 1,\dots,N. \tag{2.23}$$

holds. We present I_{21} as follows

$$\begin{aligned} I_{21} \ = & \int\limits_0^{t'} \int\limits_{\mathbb{R}^m \setminus \overline{B}(0,R_\eta^3)} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau \\ & + \int\limits_0^{t'} \int\limits_{\overline{B}(0,R_\eta^3)} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau := I_{211} + I_{212}. \end{aligned}$$

Applying (2.23) we can establish the following estimate

$$I_{211} < \frac{\eta}{16CT} \int\limits_0^{t'} \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)| + |\Gamma^i(t',x';\tau,\xi)|d\xi d\tau \leqslant \frac{\eta}{8}. \tag{2.24}$$

Obviously

$$I_{212} \leqslant M' \int_0^{t'} \int_{\overline{B}(0,R_\eta^3)} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)| d\xi d\tau. \tag{2.25}$$

Let us notice that in view of the integrability of fundamental solution, there exists $0 < \delta_{10} < \delta_9$ such that if $(t,x),(t',x') \in \overline{\Omega}$ and $t' \leqslant t,\ \tau \in (0,t')$ then for each $\overline{x} \in \mathbb{R}^m$

$$\int_0^{t'} \int_{B(\overline{x},\delta_{10})} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)| d\xi < \frac{\eta}{16M'} \text{ for } i = 1,\dots,N \tag{2.26}$$

follows. In order to estimate (2.25) we present it as follows

$$\begin{aligned}
&\int_0^{t'} \int_{\overline{B}(0,R_\eta^3)} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)| d\xi d\tau \\
&\quad = \int_0^{t'} \int_{\overline{B}(0,R_\eta^3)\cap B(\frac{x+x'}{2},\delta_{10})} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)| d\xi d\tau \\
&\quad + \int_0^{t'} \int_{\overline{B}(0,R_\eta^3)\setminus B(\frac{x+x'}{2},\delta_{10})} |\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)| d\xi d\tau.
\end{aligned} \tag{2.27}$$

(2.26) yields that the first term of (2.27) is less than $\frac{\eta}{16M'}$. Now, we assume that $\max\{|t-t'|,\ |x-x'|^m\} < \delta_{11} < \delta_{10}$ for some $\delta_{11} > 0$. By virtue of the mean-value theorem there exists (t_0,x_0) belonging to the segment connecting $(t,x),\ (t',x')$, such that

$$\begin{aligned}
&|\Gamma^i(t,x;\tau,\xi) - \Gamma^i(t',x';\tau,\xi)| \\
&\quad = \left|(t-t')\frac{\partial \Gamma^i}{\partial t}(t_0,x_0,\tau,\xi) + \sum_{k=1}^m (x_k - x_k')\frac{\partial \Gamma^i}{\partial x_k}(t_0,x_0,\tau,\xi)\right| \\
&\quad < \delta_{11}\left(\left|\frac{\partial \Gamma^i}{\partial t}(t_0,x_0,\tau,\xi)\right| + \sum_{k=1}^m \left|\frac{\partial \Gamma^i}{\partial x_k}(t_0,x_0,\tau,\xi)\right|\right).
\end{aligned} \tag{2.28}$$

If $|x_0-\xi| \geq \frac{\delta_{10}}{2}$, then the derivatives appearing in (2.28) are esimated as below (cf. [**9**], p. 427)

$$\begin{aligned}
&\left|\frac{\partial \Gamma^i}{\partial t}(t_0,x_0,\tau,\xi)\right| < C'(t_0-\tau)^{-\frac{m+2}{2}} \exp\left(-C''\frac{|x_0-\xi|^2}{t_0-\tau}\right) \\
&\qquad \leqslant C'(t_0-\tau)^{-\frac{m+2}{2}} \exp\left(-C''\frac{\delta_{10}^2}{4(t_0-\tau)}\right) \leqslant M_t, \\
&\left|\frac{\partial \Gamma^i}{\partial x_k}(t_0,x_0,\tau,\xi)\right| < C'(t_0-\tau)^{-\frac{m+1}{2}} \exp\left(-C''\frac{|x_0-\xi|^2}{t_0-\tau}\right) \\
&\qquad \leqslant C'(t_0-\tau)^{-\frac{m+1}{2}} \exp\left(-C''\frac{\delta_{10}^2}{4(t_0-\tau)}\right) \leqslant M_x \text{ for } k=1,\ldots,m.
\end{aligned} \tag{2.29}$$

Owing to (2.29), we discover that for all (t,x), $(t',x') \in \overline{\Omega}$ such that $\max\{|t-t'|,\ |x-x'|^m\} < \delta_{11}$ we obtain

$$\int_0^{t'} \int_{\overline{B}(0,R_\eta^3)\setminus B(\frac{x+x'}{2},\delta_{10})} |\Gamma^i(t,x;\tau,\xi)-\Gamma^i(t',x';\tau,\xi)|d\xi d\tau \tag{2.30}$$

$$< \delta_{11}(M_t+kM_x)m(\overline{B}(0,R_\eta^3))T < \frac{\eta}{16M'}$$

if only $\delta_{11} < \frac{\eta}{16(M_t+kM_x)m(\overline{B}(0,R_\eta^3))M'T}$.

Thus, combining (2.22), (2.24), (2.25), (2.26) and (2.30), we see that for all (t,x), $(t',x') \in \overline{\Omega}$ such that $\max\{|t-t'|,\ |x-x'|^m\} < \delta_{11}$,

$$I_2 = \left|\int_0^t \int_{\mathbb{R}^m} \Gamma^i(t,x;\tau,\xi)z^i(\tau,\xi)d\xi d\tau - \int_0^{t'} \int_{\mathbb{R}^m} \Gamma^i(t',x';\tau,\xi)z^i(\tau,\xi)d\xi d\tau\right| < \frac{\eta}{2}. \tag{2.31}$$

And finally, owing to (2.1), (2.10), (2.21) and (2.31) we conclude that for all (t,x), $(t',x') \in \overline{\Omega}$, $z \in G$, if $\max\{|t-t'|,\ |x-x'|^m\} < \delta_{11}$, then

$$\left|\mathbf{T}^i[z^i](t,x) - \mathbf{T}^i[z^i](t',x')\right| < \eta,$$

which completes the proof of Lemma 2. □

Next we will show a certain property of the operator $\mathbf{T}_N$. Roughly speaking, we will check that the operator $\mathbf{T}_N$ transfers the **(**$\star$**)**–*condition.*

Lemma 3. *Let* $N \in \mathcal{N}$ *be fixed. If* $\varphi^i \in CB^\infty(\mathbb{R}^m)$, $i=1,\ldots N$, *then the operator* $\mathbf{T}_N$ *transforms a bounded set in* $CB_N^\infty(\overline{\Omega})$, *satisfying* **(**$\star$**)**–*condition into a set satisfying* **(**$\star$**)**–*condition in* $CB_N^\infty(\overline{\Omega})$.

Proof. Let $K \subset CB_N^\infty(\overline{\Omega})$ be a bounded set which satisfies the $(\star)$–*condition.* Fix $\epsilon > 0$. There exists $R_\epsilon^1 > 0$ such that for $x \in \mathbb{R}^m \setminus \overline{B}(0, R_\epsilon^1)$

$$|\varphi^i(x)| < \frac{\epsilon}{4C} \text{ for } i = 1, \dots, N \tag{3.1}$$

follows. We select $\overline{R}_\epsilon^1 > R_\epsilon^1$ such that for $x \in \mathbb{R}^m \setminus \overline{B}(0, \overline{R}_\epsilon^1),\ \xi \in B(0, R_\epsilon^1),\ t \in [0, T]$ we can obtain the inequality

$$|\Gamma^i(t, x; 0, \xi)| < \frac{\epsilon}{4\|\varphi^i\|_{\mathbb{R}^m} m(B(0, R_\epsilon^1))} \text{ for } i = 1, \dots, N. \tag{3.2}$$

It is obvious that

$$\int\limits_{\mathbb{R}^m} |\Gamma^i(t, x; 0, \xi)||\varphi^i(\xi)| d\xi =$$

$$= \int\limits_{\overline{B}(0, R_\epsilon^1)} |\Gamma^i(t, x; 0, \xi)||\varphi^i(\xi)| d\xi + \int\limits_{\mathbb{R}^m \setminus \overline{B}(0, R_\epsilon^1)} |\Gamma^i(t, x; 0, \xi)||\varphi^i(\xi)| d\xi.$$

Therefore, by (3.1) and (3.2) for $x \in \mathbb{R}^m \setminus \overline{B}(0, \overline{R}_\epsilon^1)$, we easily obtain the following estimates

$$\int\limits_{\mathbb{R}^m \setminus \overline{B}(0, R_\epsilon^1)} |\Gamma^i(t, x; 0, \xi)||\varphi^i(\xi)| d\xi < \frac{\epsilon}{4C} \int\limits_{\mathbb{R}^m \setminus \overline{B}(0, R_\epsilon^1)} |\Gamma^i(t, x; 0, \xi)| d\xi \leqslant \frac{\epsilon}{4}, \tag{3.3}$$

(3.4)
$$\int\limits_{\overline{B}(0, R_\epsilon^1)} |\Gamma^i(t, x; 0, \xi)||\varphi^i(\xi)| d\xi < \int\limits_{\overline{B}(0, R_\epsilon^1)} \frac{\epsilon}{4\|\varphi^i\|_{\mathbb{R}^m} m(\overline{B}(0, R_\epsilon^1))} \|\varphi^i\|_{\mathbb{R}^m} d\xi = \frac{\epsilon}{4}.$$

From the $(\star)$–*condition* for K, it follows that there exists $R_\epsilon^2 > 0$ such that for each function $z \in K$ for all $t \in [0, T],\ x \in \mathbb{R}^m \setminus \overline{B}(0, R_\epsilon^2)$, we can assert

$$|z^i(t, x)| < \frac{\epsilon}{4TC} \text{ for } i = 1, \dots, N. \tag{3.5}$$

Now consider

$$\int_0^t \int\limits_{\mathbb{R}^m} |\Gamma^i(t, x; \tau, \xi)||z^i(\tau, \xi)| d\xi d\tau$$

$$= \int_0^t \int\limits_{\mathbb{R}^m \setminus \overline{B}(0, R_\epsilon^2)} |\Gamma^i(t, x; \tau, \xi)||z^i(\tau, \xi)| d\xi d\tau + \int_0^t \int\limits_{\overline{B}(0, R_\epsilon^2)} |\Gamma^i(t, x; \tau, \xi)||z^i(\tau, \xi)| d\xi d\tau.$$

Making use of (3.5) we estimate the first term as follows

(3.6)
$$\int_0^t \int\limits_{\mathbb{R}^m\setminus B(0,R_\epsilon^2)} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau < \frac{\epsilon}{4TC}\int_0^t \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)|d\xi d\tau \leqslant \frac{\epsilon}{4}.$$

In order to estimate the second one, we choose $\overline{R}_\epsilon^2 > R_\epsilon^2$ such that for all $x \in \mathbb{R}^m \setminus \overline{B}(0,\overline{R}_\epsilon^2),\ \xi \in \overline{B}(0,R_\epsilon^2),\ t,\tau \in [0,T]:\ \tau < t$, we can get the estimate

$$|\Gamma^i(t,x;\tau,\xi)| < \frac{\epsilon}{4TM'm(\overline{B}(0,R_\epsilon^2))} \text{ for } i = 1,\dots,N, \tag{3.7}$$

where $M' := \sup\limits_{z\in K} \|z\|_\Omega^N$. Then using (3.7) we readily assert

$$\int_0^t \int\limits_{\overline{B}(0,R_\epsilon^2)} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau < \frac{\epsilon}{4}. \tag{3.8}$$

Thus, for $R_\epsilon = \max\{\overline{R}_\epsilon^1, \overline{R}_\epsilon^2\},\ z \in K,\ t \in [0,T],\ x \in \mathbb{R}^m \setminus \overline{B}(0,R_\epsilon)$ combining (3.3), (3.4), (3.6) and (3.8), we find

$$\begin{aligned}
|(\mathbf{L}^i)^{-1}[z^i](t,x)| &\leqslant \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi + \int_0^t \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau \\
&= \int\limits_{\overline{B}(0,R_\epsilon^1)} |\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi + \int\limits_{\mathbb{R}^m\setminus\overline{B}(0,R_\epsilon^1)} |\Gamma^i(t,x;0,\xi)||\varphi^i(\xi)|d\xi \\
&\quad + \int_0^t \int\limits_{\overline{B}(0,R_\epsilon^2)} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau \\
&\quad + \int_0^t \int\limits_{\mathbb{R}^m\setminus\overline{B}(0,R_\epsilon^2)} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi)|d\xi d\tau < \epsilon \text{ for } i = 1,\dots,N.
\end{aligned}$$

The Lemma 3 is proved. □

Let us now prove the next auxiliary lemma. We adopt the Arzela–Ascoli compactness theorem, developing it for the case of functions defined on an unbounded domain but satisfying **(⋆)**–*condition.* The diagonal method will be our main tool in a proof of this lemma, like in the proof of the Arzela–Ascoli compactness criterion.

LEMMA 4. *Let $Y \subset CB_N^\infty(\overline{\Omega})$. If Y is a closed set of equicontinuous, uniformly bounded functions and Y satisfies* **(⋆)**–*condition, then Y is a compact set in $CB_N^\infty(\overline{\Omega})$.*

PROOF. Let $\{v_n\}_{n\in\mathcal{N}}$ be a sequence in Y. Let $\epsilon_1 = 1$.
From the **($\star$)**–*condition* it follows that there exists a constant $R_{\epsilon_1} > 0$ such that $|v_n(t,x)|^N < \frac{\epsilon_1}{2}$ follows for each $n \in \mathcal{N}$, $t \in [0,T]$, $x \in \mathbb{R}^m \setminus \overline{B}(0, R_{\epsilon_1})$.
Thus, for each $t \in [0,T]$, $x \in \mathbb{R}^m \setminus \overline{B}(0, R_{\epsilon_1})$, $n, k \in \mathcal{N}$, there is

$$|v_n(t,x) - v_k(t,x)|^N \leqslant |v_n(t,x)|^N + |v_k(t,x)|^N \leqslant \epsilon.$$

Now, let us remark that $[0,T] \times \overline{B}(0, R_{\epsilon_1}) =: \overline{B}_1$ is separable. Let $\Xi_1 := \{(t_1,x_1),(t_2,x_2),\dots\}$ be a countable and dense subset in $\overline{B}_1$.

In view of the boundness of Y, the sequence $\{v_n(t_i,x_i)\}_{n\in\mathcal{N}}$ is bounded in $\mathbb{R}^N$ for each $(t_i,x_i) \in \Xi_1$.

In particular, $\{v_n(t_1,x_1)\}_{n\in\mathcal{N}}$ is a bounded sequence in $\mathbb{R}^N$. Therefore there exists a convergent subsequence $\{v_{n,1}(t_1,x_1)\}_{n\in\mathcal{N}}$. Obviously, $\{v_{n,1}\}_{n\in\mathcal{N}}$ is a subsequence of the sequence $\{v_n\}_{n\in\mathcal{N}}$.

Since $\{v_{n,1}(t_2,x_2)\}_{n\in\mathcal{N}}$ is a bounded sequence in $\mathbb{R}^N$, then there exists a convergent subsequence $\{v_{n,2}(t_2,x_2)\}_{n\in\mathcal{N}}$ and $\{v_{n,2}\}_{n\in\mathcal{N}}$ is a subsequence of the sequence $\{v_{n,1}\}_{n\in\mathcal{N}}$.
Proceeding in this way, we will define an infinite matrix as follows:

$$\begin{array}{cccc}
v_{1,1}(t_1,x_1) & v_{2,1}(t_1,x_1) & v_{3,1}(t_1,x_1) & \dots \\
v_{1,2}(t_2,x_2) & v_{2,2}(t_2,x_2) & v_{3,2}(t_2,x_2) & \dots \\
v_{1,3}(t_3,x_3) & v_{2,3}(t_3,x_3) & v_{3,3}(t_3,x_3) & \dots \\
\dots\dots & \dots\dots & \dots\dots & \dots
\end{array}$$

From the Cantor diagonal procedure it follows that the sequence $\{v_{k,k}\}_{k\in\mathcal{N}}$ is characterized by the fact that the sequence $\{v_{k,k}(t_i,x_i)\}_{k\in\mathcal{N}}$ converges in every point $(t_i,x_i) \in \Xi_1$
Hereafter we denote $v_k^1 := v_{k,k}$.

Now, we take $\epsilon_2 = \frac{1}{2}$. There exists a constant $R_{\epsilon_2} > 0 \geq R_{\epsilon_1}$ such that $|v_k^1(t,x)|^N < \frac{\epsilon_2}{2}$ for each $k \in \mathcal{N}$, $t \in [0,T]$, $x \in \mathbb{R}^m \setminus \overline{B}(0, R_{\epsilon_2})$. In $[0,T] \times \overline{B}(0, R_{\epsilon_2}) =: \overline{B}_2$, we take a dense and countable subset Ξ_2 such that $\Xi_2 \cap \overline{B}_1 = \Xi_1$. Proceeding as above, we choose such a subsequence $\{v_k^2\}_{k\in\mathcal{N}}$ of the sequence $\{v_k^1\}_{k\in\mathcal{N}}$ which converges in every point of Ξ_2.

Repeating the previous procedure with respect to $\epsilon_i = \frac{1}{i}$ for $i = 3,4,\dots$, we obtain an infinite matrix of functions

$$\begin{array}{cccc}
v_1^1 & v_2^1 & v_3^1 & \dots \\
v_1^2 & v_2^2 & v_3^2 & \dots \\
v_1^3 & v_2^3 & v_3^3 & \dots \\
\dots\dots & \dots\dots & \dots\dots & \dots
\end{array}$$

Using the Cantor procedure once again, we choose a subsequence $\{v_k^k\}_{k\in\mathcal{N}}$ of the sequence $\{v_n\}_{n\in\mathcal{N}}$ at every point of $\bigcup_{n\in\mathcal{N}} \Xi_n$.

Now, we prove that the sequence $\{v_k^k\}_{k\in\mathcal{N}}$ is a Cauchy sequence in $C_N^\infty(\overline{\Omega})$. Let $\epsilon > 0$ be arbitrary. By virtue of an assumption that $\{v_n\}_{n\in\mathcal{N}}$ are equicontinuous, there exists $\delta > 0$ such that for all $k = 1, 2, \dots$

$$|v_k^k(t,x) - v_k^k(s,y)|^N < \frac{\epsilon}{3} \text{ if only } \max\{|t-s|,\ |x-y|^m\} < \delta.$$

There exists j_0 such that $\epsilon_{j_0} = \frac{1}{j_0} < \epsilon$.

From the **(⋆)**–*condition* it follows that there exists a constant $R_{\epsilon_{j_0}} > 0$ such that $|v_n(t,x)|^N < \frac{\epsilon_1}{2}$ follows for each $n \in \mathcal{N}$, $t \in [0,T]$, $x \in \mathbb{R}^m \setminus \overline{B}_{\epsilon_{j_0}}$.

Thus, for each $t \in [0,T]$, $x \in \mathbb{R}^m \setminus \overline{B}_{\epsilon_{j_0}}$, $p, q > k_0$, there is

$$|v_p^p(t,x) - v_q^q(t,x)|^N \leqslant |v_p^p(t,x)|^N + |v_q^q(t,x)|^N \leqslant \epsilon_{j_0} < \epsilon. \tag{4.1}$$

By B_i we denote an open ball with center at $(t_i, x_i) \in \Xi_{j_0}$ and radius $\delta > 0$. Then $\overline{B}_{j_0} \subset \bigcup_{i=1}^{\infty} B_i^\delta$ and, moreover

$$|v_k^k(t,x) - v_k^k(t_i,x_i)|^N \leqslant \frac{\epsilon}{3} \text{ for } (t,x) \in B_i^\delta. \tag{4.2}$$

By the Borel-Lebesque theorem there exists $i_0 \in \mathcal{N}$ such that $\overline{B}_{j_0} \subset \bigcup_{i=1}^{i_0} B_i^\delta$.

Since the sequence $\{v_k^k(t_i,x_i)\}_{k\in\mathcal{N}}$ converges for $i = 1, 2, \dots, i_0$, it follows that there exists a constant k_0 such that

$$|v_p^p(t_i,x_i) - v_q^q(t_i,x_i)|^N \leqslant \frac{\epsilon}{3} \text{ for } p, q > k_0,\ i = 1, \dots, i_0. \tag{4.3}$$

If $x \in \overline{B}(0, R_{\epsilon_{j_0}})$, then there exists $i \leqslant i_0$, such that $(t,x) \in B_i^\delta$. Hence, applying (4.2) and (4.3), we get

$$|v_p^p(t,x) - v_q^q(t,x)|^N \leqslant \tag{4.4}$$

$$|v_p^p(t,x) - v_p^p(t_i,x_i)|^N + |v_p^p(t_i,x_i) - v_q^q(t_i,x_i)|^N + |v_q^q(t_i,x_i) - v_q^q(t,x)|^N < \epsilon$$

for $p, q > k_0$, where k_0 is independent of (t,x).

Then owing to (4.1) and (4.4), we discover that $\{v_k^k\}_{k\in\mathcal{N}}$ is a Cauchy sequence in $C_N^\infty(\overline{\Omega})$, which in view of the completeness of the space implies the convergence of $\{v_k^k\}_{k\in\mathcal{N}}$ in $CB_N^\infty(\overline{\Omega})$. Thus, since Y is closed, the limit of $\{v_k^k\}_{k\in\mathcal{N}}$ belongs to Y, which completes the proof of Lemma 4. □

5. Proof of the theorem. In this section let us finally give a proof of the main result of the paper.

To begin with, we remark that proving that there exists a C–solution of problem (1), (2) is equivalent to showing that the operator $\mathbf{TF} : CB_S^\infty(\overline{\Omega}^\tau) \to CB_S^\infty(\overline{\Omega}^\tau)$ has a fixed point. We will prove it on the basis of the Schauder fixed point theorem.

For fixed $M > 0$ in $CB^{\infty}_{S}(\overline{\Omega}^{\tau})$ we consider a closed ball $\overline{B}^{\tau}(0, M) =: \overline{B}^{\tau}$ with center at 0 and radius M. Obviously, $\overline{B}^{\tau}$ is a closed and convex set $\tau \in (0, T]$.

Let us first demonstrate that $\mathbf{TF}(\overline{B}^{\tau}) \subset \overline{B}^{\tau}$ for a sufficiently small τ. Let $w \in \overline{B}^{\tau}$. It is easy to see that

$$|\mathbf{T}^i\mathbf{F}^i[w](t,x)| = \left| \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;0,\xi)\varphi^i(\xi)d\xi + \int\limits_0^t \int\limits_{\mathbb{R}^m} \Gamma^i(t,x;\tau',\xi)\mathbf{F}^i[w](\tau',\xi)d\xi d\tau' \right|$$

$$\leqslant \|\varphi^i\|_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;0,\xi)|d\xi + \|\mathbf{F}^i[w]\|_{\Omega^\tau} \int\limits_0^t \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau',\xi)|d\xi d\tau'$$

$$\leqslant C\|\varphi^i\|_{\mathbb{R}^m} + C\tau\|\mathbf{F}^i[w]\|_{\Omega^\tau}.$$

This immediately implies

$$\|\mathbf{T}^i\mathbf{F}^i[w]\|_{\Omega^\tau} \leqslant C\|\varphi^i\|_{\mathbb{R}^m} + C\tau\|\mathbf{F}^i[w]\|_{\Omega^\tau}.$$

Next, we note that $\|\varphi\|^{\Sigma}_{\mathbb{R}^m} < \infty$ and $\|\mathbf{F}[w]\|^{\Sigma}_{\Omega^\tau} < \infty$ for each $w \in \overline{B}^{\tau}$. This is a direct consequence of assumptions (F_1), (F_2), (φ_1) and (φ_2). These facts instantly lead to the following estimate

$$\|\mathbf{TF}[w]\|^{\Sigma}_{\Omega^\tau} = \sum_{i=1}^{\infty} \frac{1}{Q^i}\|\mathbf{T}^i\mathbf{F}^i[w]\|_{\Omega^\tau}$$

$$\leqslant C\sum_{i=1}^{\infty} \frac{1}{Q^i}\|\varphi^i\|_{\mathbb{R}^m} + C\tau\sum_{i=1}^{\infty} \frac{1}{Q^i}\|\mathbf{F}^i[w]\|_{\Omega^\tau} = C\|\varphi\|^{\Sigma}_{\mathbb{R}^m} + C\tau\|\mathbf{F}[w]\|^{\Sigma}_{\Omega^\tau}.$$

If

$$0 < \tau \leqslant \min\left\{ \frac{M - C\|\varphi\|^{\Sigma}_{\mathbb{R}^m}}{CM_1}, T \right\} := \tau^*,$$

where $M_1 := \sup\limits_{w \in \overline{B}^T} \|\mathbf{F}[w]\|^{\Sigma}_{\Omega^\tau}$ (due to assumption (F_2), we see that $M_1 < \infty$), then we readily obtain $\|\mathbf{TF}[w]\|^{\Sigma}_{\Omega^\tau} \leqslant M$.

Now, let us check that $\mathbf{TF} : \overline{B}^{\tau} \to \overline{B}^{\tau}$ is the continuous operator. Since $\mathbf{F}$ is a continuous operator in $\overline{B}^{\tau}$, it is enough to prove that $\mathbf{T}$ is continuous on $\mathbf{F}(\overline{B}^{\tau})$. To verify this assertion, fix $\epsilon > 0$ and $z = (z^1, z^2, \dots) \in \mathbf{F}(\overline{B}^{\tau})$. Let $\overline{z} = (\overline{z}^1, \overline{z}^2, \dots) \in \mathbf{F}(\overline{B}^{\tau})$ be such that $\|z - \overline{z}\|^{\Sigma}_{\Omega^\tau} < \delta := \frac{\epsilon}{C\tau}$. Then

$$|\mathbf{T}^i[z^i](t,x) - \mathbf{T}^i[\overline{z}^i](t,x)|$$

$$\leqslant \int\limits_0^t \int\limits_{\mathbb{R}^m} |\Gamma^i(t,x;\tau,\xi)||z^i(\tau,\xi) - \overline{z}^i(\tau,\xi)|d\xi d\tau \leqslant C\tau\|z^i - \overline{z}^i\|_{\Omega^\tau}$$

for $(t,x)\in\overline{\Omega}^{\tau}$, that is

$$\|\mathbf{T}^i[z^i]-\mathbf{T}^i[\overline{z}^i]\|_{\Omega^\tau}\leqslant C\tau\|z^i-\overline{z}^i\|_{\Omega^\tau}.$$

Thus

$$\|\mathbf{T}[z]-\mathbf{T}[\overline{z}]\|^{\Sigma}_{\Omega^\tau}=\sum_{i=1}^{\infty}\frac{1}{Q^i}\|\mathbf{T}^i[z^i]-\mathbf{T}^i[\overline{z}^i]\|_{\Omega^\tau}$$

$$\leqslant C\tau\sum_{i=1}^{\infty}\frac{1}{Q^i}\|z^i-\overline{z}^i\|_{\Omega^\tau}=C\tau\|z-\overline{z}\|^{\Sigma}_{\Omega^\tau}<\epsilon.$$

In the next step of our proof, we show that $\mathbf{TF}(\overline{B}^{\tau})$ is a precompact set in $CB^{\infty}_{S}(\overline{\Omega}^{\tau})$. To confirm this, we apply Fréchet compacteness theorem. First we denote $Y:=\mathbf{TF}(\overline{B}^{\tau})$.

By the Fréchet theorem it follows that $\overline{Y}$ is a compact subset of $CB^{\infty}_{S}(\overline{\Omega}^{\tau})$, if and only if for each ϵ there exists a compact set $Y_\epsilon\subset CB^{\infty}_{S}(\overline{\Omega}^{\tau})$ such that for each $y\in\overline{Y}$ there exists $y_\epsilon\in Y_\epsilon$ such that $\|y-y_\epsilon\|^{\Sigma}_{\Omega^\tau}<\epsilon$.

Fix ϵ and ϵ' such that $0<\epsilon'<\epsilon$. By virtue of the finiteness of the norm $\|\varphi\|^{\Sigma}_{\mathbb{R}^m}$, on the basis of assumption (F_2), there exists $N_\epsilon\in\mathcal{N}$ such that

$$\sum_{i=N_\epsilon+1}^{\infty}\frac{1}{Q^i}\|v^i\|_{\Omega^\tau}<\epsilon'<\epsilon$$

for each $v\in Y$.

Let $v\in\overline{Y}$ be arbitrary. There exists a sequence $\{v^i_n\}_{n\in\mathcal{N}}\subset Y$ which converges to v in $CB^{\infty}_{S}(\overline{\Omega}^{\tau})$.

Then

$$\sum_{i=N_\epsilon+1}^{\infty}\frac{1}{Q^i}\|v^i\|_{\Omega^\tau}=\sum_{i=N_\epsilon+1}^{\infty}\frac{1}{Q^i}\|\lim_{n\to\infty}v^i_n\|_{\Omega^\tau}=\lim_{n\to\infty}\sum_{i=N_\epsilon+1}^{\infty}\frac{1}{Q^i}\|v^i_n\|_{\Omega^\tau}\leqslant\epsilon'<\epsilon.$$

We define Y_ϵ as a set of functions $v_\epsilon=(v^1,\dots,v^{N_\epsilon},0,\dots)$ such that $v=(v^1,v^2,\dots)\in\overline{Y}$. Therefore,

$$\|v_\epsilon-v\|^{\Sigma}_{\Omega^\tau}=\sum_{n=N_\epsilon+1}^{\infty}\frac{1}{Q^n}\|v^n\|_{\Omega^\tau}<\epsilon.$$

In order to show the compactness of Y_ϵ, we denote

$$Y^{'}_{\epsilon}:=\{\ v^{'}_{\epsilon}=(v^1_\epsilon,\dots,v^{N_\epsilon}_\epsilon)\ :\ v_\epsilon=(v^1_\epsilon,\dots,v^{N_\epsilon}_\epsilon,0,\dots)\in Y_\epsilon\ \}.$$

Obviously, the set Y'_ϵ is bounded (i.e. functions from Y'_ϵ are equi-bounded) and closed in $CB^{\infty}_{N_\epsilon}(\overline{\Omega}^{\tau})$.
We claim that Y'_ϵ is the set of equicontinuous functions and satisfies $(\star)$–*condition.*

To verify this assertion, fix η and η', such that $0 < \eta' < \eta$. Let $v_\epsilon^{'} \in Y_\epsilon'$, $v_\epsilon^{'} = (v_\epsilon^1, \ldots, v_\epsilon^{N_\epsilon})$.

There exists $v \in \overline{Y}$ such that $v^i \equiv v_\epsilon^i$ for $i = 1, \ldots, N_\epsilon$.

Let $\{v_n\}_{n\in\mathcal{N}} \subset Y$ be a sequence which converges to v in $CB_S^\infty(\overline{\Omega}^\tau)$.

There exists $\{z_n\}_{n\in\mathcal{N}} \subset \mathbf{F}(\overline{B}^\tau)$ such that $v_n = \mathbf{T}[z_n]$.

We already know from Lemma 2 that there exists $\delta > 0$ such that, if $\max\{|t-t'|,\ |x-x'|^m\} < \delta$, then $|v_n^i(t,x) - v_n^i(t',x')| < \eta' < \eta$ for $i = 1, \ldots, N_\epsilon$.

Thus we conclude

$$|v_\epsilon^i(t,x) - v_\epsilon^i(t',x')| = |\lim_{n\to\infty} v_n^i(t,x) - \lim_{n\to\infty} v_n^i(t',x')|$$

$$= \lim_{n\to\infty} |v_n^i(t,x) - v_n^i(t',x')| \leqslant \eta' < \eta \text{ for } i = 1, \ldots, N_\epsilon.$$

Let us remark that δ is independent of $v_\epsilon' \in Y_\epsilon'$, that is, Y_ϵ' is a set of equicontinuous functions, as asserted. Owing to Lemma 3, there exists R_η such that for $t \in [0,\tau],\ x \in \mathbb{R}^m \setminus B(0, R_\eta)$ the following estimate holds

$$|v^i(t,x)| = |\lim_{n\to\infty} v_n^i(t,x)| = \lim_{n\to\infty} |v_n^i(t,x)| \leqslant \eta' < \eta,\ i = 1, \ldots, N_\epsilon.$$

We point out that R_η does not depend on $v_\epsilon' \in Y_\epsilon'$ and we conclude that Y_ϵ' satisfies **($\star$)**–*condition.*

Thus, due to Lemma 4, the set $Y_\epsilon^{'}$ is compact in $CB_{N_\epsilon}^\infty(\overline{\Omega}^\tau)$. It is easy to see that the set Y_ϵ is compact, since it is the image of the compact set $Y_\epsilon^{'}$ by the following continuous mapping

$$CB_{N_\epsilon}^\infty(\overline{\Omega}^\tau) \ni (v^1, \ldots, v^{N_\epsilon}) \hookrightarrow (v^1, \ldots, v^{N_\epsilon}, 0, \ldots) \in C_S^\infty(\overline{\Omega}^\tau).$$

Having obtained this, the Fréchet theorem yields the precompactness of the set Y. Therefore, by the Schauder theorem, **TF** has a fixed point in $\overline{B}^\tau$, $0 < \tau \leqslant \tau^* \leqslant T$. Thus, there exists at least one C–solution of problem (1), (2) in $CB_S^\infty(\overline{\Omega}^\tau)$ for $0 < \tau \leqslant \tau^* \leqslant T$. Theorem is proved. □

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