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A SOLUTION OF THE CAUCHY PROBLEM IN THE CLASS OF ABSOLUTELY CONTINUOUS DISTRIBUTION–VALUED FUNCTIONS

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Abstract. The aim of the paper is to give an exact formula for the solution of an evolution problem with matrix coefficients, and initial condition and external forces being tempered distributions.

1. Introduction. Let $\mathcal{S}$ be the Schwartz space,

$$\mathcal{S} := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}) : \quad P \cdot D^\alpha \varphi \text{ is bounded } \ \forall \alpha \in \mathbb{N}^n \ \forall P \in \mathcal{P}(\mathbb{R}^n)\},$$

where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials $\mathbb{R}^n \to \mathbb{R}$. $\mathcal{S}$ is a Fréchet space with the topology induced by the family of seminorms

$$q_m(\varphi) := \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \le m} (1 + |x|^2)^m |D^\alpha \varphi(x)|, \qquad m = 0, 1, 2, \dots$$

Write $\operatorname{sn} \mathcal{S}$ for the cone of all continuous seminorms on $\mathcal{S}$. Let B be a complex Banach space and $\mathcal{D}(\mathbb{R}^n)$ stand for the space of all complex test functions on $\mathbb{R}^n$. A distribution $T : \mathcal{D}(\mathbb{R}^n) \to B \quad (T \in \mathcal{D}'(\mathbb{R}^n; B))$ is tempered when it has the unique continuous extension $\overline{T} : \mathcal{S} \to B \quad (T \in \mathcal{D}'_{temp})$.

Let J be an interval in $\mathbb{R}$. We will consider the Cauchy problem

$$(1) \qquad \begin{cases} \frac{d}{dt} u(t) &= \sum\limits_\alpha A_\alpha(t) \circ D^\alpha u(t) + f(t) \quad \text{for a.e. } t \in J \\ u(t_0) &= u_0 \end{cases}$$

with given $t_0 \in J$, initial condition $u_0 \in \mathcal{D}'_{temp}$ and external forces $f : J \mapsto \mathcal{D}'_{temp}$. In order to prove the main theorem (Th. 17) on the existence and uniqueness of (1) we use the method used by K. Holly in the case of scalar coefficients $A_\alpha : J \mapsto \mathbb{C}$, [**4**]. This method is based on the integration of functions whose values are tempered distributions. In the second part of Section 2 we give a brief exposition of Holly's theory ([**3**]) of absolutely continuous distribution-valued functions and their integrals (Def. 7 – Th. 16). In the first part of Section 2 we will be concerned with the notion of the product of a distribution and a vector-valued function. Having this product and a product integral in a Fréchet space (see [**6**]), we obtain the exact formula (11) for the solution of problem (1), being an extension of a similar formula in the scalar case.

In the case of time-independed coefficients A_α, we rewrite the main theorem as Theorem 22, replacing abstract assumption (9) with easy to check spectral condition (25).

In Section 4, some applications are indicated.

2. Preliminaries. We begin by recalling the notion of the product of a distribution and a scalar-valued function. Let $\Omega \in \operatorname{top}\mathbb{R}^n$, $T \in \mathcal{D}'(\Omega; B)$, $\psi \in \mathcal{C}^\infty(\Omega)$. Then

$$\psi T : \qquad \mathcal{D}(\Omega) \ni \varphi \overset{\text{def}}{\longmapsto} T(\varphi\psi) \in B.$$

Now we extend the product to the case of a vector-valued function. Assume that B is a complex vector space. Let $End\, B$ denote the space of all linear endomorphisms on B and $\{b_1, \ldots, b_k\}$ be the basis of $End\, B$.

Definition 1. Let $T \in \mathcal{D}'(\Omega; B)$, $\eta \in \mathcal{C}^\infty(\Omega, End\, B)$. Then

$$\eta \cdot T := \sum_{j=1}^{k} (b_j^\star \circ \eta)\,(b_j \circ T).$$

The above definition does not depend on the choice of a basis of $End\, B$. Moreover, $\eta \cdot T \in \mathcal{D}'(\Omega; B))$ and for $u \in L^1_{loc}(\Omega, B)$ there is $\eta \cdot [u] = [\eta u]$, where $[u]$ denotes the regular distribution

$$[u](\varphi) := \int_\Omega u(x)\varphi(x)dx \qquad \text{for } \varphi \in \mathcal{D}(\Omega).$$

The product has properties analogous to those in the scalar case. We have pointed out two of them which are strickly connected to a vector-valued function.

Remark 2. Let $T \in \mathcal{D}'(\Omega; B)$, $\eta \in \mathcal{C}^\infty(\Omega, End\, B)$ and $\psi \in \mathcal{C}^\infty(\Omega, \mathbb{K})$. Then

$$\eta \cdot T = \sum_{j=1}^{k} b_j \circ ((b_j^\star \circ \eta) T),$$

$$\psi(\eta T) = (\psi\eta)T = \eta(\psi T).$$

Proof. We prove the first equality. The proof of the second is analogous. Let $\varphi \in \mathcal{D}(\Omega)$.

$$\begin{aligned}(\eta T)(\varphi) &= \left(\sum_{j=1}^{k}(b_j^\star \circ \eta)(b_j \circ T)\right)(\varphi) = \sum_{j=1}^{k}(b_j \circ T)((b_j^\star \circ \eta)\varphi) = \\ &= \sum_{j=1}^{k} b_j(((b_j^\star \circ \eta)T)(\varphi)) = \sum_{j=1}^{k}(b_j \circ (b_j^\star \circ \eta)T)(\varphi).\end{aligned}$$

□

As in the scalar case the following theorem holds.

Theorem 3. *Let $T \in \mathcal{D}'_{temp}$, $\eta \in \mathcal{C}^\infty(\mathbb{R}^n, End\, B)$ be polynomially bounded together with all of its derivatives. Then $\eta T \in \mathcal{D}'_{temp}$ and*

$$\overline{\eta T} = \sum_{j=1}^{k}(b_j^\star \circ \eta)\,(b_j \circ \overline{T}).$$

Proof. Since $b_j \circ \overline{T} : \mathcal{S} \to B$ is linear continuous and $b_j^\star \circ \eta : \mathbb{R}^n \to \mathbb{K}$ is polynomially bounded together with all of its derivatives for any $j \in \{1, \dots, k\}$, $\sum_{j=1}^{k}(b_j^\star \circ \eta)\,(b_j \circ \overline{T}) : \mathcal{S} \to B$ is also linear continuous. Clearly,

$$\sum_{j=1}^{k}(b_j^\star \circ \eta)\,(b_j \circ \overline{T})_{|\mathcal{D}(\mathbb{R}^n)} = \eta T.$$

□

Let $\mathcal{H}ol := \mathcal{H}ol(\mathbb{C}^n, End\, B)$ stand for the space of all $End\, B$-valued holomorphic functions with the topology of local uniform convergence. Setting for any $h, f \in \mathcal{H}ol$: $h \cdot f : \Omega \ni z \overset{\text{def}}{\longmapsto} h(z) \circ f(z) \in End\, B$, $\mathcal{H}ol$ is a Fréchet algebra (see [**6**]).

Definition 4. Let $\eta \in \mathcal{H}ol$, $T \in \mathcal{D}'_{temp}$. Then

$$\eta \cdot T := \eta_{|\mathbb{R}^n} \cdot T.$$

Let K be a compact subset of $\mathbb{R}^n$. Fix $N \in \mathbb{N}$ and consider the space

$$X := \{T \in \mathcal{D}'_{temp} : \quad \operatorname{supp} T \subset K,\, \overline{T} \text{ is } q_N\text{-continuous }\}. \tag{2}$$

Then X is a Banach space with the norm $|T|_X := |\overline{T}|_{q_N}$.

THEOREM 5.

(i) *For any $h \in \mathcal{H}ol$, the mapping*

$$\tilde{h} : X \ni T \overset{\text{def}}{\longmapsto} \sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!} D^\beta h(0) \circ x^\beta T \in X \tag{3}$$

is correctly defined, linear and continuous.

(ii) *The map*

$$\tilde{\ } : \mathcal{H}ol \ni h \overset{\text{def}}{\longmapsto} \tilde{h} \in End\, X$$

is a continuous homomorphism of algebras.

PROOF. Let $h \in \mathcal{H}ol$. Fix $\lambda \in \mathcal{D}(\mathbb{R}^n)$ such that $\lambda = 1$ in the neighbourhood of K. There is $r > 0$ such that $\operatorname{supp} \lambda \subset \{|x| < r\}$. Fix $r < R < \infty$. Set $a_\beta := \dfrac{1}{\beta!} D^\beta h(0) \in End\, B$. Let $T \in X$ and $\varphi \in \mathcal{S}$.

$$|\overline{a_\beta \circ x^\beta T}(\varphi)| = |(a_\beta \circ x^\beta \overline{T})(\varphi)| \leq |a_\beta| \cdot |\overline{T}|_{q_N} \cdot q_N(\lambda x^\beta \varphi). \tag{4}$$

By the Cauchy inequalities,

$$|a_\beta| \leq \frac{p_R(h)}{R^{|\beta|}}, \tag{5}$$

where $p_R(h) := \sup\limits_{|z_i|<R,\, \forall i=1,\dots,n} \|h(z)\|_{End\, B} < \infty$. And by Leibniz's formula,

$$q_N(\lambda x^\beta \varphi) \leq \max_{|\alpha| \leq N} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \|D^\gamma(\lambda x^\beta)\|_{\mathcal{L}^\infty(\operatorname{supp} \lambda)} \cdot q_N(\varphi). \tag{6}$$

Combining (4)–(6) we conclude that $\sum\limits_{\beta \in \mathbb{N}^n} \dfrac{1}{\beta!} \left(D^\beta h(0) \circ x^\beta \overline{T} \right)(\varphi)$ is convergent for all $\varphi \in \mathcal{S}$ and

$$\left| \sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!} \left(D^\beta h(0) \circ x^\beta \overline{T} \right)(\varphi) \right| \leq M \cdot p_R(h) \cdot |\overline{T}|_{q_N} \cdot q_N(\varphi),$$

where the constant M depends on N, R and r. This gives $\tilde{h}(T) \in X$ and $|\tilde{h}(T)|_X \leq M \cdot p_R(h) \cdot |T|_X$. In cosequence the map

$$X \ni T \mapsto \tilde{h}(T) \in X$$

is continuous and $|\tilde{h}|_{End\, X} \leq M \cdot p_R(h)$, which proves that also $h \mapsto \tilde{h}$ is continuous.

An easy computation shows that if $h_1, h_2 \in \mathcal{H}ol$ then $(h_1 \cdot h_2)\tilde{\ } = \tilde{h}_1 \cdot \tilde{h}_2$. □

Theorem 6. *If $T \in \mathcal{D}'(\mathbb{R}^n; B)$, $\operatorname{supp} T$ is compact and if $\eta \in \mathcal{H}ol$, $\eta_{|\mathbb{R}^n}$ is polynomially bounded together with all of its derivatives, then*

$$\eta \cdot T = \tilde{\eta}(T)$$

Proof. Let $(b_j)_{j=1,\ldots,k}$ be a basis of $End\, B$. Then $b_j^\star \circ \eta \in \mathcal{H}ol(\mathbb{C}^n, \mathbb{C})$ and

$$(b_j^\star \circ \eta)(x) = \sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!}(b_j^\star \circ D^\beta \eta(0)) x^\beta$$

for any $x \in \mathbb{R}^n$. Let $\varphi \in \mathcal{S}$. Since $\eta_{|\mathbb{R}^n}$ is polynomially bounded with all of its derivatives and $\operatorname{supp} T$ is compact, it follows that ηT is tempered and

$$\overline{\eta T}(\varphi) = \sum_{j=1}^{k} \sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!}(b_j^\star \circ D^\beta \eta(0)) \cdot b_j(\overline{T}(x^\beta \varphi)). \tag{7}$$

On the other hand, in the proof of Theorem 5, we have proved that $\sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!} |(D^\beta \eta(0) \circ x^\beta \overline{T})(\varphi)|$ is convergent, and, in consequence,

$$\sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!}(D^\beta \eta(0) \circ x^\beta \overline{T})(\varphi) = \sum_{j=1}^{k} \sum_{\beta \in \mathbb{N}^n} \frac{1}{\beta!}(b_j^\star \circ D^\beta \eta(0)) \cdot b_j(\overline{T}(x^\beta \varphi)). \tag{8}$$

Combining (7) and (8), by the arbitrariness of $\varphi \in \mathcal{S}$, we finally obtain

$$\eta \cdot T = \tilde{\eta}(T).$$

□

Now turn to distribution-valued functions. Let B be a complex Banach space. Let $\mathcal{L}(\mathcal{S}, B)$ denote the space of all $\mathbb{C}$-linear continuous operators of $\mathcal{S}$ into B. Fixing $q \in \operatorname{sn} \mathcal{S}$, we consider the space $\mathcal{L}((\mathcal{S}, q), B)$ of all $\mathbb{C}$-linear, q-continuous $\mathcal{S} \to B$ mappings. Note that it is a Banach space with the norm $|T|_q := \sup_{q(\varphi) \le 1} |T(\varphi)|$. Clearly,

$$\mathcal{L}(\mathcal{S}, B) = \bigcup_{q \in \operatorname{sn} \mathcal{S}} \mathcal{L}((\mathcal{S}, q), B) = \bigcup_{m=0}^{\infty} \mathcal{L}((\mathcal{S}, q_m), B).$$

We equipped the space $\mathcal{L}(\mathcal{S}, B)$ with an inductive topology, i.e. the strongest locally convex topology such that for all $q \in \operatorname{sn} \mathcal{S}$, canonical inclusions $\mathcal{L}((\mathcal{S}, q), B) \hookrightarrow \mathcal{L}(\mathcal{S}, B)$ are continuous.

Let I be a closed interval in $\mathbb{R}$.

Definition 7. A function $\chi : I \to \mathcal{L}(\mathcal{S}, B)$ is summable iff there is $q \in \operatorname{sn} \mathcal{S}$ such that $\chi(I) \subset \mathcal{L}((\mathcal{S}, q), B)$ and the function $\chi : I \to \mathcal{L}((\mathcal{S}, q), B)$ is summable in the Bochner sense.

Definition 8. Let $\chi : I \to \mathcal{L}(\mathcal{S}, B)$ be summable and $q \in \operatorname{sn}\mathcal{S}$ be the seminorm from Definition 7. If $\left(\int_I \chi(t)dt\right)_q$ denotes the Bochner integral of $\chi : I \to \mathcal{L}((\mathcal{S}, q), B)$, then

$$\mathcal{L}(\mathcal{S}, B) \ni \int_I \chi(t)dt := \left(\int_I \chi(t)dt\right)_q .$$

Obviously, the definition does not depend on the choice of q.

Definition 9. A function $\chi : I \to \mathcal{D}'_{temp}$ is summable iff

$$\overline{\chi} : I \ni t \mapsto \overline{\chi(t)} \in \mathcal{L}(\mathcal{S}, B)$$

is summable.

Clearly, $\int_I \chi(t)dt \in \mathcal{D}'_{temp}$ and $\int_I \chi(t)dt \subset \int_I \overline{\chi(t)}dt$.

Let $\widehat{(\cdot)} = \mathcal{F} : \mathcal{D}'_{temp} \to \mathcal{D}'_{temp}$ denote the Fourier transform.

Theorem 10. *Let $\chi : I \to \mathcal{D}'_{temp}$ be summable. Consider a multi-index $\alpha \in \mathbb{N}^n$ and a function $\eta \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ which is polynomially bounded with all of its derivatives. Then the functions $D^\alpha\chi : J \ni t \mapsto D^\alpha\chi(t) \in \mathcal{D}'_{temp}$, $\eta\chi : J \ni t \mapsto \eta\chi(t) \in \mathcal{D}'_{temp}$, $\hat{\chi} : J \ni t \mapsto \widehat{\chi(t)} \in \mathcal{D}'_{temp}$ are summable and*

- **a):** $\int_I D^\alpha\chi(t)dt = D^\alpha \int_I \chi(t)dt \subset \int_I \overline{D^\alpha\chi(t)}dt,$
- **b):** $\int_I (\eta\chi)(t)dt = \eta \int_I \chi(t)dt \subset \int_I \overline{\eta\chi(t)}dt,$
- **c):** $\int_I \hat{\chi}(t)dt = \mathcal{F}\left(\int_I \chi(t)dt\right) \subset \int_I \overline{\hat{\chi}(t)}dt.$

Definition 11. A function $u : I \to \mathcal{L}(\mathcal{S}, B)$ is absolutely continuous iff there is a locally summamble $\chi : I \to \mathcal{L}(\mathcal{S}, B)$ such that

$$\forall t_1, t_2 \in I \quad u(t_2) - u(t_1) = \int_{t_1}^{t_2} \chi(\tau)d\tau.$$

Theorem 12. *Let $u : I \to \mathcal{L}(\mathcal{S}, B)$. The following conditions are equivalent*

(i) *u is absolutely continuous;*
(ii) *there is $q \in \operatorname{sn}\mathcal{S}$ such that $u(I) \subset \mathcal{L}((\mathcal{S}, q), B)$ and $u : I \to \mathcal{L}((\mathcal{S}, q), B)$ is absolutely continuous and a.e. differentiable;*
(iii) *there is $n \in \mathbb{N}$ such that $u(I) \subset \mathcal{L}((\mathcal{S}, q_N), B)$ and $u : I \to \mathcal{L}((\mathcal{S}, q_N), B)$ is absolutely continuous and a.e. differentiable.*

Definition 13. A function $u : I \to \mathcal{D}'_{temp}$ is absolutely continuous iff

$$\overline{u} : I \ni t \mapsto \overline{u(t)} \in \mathcal{L}(\mathcal{S}, B)$$

is absolutely continuous.

Let J be an arbitrary interval in $\mathbb{R}$.

Definition 14. A function $u : J \to \mathcal{D}'_{temp}$ is absolutely continuous iff the restriction u_I is absolutely continuous for every closed subinterval $I \subset J$.

Remark 15. A function $u : J \to \mathcal{D}'_{temp}$ is absolutely continuous iff there are $t_0 \in J$ and locally summable $\chi : J \to \mathcal{D}'_{temp}$ such that

$$u(t) = u(t_0) + \int_{t_0}^{t} \chi(\tau) d\tau \qquad \forall t \in J.$$

Moreover, the set $\{t \in \operatorname{dom} u' : \quad u'(t) \neq \chi(t)\}$ is Borelian and of measure zero.

Theorem 16. *Let $u : J \to \mathcal{D}'_{temp}$ be absolutely continuous. Consider a multi-index $\alpha \in \mathbb{N}^n$ and a function $\eta \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ polynomially bounded with all its derivatives. Then the functions $D^\alpha u : J \ni t \mapsto D^\alpha u(t) \in \mathcal{D}'_{temp}$, $\eta u : J \ni t \mapsto \eta u(t) \in \mathcal{D}'_{temp}$, $\hat{u} : J \ni t \mapsto \widehat{u(t)} \in \mathcal{D}'_{temp}$ are absolutely continuous and for a.e. $t \in J$:*

a) $(D^\alpha u)'(t) = D^\alpha u'(t)$,
b) $(\eta u)'(t) = \eta u'(t)$,
c) $\hat{u}'(t) = \widehat{u'(t)}$.

3. Results. Let B be a complex vector space and J be an interval in $\mathbb{R}$.

Theorem 17. *Given are $t_0 \in J$, $u_0 \in \mathcal{D}'_{temp}$ and a locally summable $f : J \to \mathcal{D}'_{temp}$. Consider a family of locally summable coefficients $A_\alpha : J \to End\, B$, ($\alpha \in \mathbb{N}^n$), such that $\#\{\alpha : A_\alpha \neq 0\} < \infty$ and*

(9) $\forall t_1 \in J \qquad \mathbb{R}^n \ni \xi \mapsto \prod_s^t e^{\sum_\alpha (i\xi)^\alpha A_\alpha(\tau) d\tau} \in End\, B$ *is polynomially bounded together with all of its derivatives, uniformly in $(s,t) \in [t_0, t] \times [t_0, t_1]$.*

Then there is the unique absolutely continuous function $u : J \to \mathcal{D}'_{temp}$ such that

$$\begin{cases} \frac{d}{dt} u(t) &= \sum_\alpha A_\alpha(t) \circ D^\alpha u(t) + f(t) \quad \text{for a.e. } t \in J \\ u(t_0) &= u_0. \end{cases} \tag{10}$$

Moreover, for each $t \in J$
(11)
$$u(t) = \mathcal{F}^{-1}\left(\prod_{t_0}^{t} e^{\sum_\alpha (i\xi)^\alpha A_\alpha(\tau)d\tau} \cdot \widehat{u_0}\right) + \int_{t_0}^{t} \mathcal{F}^{-1}\left(\prod_{s}^{t} e^{\sum_\alpha (i\xi)^\alpha A_\alpha(\tau)d\tau} \cdot \widehat{f}(s)\right) ds.$$

$\prod_{s}^{t} e^{\sum_\alpha (i\xi)^\alpha A_\alpha(\tau)d\tau}$ denotes the product integral. Let us denote

$$J^2(t_0) := \{(s,t) \in J \times J : \quad s \in [t_0, t]\}. \tag{12}$$

By the Leibniz formula, the following lemmas hold:

LEMMA 18. *Let $\eta \in \mathcal{C}^\infty(\mathbb{R}^n, End\, B)$ be polynomially bounded with all its derivatives. For any $N \in \mathbb{N}$ there is $\overline{N}$ such that for all $j = 1, \ldots, k$ the map*

$$(\mathcal{S}, q_{\overline{N}}) \ni \varphi \overset{\text{def}}{\longmapsto} (b_j^\star \circ \eta)\varphi \in (\mathcal{S}, q_N)$$

is linear and continuous.

LEMMA 19. *Fix $(s,t) \in J^2(t_0)$. Let us consider $\eta_s^t \in \mathcal{C}^\infty(\mathbb{R}^n, End\, B)$ such that*

$$\begin{array}{ll} \forall t_1 \in J & \mathbb{R}^n \ni \xi \mapsto \eta_s^t(\xi) \in End\, B \\ & \textit{is polynomially bounded together with all of its} \\ & \textit{derivatives, uniformly in } (s,t) \in [t_0,t] \times [t_0,t_1]. \end{array} \tag{13}$$

Additionally assume that

$$\begin{array}{l} \textit{if } (s_\nu, t_\nu) \overset{\nu \to \infty}{\longrightarrow} (s,t) \textit{ in } J^2(t_0), \textit{ then for every } \gamma \in \mathbb{N}^n \\ D^\gamma \eta_{s_\nu}^{t_\nu} \overset{\nu \to \infty}{\longrightarrow} D^\gamma \eta_s^t \textit{ almost uniformly in } \mathbb{R}^n. \end{array} \tag{14}$$

Then for any $N \in \mathbb{N}$ there is $\overline{N} \in \mathbb{N}$ such that for all $j = 1, \ldots, k$ the map

$$\begin{aligned} l_j : J^2(t_0) \ni (s,t) \overset{\text{def}}{\longmapsto} \; & l_j^{s,t} \in \mathcal{L}\left((\mathcal{S}, q_{\overline{N}}), (\mathcal{S}, q_N)\right), \\ & l_j^{s,t}(\varphi) := (b_j^\star \circ \eta_s^t)\varphi \;\; \textit{for } \; \varphi \in \mathcal{S} \end{aligned}$$

is continuous.

By the continuous dependence on the limits of integration for a product integral (see [**6**]) we obtain

REMARK 20. Let $A_\alpha : J \longrightarrow End\, B$ $(\alpha \in \mathbb{N}^n)$ be a finite family of locally summable functions and

$$\eta_s^t : \mathbb{C} \ni z \overset{\text{def}}{\longmapsto} \prod_{s}^{t} e^{\sum_\alpha (iz)^\alpha A_\alpha(\tau)d\tau} \in \mathcal{H}ol \qquad \forall (s,t) \in J^2(t_0).$$

Then $\eta_s^t|_{\mathbb{R}^n}$ satisfies assumption (14).

Corollary 21. *If* $A_\alpha \in End\, B$ $(\alpha \in \mathbb{N}^n)$ *then*

$$\eta_s^t(\xi) := e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha} \qquad \forall \xi \in \mathbb{R}^n \quad \forall (s,t) \in J^2(t_0)$$

satisfies (14).

Proof of Theorem 17.

Step 1° We begin by proving that the right-hand side of (11) makes sense. First, note that for any $t \in J$, $s \in [t_0, t]$ the mapping

$$\eta_s^t : \quad \mathbb{R}^n \ni \xi \overset{\text{def}}{\longmapsto} \prod_s^t e^{\sum_\alpha (i\xi)^\alpha A_\alpha(\tau) d\tau} \qquad \in End\, B \tag{15}$$

is polynomially bounded with all its derivatives, and so $\eta_{t_0}^t \cdot \hat{u}_0$, $\eta_s^t \cdot \hat{f}(s)$ are tempered distributions.

Let $t \in J$. The function $f_{[t_0,t]} : [t_0, t] \to \mathcal{D}'_{temp}$ is summable, thus by Theorem 10, so is $\hat{f}_{[t_0,t]} : [t_0, t] \to \mathcal{D}'_{temp}$. According to Definition 9 and Definition 7 there is $N \in \mathbb{N}$ such that $\overline{\hat{f}}([t_0, t]) \subset \mathcal{L}((\mathcal{S}, q_N), B)$ and $\overline{\hat{f}}_{[t_0,t]} : [t_0, t] \to \mathcal{L}((\mathcal{S}, q_N), B)$ is summable.

Fix $j \in \{1, \ldots, k\}$. Due to Remark 20 and Lemma 19, there is $\overline{N}$ such that $l_j(J^2(t_0)) \subset \mathcal{L}((\mathcal{S}, q_{\overline{N}}), (\mathcal{S}, q_N))$ and the map $l_j : J^2(t_0) \to \mathcal{L}((\mathcal{S}, q_{\overline{N}}), (\mathcal{S}, q_N))$ is continuous.

Consider the continuous bilinear mapping

$$\mathcal{G}_{\overline{N},N} : \begin{cases} \mathcal{L}((\mathcal{S}, q_{\overline{N}}), (\mathcal{S}, q_N)) \times \mathcal{L}((\mathcal{S}, q_N), B) & \longrightarrow \quad \mathcal{L}((\mathcal{S}, q_{\overline{N}}), B) \\ (E, T) & \overset{\text{def}}{\longmapsto} \quad T \circ E. \end{cases} \tag{16}$$

Then the function

$$[t_0, t] \ni s \mapsto \mathcal{G}_{\overline{N},N}(l_j^{s,t}, \overline{\hat{f}}(s)) \in \mathcal{L}((\mathcal{S}, q_{\overline{N}}), B)$$

is summable, and for $s \in [t_0, t]$, $\varphi \in \mathcal{S}$ there is

$$\left(\mathcal{G}_{\overline{N},N}\left(l_j^{s,t}, \overline{\hat{f}}(s)\right)\right)(\varphi) = \left(\overline{\hat{f}}(s) \circ l_j^{s,t}\right)(\varphi) = \overline{\hat{f}}(s)((b_j^\star \circ \eta_s^t)\varphi) = \overline{(b_j^\star \circ \eta_s^t)\hat{f}(s)}(\varphi).$$

Consequently,

$$[t_0, t] \ni s \mapsto \overline{\sum_{j=1}^k b_j \circ (b_j^\star \circ \eta_s^t)\hat{f}(s)} \in \mathcal{L}((\mathcal{S}, q_{\overline{N}}), B)$$

is summable, and by Definitions 7, 9, so is

$$[t_0, t] \ni s \mapsto \eta_s^t \cdot \hat{f}(s) \in \mathcal{D}'_{temp}(\mathbb{R}^n; B).$$

Finally according to the Schwartz theorem and Theorem 10, the function

$$[t_0, t] \ni s \mapsto \mathcal{F}^{-1}(\eta_s^t \cdot \hat{f}(s)) \in \mathcal{D}'_{temp}$$

is well defined and summable.

Step 2° Uniqueness.
Let $u : J \to \mathcal{D}'_{temp}$ be an absolutely continuous solution of problem (10). Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$. Then on account of Theorem 16 the function $v : J \ni t \overset{\text{def}}{\longmapsto} \psi\hat{u}(t) \in \mathcal{D}'_{temp}$ is also absolutely continuous and for a.e. $t \in J$

$$(17) \qquad v'(t) = \sum_{\alpha} A_\alpha(t) \circ (i\xi)^\alpha v(t) + \psi\hat{f}(t).$$

Fix $t_\star \in J$ and denote $I := [t_0, t_\star]$. Similarly as in Step 1^o, there is $N \in \mathbb{N}$ such that

$\overline{v}(I) \subset \mathcal{L}((\mathcal{S}, q_N), B)$ and $\overline{v}_I : I \to \mathcal{L}((\mathcal{S}, q_N), B)$ is absolutely continuous, a.e. differentiable,

$\overline{\psi\hat{f}}(I) \subset \mathcal{L}((\mathcal{S}, q_N), B)$ and $\overline{\psi\hat{f}} : I \to \mathcal{L}((\mathcal{S}, q_N), B)$ is summable,

$\overline{\psi\hat{u}_0} \in \mathcal{L}((\mathcal{S}, q_N), B)$.

Consider the Banach space

$$X := \{T \in \mathcal{D}'_{temp} : \quad \operatorname{supp} T \subset \operatorname{supp} \psi,\ \overline{T} \text{ is } q_N - \text{continuous}\}$$

with the norm $|T|_X := |\overline{T}|_{q_N}$ (see (2)). Let $\iota : X \ni T \mapsto \overline{T} \in \mathcal{L}((\mathcal{S}, q_N), B)$ be a canonical injection. $\overline{\iota(X)}$ is a closed subspace of $\mathcal{L}((\mathcal{S}, q_N), B)$. $v(I) \subset X$ hence $\overline{v}(I) \subset \iota(X)$. Similarly $\overline{\psi\hat{f}(I)} \subset \iota(X)$. Thus $\overline{v}_I : I \to \iota(X)$ is absolutely continuous, a.e. differentiable and $\overline{\psi\hat{f}}_I : I \to \iota(X)$ is summable.

Setting $x := v_I$, we derive that the function $x : I \to X$ is absolutely continuous, a.e. differentiable. Similarly $\psi\hat{f}_I : I \to X$ is summable and $\psi\hat{u}_0 \in X$. Clearly,

$$(18) \qquad \dot{x}(t) = v'(t)$$

a.e. in I, where the derivative on the left-hand side is a derivative in $\operatorname{top} X$ while the one on the right-hand side is a derivative in $\mathcal{D}'_{temp}$. Therefore by (17), for a.e. $t \in I$, there is

$$\dot{x}(t) = \sum_{\alpha} A_\alpha(t) \circ (i\xi)^\alpha x(t) + \psi\hat{f}(t) = \tilde{A}_t x(t) + \psi\hat{f}(t),$$

where $A_t(z) := \sum_{\alpha} (iz)^\alpha A_\alpha(t)$ for $z \in \mathbb{C}^n$. Obviously, $A_t \in \mathcal{H}ol$. The mapping $\tilde{}: \mathcal{H}ol \to End\, X$ is defined as in Theorem 5 (ii), in particular $\tilde{A}_t T = \sum_{\alpha} A_\alpha(t) \circ (i\xi)^\alpha T$ for $T \in X$.

Thus we have obtained the Cauchy problem in the Banach space X, treated as a module over the Banach algebra $End\, X$:

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & \tilde{A}_t x(t) + \psi \hat{f}(t)\\ x(t_0) & = & \psi \hat{u}_0.\end{array}\right. \tag{19}$$

On account of Theorem 41, [**6**]

$$x(t) = \prod_{t_0}^{t} e^{\tilde{A}_\tau d\tau}(\psi \hat{u}_0) + \int_{t_0}^{t} \prod_{s}^{t} e^{\tilde{A}_\tau d\tau}(\psi \hat{f}(s))ds$$

is the unique absolutely continuous solution of problem (19). By Lemma 44, [**6**] (comp. Th. 5 (ii)) and using notation (15)

$$\begin{aligned} x(t) &= \left(\prod_{t_0}^{t} e^{A_\tau d\tau}\right)^{\sim}(\psi \hat{u}_0) + \int_{t_0}^{t} \left(\prod_{s}^{t} e^{A_\tau d\tau}\right)^{\sim}(\psi \hat{f}(s))ds = \\ &= \widetilde{\eta_{t_0}^t}(\psi \hat{u}_0) + \int_{t_0}^{t} \widetilde{\eta_s^t}(\psi \hat{f}(s))ds. \end{aligned}$$

Functions $\eta_{t_0}^t$, η_s^t are polynomially bounded together with all its derivatives, distributions $\psi \hat{u}_0$, $\psi \hat{f}(s)$ have compact supports, so by Theorem 6

$$x(t) = \eta_{t_0}^t \cdot (\psi \hat{u}_0) + \int_{t_0}^{t} \eta_s^t \cdot (\psi \hat{f}(s))ds,$$

with $\int_{t_0}^{t} \eta_s^t \cdot (\psi \hat{f}(s))ds$ being an integral in X. The function

$$[t_0, t] \ni s \mapsto \iota(\eta_s^t \cdot (\psi \hat{f}(s))) = \overline{\eta_s^t \cdot (\psi \hat{f}(s))} \in \mathcal{L}((\mathcal{S}, q_N), B)$$

is summable in the sense of Bochner and

$$\iota\left(\int_{t_0}^{t} \eta_s^t \cdot (\psi \hat{f}(s))ds\right) = \int_{t_0}^{t} \overline{\eta_s^t \cdot (\psi \hat{f}(s))}ds.$$

In consequence, $[t_0, t] \ni s \mapsto \eta_s^t \cdot (\psi \hat{f}(s)) \in \mathcal{D}'_{temp}$ is summable and the integral $\int_{t_0}^{t} \eta_s^t \cdot (\psi \hat{f}(s))ds$ in the space of tempered distributions $\mathcal{D}'_{temp}$ is equal to the integral in X.

In Step 1° we have proved that the function

$$[t_0, t] \ni s \mapsto \eta_s^t \cdot \hat{f}(s) \in \mathcal{D}'_{temp}$$

is summable. By the arbitrariness of $\psi \in \mathcal{D}(\mathbb{R}^n)$ and Remark 2, there is

$$\hat{u}(t) = \eta_{t_0}^t \cdot \hat{u}_0 + \int_{t_0}^{t} \eta_s^t \cdot \hat{f}(s)ds$$

for all $t \in I$ (in particular for $t_\star$). Finally, by the Schwartz theorem,

$$u(t_\star) = \mathcal{F}^{-1}\left(\eta_{t_0}^{t_\star} \cdot \hat{u}_0\right) + \mathcal{F}^{-1}\left(\int_{t_0}^{t_\star} \eta_s^{t_\star} \cdot \hat{f}(s)ds\right).$$

As $t_\star \in J$ is arbitrary and on account of Theorem 10

$$u(t) = \mathcal{F}^{-1}\left(\eta_{t_0}^{t} \cdot \hat{u}_0\right) + \int_{t_0}^{t} \mathcal{F}^{-1}\left(\eta_s^{t} \cdot \hat{f}(s)ds\right) \qquad \forall t \in J.$$

Since every solution of (10) is of the form (11), we have proved the uniqueness.

Step 3° Existence.
We prove that the function

$$u : J \ni t \overset{\text{def}}{\longmapsto} \mathcal{F}^{-1}\left(\eta_{t_0}^{t} \cdot \hat{u}_0\right) + \int_{t_0}^{t} \mathcal{F}^{-1}\left(\eta_s^{t} \cdot \hat{f}(s)\right) ds \in \mathcal{D}'_{temp} \tag{20}$$

is an absolutely continuous solution of (10).

Let $I \subset J$ be a closed interval such that $t_0 \in I$. Clearly, $u(t_0) = u_0$.

In fact, we shall prove that for all $t \in I$ $\hat{u}(t) = \hat{u}_0 + \int_{t_0}^{t} \chi(\tau)d\tau$, where

$$\chi : I \ni t \overset{\text{def}}{\longmapsto} A_t \cdot \hat{u}(t) + \hat{f}(t) \in \mathcal{D}'_{temp}. \tag{21}$$

Applying the argument from *Step* 1°, one can deduce that χ is summable.

Let $t \in I$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$. There is $N \in \mathbb{N}$ such that $\hat{u}_0 \in \mathcal{L}((\mathcal{S}, q_N), B)$ and the function $I \ni t \mapsto \hat{f}(t) \in \mathcal{L}((\mathcal{S}, q_N), B)$ is summable. Having N and ψ we consider the space X, as in Step 1°. Then $\psi\hat{u}_0 \in X$ and $I \ni t \mapsto \psi\hat{f}(t) \in X$ is summable.

Similarly as in *Step* 2°, the problem

$$\begin{cases} \dot{x}(t) &= \tilde{A}_t x(t) + \psi\hat{f}(t) \\ x(t_0) &= \psi\hat{u}_0 \end{cases} \tag{22}$$

has the unique solution given for all $t \in I$ by the formula

$$x(t) = \left(\prod_{t_0}^{t} e^{A_\tau d\tau}\right) \cdot (\psi\hat{u}_0) + \int_{t_0}^{t} \left(\prod_{s}^{t} e^{A_\tau d\tau}\right) \cdot (\psi\hat{f}(s))ds.$$

Thus remembering Remark 2 and definition (20) we obtain

$$x(t) = \psi\hat{u}(t). \tag{23}$$

On the other hand, combining (22), (23) and (21) we obtain

$$\dot{x}(t) = A_t \cdot x(t) + \psi\hat{f}(t) = \psi\chi(t). \tag{24}$$

The function $x : I \to X$ is absolutely continuous, a.e. differentiable, so by (23), (24), for all $t \in I$, there holds

$$\psi\hat{u}(t) = x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)d\tau = \psi\hat{u}_0 + \int_{t_0}^{t} \psi\chi(\tau)d\tau.$$

Consequently,

$$\forall t \in I : \qquad \hat{u}(t) = \hat{u}_0 + \int_{t_0}^{t} \chi(\tau)d\tau,$$

since $\psi \in \mathcal{D}(\mathbb{R}^n)$ is arbitrary. Therefore, by Remark 15, the function $\hat{u}_I : I \to \mathcal{D}'_{temp}$ is absolutely continuous and for a.e. $t \in I$

$$(\hat{u}_I)'(t) = \chi(t).$$

In consequence, so is $\hat{u} : J \to \mathcal{D}'_{temp}$, and for a.e. $t \in J$,

$$\hat{u}'(t) = A_t \cdot \hat{u}(t) + \hat{f}(t) = \sum_{\alpha} A_\alpha(t) \circ (i\xi)^\alpha \hat{u}(t) + \hat{f}(t),$$

since I is arbitrary. On account of Theorem 16, we finally have that $u : J \to \mathcal{D}'_{temp}$ is absolutely continuous and for a.e. $t \in J$

$$u'(t) = \mathcal{F}^{-1}(\hat{u}'(t)) = \sum_{\alpha} A_\alpha(t) \circ D^\alpha u(t) + f(t).$$

Thus u defined by (20) is a solution of (10).

□

In the case of time-independent coefficients $A_\alpha \in End\, B$, let us define

$$\Lambda_1 := \sup \left\{ \operatorname{Re} \lambda : \quad \lambda \in \sigma\left(\sum_{\alpha} (i\xi)^\alpha A_\alpha \right),\ \xi \in \mathbb{R}^n \right\},$$

$$\Lambda_{-1} := \inf \left\{ \operatorname{Re} \lambda : \quad \lambda \in \sigma\left(\sum_{\alpha} (i\xi)^\alpha A_\alpha \right),\ \xi \in \mathbb{R}^n \right\},$$

with $\sigma(A)$ denoting the spectrum of $A \in End\, B$.

Theorem 22. *Given are $t_0 \in J$, $u_0 \in \mathcal{D}'_{temp}$, locally summable $f : J \to \mathcal{D}'_{temp}$ and a finite family of coefficients $A_\alpha \in End\, B$ $(\alpha \in \mathbb{N}^n)$. Suppose that*

$$\text{for all } t \in J \setminus \{t_0\} \qquad \Lambda_{\operatorname{sgn}(t-t_0)} \in \mathbb{R}. \tag{25}$$

Then there is the unique absolutely continuous function $u : J \to \mathcal{D}'_{temp}$ such that

$$\begin{cases} \frac{d}{dt}u(t) &= \sum_{\alpha} A_\alpha \circ D^\alpha u(t) + f(t) \qquad \text{for a.e. } t \in J \\ u(t_0) &= u_0. \end{cases} \tag{26}$$

Moreover for each $t \in J$

(27)
$$u(t) = \mathcal{F}^{-1}\left(e^{(t-t_0)\sum\limits_{\alpha}(i\xi)^{\alpha}A_{\alpha}} \cdot \widehat{u_0}\right) + \int_{t_0}^{t} \mathcal{F}^{-1}\left(e^{(t-s)\sum\limits_{\alpha}(i\xi)^{\alpha}A_{\alpha}} \cdot \widehat{f}(s)\right) ds.$$

Proof. First note that the theorem is a corollary of Theorem 17. Namely, denoting as in the proof of Theorem 17, $A_t \equiv A \in \mathcal{H}ol$ for $t \in J$, with $A(z) := \sum\limits_{\alpha}(iz)^{\alpha}A_{\alpha}$ for $z \in \mathbb{C}^n$, it suffices to note that $A_t \circ A_s = A_s \circ A_t$ for all $s, t \in J$ and in consequence

$$\prod_{s}^{t} e^{A_\tau d\tau} = e^{\int_s^t A_\tau d\tau} = e^{(t-s)A} \qquad \forall s, t \in J.$$

The task is now to show that the family A_α satisfies condition (9). The proof will be devided into 3 steps.

Step 1° First we prove that, for any $t_1 \in J$, the function

$$\mathbb{R}^n \ni \xi \mapsto e^{(t-s)\sum_{\alpha}(i\xi)^{\alpha}A_{\alpha}} \in End\, B \tag{28}$$

is polynomially bounded, uniformly in $(s,t) \in [t_0, t] \times [t_0, t_1]$.

Fix $t_1 \in J$. Let $(s,t) \in [t_0, t] \times [t_0, t_1]$, $\xi \in \mathbb{R}^n$. According to the lemma in [**2**], p. 78, there is

$$\left|e^{(t-s)\sum_{\alpha}(i\xi)^{\alpha}A_{\alpha}}\right| \le e^{\Lambda_{(t-s),\xi}} \cdot \sum_{j=0}^{\dim B-1}\left(2|t-s|\left|\sum_{\alpha}(i\xi)^{\alpha}A_{\alpha}\right|\right)^{j}, \tag{29}$$

with $\Lambda_{(t-s),\xi} := \sup\left\{\operatorname{Re}\lambda : \quad \lambda \in \sigma\left((t-s)\sum\limits_{\alpha}(i\xi)^{\alpha}A_{\alpha}\right)\right\}$.

Since

$$\Lambda_{(t-s),\xi} \le |t_1 - t_0| \max\{|\Lambda_1|, |\Lambda_{-1}|\} < \infty,$$

there is a polynomial $Q : \mathbb{R}^n \to]0, \infty[$ such that $\forall (s,t) \in [t_0, t] \times [t_0, t_1]$ $\forall \xi \in \mathbb{R}^n$ $\left|e^{(t-s)\sum_{\alpha}(i\xi)^{\alpha}A_{\alpha}}\right| \le Q(\xi)$.

Step 2° We will prove polynomial boundedness of derivatives of function (28) by induction. Let $j = 1, \ldots, n$. We will estimate $\left|\frac{\partial}{\partial \xi_j} e^{(t-s)\sum_{\alpha}(i\xi)^{\alpha}A_{\alpha}}\right|$. Let $\xi \in \mathbb{R}^n$. There is $R > 0$ such that $|\xi| < R$. The mapping

$$A : [s,t] \times G \ni (\tau, \xi) \overset{\text{def}}{\longmapsto} \sum_{\alpha}(i\xi)^{\alpha}A_{\alpha} \in End\, B$$

is τ-summable for any $\xi \in G := \{\xi \in \mathbb{R}^n : \ |\xi| < R\}$. Its derivative $\frac{\partial}{\partial \xi_j} A(\tau, \xi)$ is dominated by summable function φ, namely

$$\varphi : [s,t] \ni \tau \mapsto \sup_{|\xi| \le R} \left| \frac{\partial}{\partial \xi_j} \sum_\alpha (i\xi)^\alpha A_\alpha \right|.$$

Therefore, according to Theorem 45, [**6**]

(30)
$$\frac{\partial}{\partial \xi_j}\left(e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha}\right) = \int_s^t e^{(t-r)\sum_\alpha (i\xi)^\alpha A_\alpha} \circ \frac{\partial}{\partial \xi_j} \sum_\alpha (i\xi)^\alpha A_\alpha \circ e^{(r-s)\sum_\alpha (i\xi)^\alpha A_\alpha} dr.$$

Since both (r,t) and (s,r) are elements of $[t_0,t] \times [t_0,t_1]$, it follows from Step 1° that both $\xi \mapsto e^{(t-r)\sum_\alpha (i\xi)^\alpha A_\alpha}$ and $\xi \mapsto e^{(r-s)\sum_\alpha (i\xi)^\alpha A_\alpha}$ are polynomially bounded. Finally, there is a polynomial $Q_{e_j} : \mathbb{R}^n \to]0,\infty[$ such that $\forall (s,t) \in [t_0,t] \times [t_0,t_1] \ \forall \xi \in \mathbb{R}^n$ $\left| \frac{\partial}{\partial \xi_j} e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha} \right| \le Q_{e_j}(\xi)$.

Step 3° Assume that the function $\mathbb{R}^n \ni \xi \mapsto D^\gamma\left(e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha}\right) \in End\, B$ is polynomially bounded, uniformly in $(s,t) \in [t_0,t] \times [t_0,t_1]$ for $\gamma \in \mathbb{N}^n$ such that $|\gamma| \le k$. Let $\beta \in \mathbb{N}^n$ be a multi-index such that $|\beta| = k+1$. On account of Theorem 49, [**6**]

(31)
$$D^\beta e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha} = \sum_{\beta \ne \gamma \le \beta} \binom{\beta}{\gamma} \int_s^t e^{(t-r)\sum_\alpha (i\xi)^\alpha A_\alpha} \circ D^{\beta-\gamma} \sum_\alpha (i\xi)^\alpha A_\alpha \circ D^\gamma e^{(r-s)\sum_\alpha (i\xi)^\alpha A_\alpha} dr.$$

For every $\gamma \le \beta$ and $\gamma \ne \beta$, there is a polynomial Q_γ such that

$$\left| D^\gamma e^{(r-s)\sum_\alpha (i\xi)^\alpha A_\alpha} dr \right| \le Q_\gamma(\xi)$$

for any $(s,r) \in [t_0,t] \times [t_0,t_1]$ and $\xi \in \mathbb{R}^n$. Moreover, as in Step 1^o, there is Q_0 such that $\left| e^{(t-r)\sum_\alpha (i\xi)^\alpha A_\alpha} \right| \le Q_0(\xi)$ for all $(r,t) \in [t_0,t] \times [t_0,t_1]$ and $\xi \in \mathbb{R}^n$. Thus

$$\left| D^\beta e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha} \right| \le \sum_{\beta \ne \gamma \le \beta} \binom{\beta}{\gamma} Q_0(\xi) \cdot |D^{\beta-\gamma} \sum_\alpha (i\xi)^\alpha A_\alpha| \cdot Q_\gamma(\xi) \cdot |t_1 - t_0|.$$

By induction we finally conclude that $\forall t_1 \in J\ \forall \beta \in \mathbb{N}^n\ \exists Q_\beta \in \mathcal{P}(\mathbb{R}^n), Q_\beta > 0$ $\forall (s,t) \in [t_0, t] \times [t_0, t_1]\ \ \forall \xi \in \mathbb{R}^n$

$$\left| D^\beta e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha} \right| \leq Q_\beta(\xi).$$

□

4. Examples.

4.1. *Parabolic systems in the sense of Pietrowski.*

Let B be a complex vector space and J be an interval in $\mathbb{R}$. Let $t_0 \in J$ and $b \in \mathbb{N}$. Let us recall that the system

$$\frac{d}{dt}u(t) = \sum_{|\alpha| \leq 2b} A_\alpha \circ D^\alpha u(t) + f(t) \tag{32}$$

is parabolic in the sense of Pietrowski iff $\exists \delta > 0\ \forall \zeta \in \mathbb{R}^n$, $|\zeta| = 1$:

$$\operatorname{Re} \lambda(\zeta) \leq -\delta, \qquad \text{where } \lambda(\zeta) \in \sigma \left(\sum_{|\alpha|=2b} (i\zeta)^\alpha A_\alpha \right),$$

see, e.g., [**1**].
We will prove that every system which is parabolic in the sense of Pietrowski satisfies condition (25).

$$\sum_{|\alpha| \leq 2b} (i\xi)^\alpha A_\alpha = |\xi|^{2b} \left(\sum_{|\alpha|<2b} \frac{(i\xi)^\alpha}{|\xi|^{2b}} A_\alpha + \sum_{|\alpha|=2b} \frac{(i\xi)^\alpha}{|\xi|^{2b}} A_\alpha \right).$$

There is $\delta > 0$ such that $\operatorname{Re}\lambda < -\delta$ for any $\lambda \in \sigma \left(\sum_{|\alpha|=2b} \frac{(i\xi)^\alpha}{|\xi|^{2b}} A_\alpha \right)$. On the other hand, there is $R > 0$ such that $\sigma \left(\sum_{|\alpha| \leq 2b} \frac{(i\xi)^\alpha}{|\xi|^{2b}} A_\alpha \right) \subset \{\operatorname{Re}\lambda < -\delta\} \subset \{\operatorname{Re}\lambda \leq 0\}$ for $|\xi| \geq R$ (see, e.g., [**5**], Th. 10.20). Therefore $\Lambda_1 < \infty$, and according to Theorem 22, there is the unique solution of system (32).

4.2. *Dirac equation.*

Consider the Dirac equation in Weyl's representation

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = \sum_{j=1}^{3} \gamma_j \left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j(t) \right) \psi + m_0 \beta \psi + V(t)\psi, \tag{33}$$

where the matrices

$$\gamma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

satisfy the relations

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}I_4 \quad j,k = 1,2,3,4, \quad \gamma_4 = \beta,$$

I_N is the $N \times N$ identity matrix,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices, m_0 is the rest mass, A_1, A_2, A_3, V are components of an external electromagnetic potential. Units were chosen so that $c = h = 1$.

Consider equation (33) together with the initial condition

$$\psi(t_0) = \psi_0 \tag{34}$$

where $\psi_0 \in \mathcal{D}'_{temp}(\mathbb{R}^3; \mathbb{C}^4)$. Observe that

$$A(\xi) = \begin{bmatrix} i\xi_3 & i\xi_1 + \xi_2 & im_0 & 0 \\ i\xi_1 - \xi_2 & -i\xi_3 & 0 & im_0 \\ im_0 & 0 & -i\xi_3 & -i\xi_1 - \xi_2 \\ 0 & im_0 & -i\xi_1 + \xi_2 & i\xi_3 \end{bmatrix}$$

and compute

$$\det(A(\xi) - \lambda I) = \left(|\xi|^2 + m_0^2 + \lambda^2\right)^2.$$

Thus eigenvalues of $A(\xi)$ are

$$\lambda = \pm i\sqrt{m_0^2 + |\xi|^2}$$

and $\forall \xi \in \mathbb{R}^n$ $\operatorname{Re}\lambda = 0$. Therefore, $\Lambda_1, \Lambda_{-1} \in \mathbb{R}$.
Suppose first that $A_j \equiv 0 \equiv V$ for $j = 1,2,3$. Then, according to Theorem 22, the only solution of problem $\{(33), (34)\}$ is

$$\psi(t) = \mathcal{F}^{-1}(e^{(t-t_0)A(\xi)} \cdot \hat{\psi}_0).$$

In the case of non-zero potentials, supposing that $A_j, V : J \to \mathbb{C}$ are locally summable ($j = 1,2,3$), Theorem 22 gives the following integral formula for ψ:

$$\psi(t) = \mathcal{F}^{-1}(e^{(t-t_0)A(\xi)} \cdot \hat{\psi}_0) + \int_{t_0}^{t} \mathcal{F}^{-1}\left(e^{(t-s)A(\xi)} \cdot i(V(s)\widehat{\psi}(s) - \sum_{j=1}^{3}\gamma_j A_j(s)\widehat{\psi}(s))\right) ds.$$

4.3. *Second-order time equations.*

Let B, B_1, B_2 be complex Banach spaces. J is an interval in $\mathbb{R}$. For $T_1 \in \mathcal{D}'(\mathbb{R}^n; B_1)$, $T_2 \in \mathcal{D}'(\mathbb{R}^n; B_2)$, let us consider the distribution $T_1 \triangle T_2 \in \mathcal{D}'(\mathbb{R}^n; B_1 \times B_2)$

$$T_1 \triangle T_2 : \quad \mathcal{D}(\mathbb{R}^n) \ni \varphi \overset{\text{def}}{\longmapsto} (T_1(\varphi), T_2(\varphi)) \in B_1 \times B_2.$$

Clearly, if $T_1 \in \mathcal{D}'_{temp}(\mathbb{R}^n; B_1)$, $T_2 \in \mathcal{D}'_{temp}(\mathbb{R}^n; B_2)$ then $T_1 \triangle T_2$ is tempered and

$$\overline{T_1 \triangle T_2} = \overline{T_1} \triangle \overline{T_2}.$$

Let functions $x : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B_1)$, $y : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B_2)$ be absolutely continuous. Then

$$x \triangle y : \quad J \ni t \overset{\text{def}}{\longmapsto} x(t) \triangle y(t) \in \mathcal{D}'_{temp}(\mathbb{R}^n; B_1 \times B_2)$$

is absolutely continuous and for any $t \in \operatorname{dom} \dot{x} \cap \operatorname{dom} \dot{y}$ (the set of full measure in J) there is

$$(x \triangle y)'(t) = \dot{x}(t) \triangle \dot{y}(t).$$

Definition 23. Given a map $F : \mathbb{R} \times \mathcal{D}'_{temp}(\mathbb{R}^n; B \times B) \multimap \mathcal{D}'_{temp}(\mathbb{R}^n; B)$. A function $x : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$ is a solution of the equation

$$\ddot{x} = F(t, x \triangle \dot{x}) \tag{35}$$

iff

1° $x : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$ is absolutely continuous, differentiable for all $t \in J$ and $\dot{x} : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$ is absolutely continuous,

2° $(t, x(t) \triangle \dot{x}(t)) \in \mathrm{domF}$ for any $t \in J$,

3° $\ddot{x}(t) = F(t, x(t) \triangle \dot{x}(t))$ for a.e. $t \in J$.

Theorem 24. *Let $x : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$ satisfy condition 1°. Denote $z := x \triangle \dot{x}$. x is a solution of (35) iff z is a solution of the equation $\dot{z} = v(t, z)$, where $v : \mathrm{domF} \to \mathcal{D}'_{\mathrm{temp}}(\mathbb{R}^{\mathrm{n}}; \mathrm{B} \times \mathrm{B})$ and $v(t, y \triangle \dot{y}) := \dot{y} \triangle F(t, y \triangle \dot{y})$.*

Proof. If $x : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$ is a solution of (35) then $z : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B \times B)$ is absolutely continuous, and for a.e. $t \in J$, $\dot{z}(t) = \dot{x}(t) \triangle \ddot{x}(t)$. Moreover,

$$v(t, x(t) \triangle \dot{x}(t)) = \dot{x}(t) \triangle F(t, x(t) \triangle \dot{x}(t)) = \dot{x}(t) \triangle \ddot{x}(t) = \dot{z}(t)$$

for a.e. $t \in J$.

Conversely, if z is a solution of the equation $\dot{z} = v(t, z)$ then $(t, z(t)) \in \mathrm{dom} v$ for any $t \in J$. Thus $(t, x(t) \triangle \dot{x}(t)) \in \mathrm{dom} F$ for any $t \in J$. Let us denote $Q(T) := T_2$ for $T = T_1 \triangle T_2 \in \mathcal{D}'_{temp}(\mathbb{R}^n; B \times B)$. Then

$$\ddot{x}(t) = Q(\dot{z}(t)) = Q(\dot{x}(t) \triangle F(t, x(t) \triangle \dot{x}(t))) = F(t, x(t) \triangle \dot{x}(t))$$

for a.e. $t \in J$.

□

Let us denote $P(T) := T_1$, $Q(T) := T_2$ for $T = T_1 \triangle T_2 \in \mathcal{D}'_{temp}(\mathbb{R}^n; B \times B)$.

COROLLARY 25. *Let* $v(t, x\triangle y) := y \triangle F(t, x\triangle y)$. *Suppose that* $z : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B \times B)$ *is a solution of the equation* $\dot{z} = v(t, z)$ *and* $x = P(z)$. *Then* $z = x\triangle\dot{x}$ *and* x *is a solution of* (35).

PROOF. Since $z : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B \times B)$ is absolutely continuous, so are $x = P \circ z$, $y := Q \circ z : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$. Let $t \in \operatorname{dom}\dot{z}$.

$$\dot{x}(t) = P(\dot{z}(t)) = P(v(t, z(t))) = P(y(t)\triangle F(t, z(t))) = y(t).$$

Thus x is differentiable for any $t \in J$, $\dot{x}$ is absolutely continuous and $z = x\triangle\dot{x}$. On account of Theorem 24, x is a solution of (35).

□

4.4. *Wave equation.*

Let $c > 0$, $t_0 = 0 \in J$. Given $u_0, u_1 \in \mathcal{D}'_{temp}(\mathbb{R}^n; \mathbb{C})$ and locally summable $f : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; \mathbb{C})$. Let us consider the Cauchy problem for the wave equation

$$(36) \qquad \begin{cases} \dfrac{d^2}{dt^2} u(t) &= c^2 \Delta u(t) + f(t) \quad \text{for a.e. } t \in J \\ u(0) &= u_0 \\ u'(0) &= u_1. \end{cases}$$

Setting

$$w\ (= w_1 \triangle w_2) : \ J \ni t \overset{\text{def}}{\longmapsto} u(t)\triangle u'(t) \in \mathcal{D}'_{temp}(\mathbb{R}^n; \mathbb{C}^2)$$

we can rewrite (36) as the following first-order Cauchy problem:

$$(37) \qquad \begin{cases} w'(t) &= \displaystyle\sum_{|\alpha| \le 2} A_\alpha \circ D^\alpha w(t) + \tilde{f}(t) \quad \text{for a.e. } t \in J \\ w(0) &= w_0, \end{cases}$$

where $\tilde{f}(t) := 0 \triangle f(t)$, $w_0 := u_0 \triangle u_1 \in \mathcal{D}'_{temp}(\mathbb{R}^n; \mathbb{C}^2)$,

$$A_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{2e_j} := c^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$(e_j)_{j=1}^n \text{ is the canonical basis in } \mathbb{R}^n,$$

$A_\alpha := 0$ for all other multi-indices α. The eigenvalues of the matrix

$$\sum_{|\alpha| \le 2} (i\xi)^\alpha A_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - c^2 |\xi|^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are: $-ic|\xi|$, $ic|\xi|$, thus Λ_1, $\Lambda_{-1} = 0 \in \mathbb{R}$ and according to Theorem 22, there is the unique solution of problem (37) given by the formula

$$w(t) = \mathcal{F}^{-1}\left(e^{t\sum_\alpha (i\xi)^\alpha A_\alpha} \cdot \hat{w}_0\right) + \int_0^t \mathcal{F}^{-1}\left(e^{(t-s)\sum_\alpha (i\xi)^\alpha A_\alpha} \cdot \widehat{\tilde{f}}(s)\right) ds.$$

Note that

$$\left(\sum_\alpha (i\xi)^\alpha A_\alpha\right)^{2\nu} = (-c^2|\xi|^2)^\nu \cdot I,$$

$$\left(\sum_\alpha (i\xi)^\alpha A_\alpha\right)^{2\nu+1} = (-c^2|\xi|^2)^\nu \cdot \left(\sum_\alpha (i\xi)^\alpha A_\alpha\right)$$

for $\nu = 0, 1, 2, \ldots$, hence

$$e^{\tau\sum_\alpha (i\xi)^\alpha A_\alpha} = \begin{pmatrix} \cos(\tau c|\xi|) & \dfrac{\sin(\tau c|\xi|)}{c|\xi|} \\ -c|\xi|\sin(\tau c|\xi|) & \cos(\tau c|\xi|) \end{pmatrix}.$$

On account of Cor. 25, we obtain the formula for the solution of (36):

$$\begin{aligned} u(t) &= \mathcal{F}^{-1}\left(\cos(tc|\xi|)\cdot \hat{u}_0\right) + \mathcal{F}^{-1}\left(\frac{\sin(tc|\xi|)}{c|\xi|}\cdot \hat{u}_1\right) + \\ &+ \int_0^t \mathcal{F}^{-1}\left(\frac{\sin((t-s)c|\xi|)}{c|\xi|}\cdot \hat{f}(s)\right) ds. \end{aligned} \tag{38}$$

Fixing $f \equiv 0$, $u_0 = 0$, $u_1 = \delta$ in (38), we obtain the formula for the fundamental solution of the d'Alembert operator:

$$u(t) = \mathcal{F}^{-1}\left(\frac{\sin(tc|\xi|)}{c|\xi|}\cdot \hat{\delta}\right) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \mathcal{F}^{-1}\left[\frac{\sin(tc|\xi|)}{c|\xi|}\right],$$

where $[\psi]$ denotes the regular distribution generated by $\psi \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, i.e.,

$$[\psi](\varphi) := \int_{\mathbb{R}^n} \psi(x)\varphi(x)dx \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

4.5. *Navier–Lamé equation.*

Let $t_0 \in J$; positive numbers λ, μ are the Lamé constants. Given u^0, $u^1 \in \mathcal{D}'_{temp}(\mathbb{R}^3;\mathbb{C}^3)$ and locally summable $f : J \to \mathcal{D}'_{temp}(\mathbb{R}^3;\mathbb{C}^3)$. We are looking for an absolutely continuous $u : J \to \mathcal{D}'_{temp}(\mathbb{R}^3;\mathbb{C}^3)$ such that

$$\left\{\begin{aligned} \frac{d^2}{dt^2}u(t) &= \mu\Delta u(t) + (\lambda+\mu)\nabla \operatorname{div} u(t) + f(t) \quad \text{for a.e. } t \in J \\ u(t_0) &= u^0 \\ u'(t_0) &= u^1. \end{aligned}\right. \tag{39}$$

For $t \in J$, denote

$$u(t) = u_1(t) \triangle u_2(t) \triangle u_3(t), \quad f(t) = f_1(t) \triangle f_2(t) \triangle f_3(t).$$

The Navier–Lamé equation can now be written in the form

$$\frac{d^2}{dt^2} u_j(t) = \mu \Delta u_j(t) + (\lambda + \mu) \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{3} \frac{\partial}{\partial x_k} u_k(t) \right) + f_j(t) \tag{40}$$

for $j = 1, 2, 3$ and a.e. $t \in J$. Settings

$$w_j^1(t) := u_j(t) \quad w_j^2(t) := u_j'(t), \quad w(t) := (w_1^1 \triangle w_2^1 \triangle w_3^1 \triangle w_1^2 \triangle w_2^2 \triangle w_3^2)(t)$$

for $t \in J$ transform (40) into the following first-order system:

$$\begin{cases} \dfrac{d}{dt} w_j^1(t) & = & w_j^2 \\ \dfrac{d}{dt} w_j^2(t) & = & \mu \Delta w_j^1(t) + (\lambda + \mu) \dfrac{\partial}{\partial x_j} \left(\displaystyle\sum_{k=1}^{3} \dfrac{\partial}{\partial x_k} w_k^1(t) \right) + f_j(t) \end{cases} \tag{41}$$

for $j = 1, 2, 3$ and a.e. $t \in J$. Denoting $\tilde{f}(t) := 0 \triangle f(t) \in \mathcal{D}'_{temp}(\mathbb{R}^3; \mathbb{C}^6)$ $(0 \in \mathcal{D}'_{temp}(\mathbb{R}^n; \mathbb{C}^3))$,

$$A_0 := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{jk} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} . \\ . \\ . \\ \ldots (3+j)\text{-th row} \\ . \\ . \end{matrix}$$

$\vdots$
k-th column

$A_{e_j+e_k} := A_{jk}$ for $j \neq k$ and $A_{2e_j} := \mu A_0^T + (\lambda + \mu) A_{jj}$, where A_0^T is transposed to A_0, $(e_j)_{j=1}^3$ is the canonical basis in $\mathbb{R}^3$, $A_\alpha = 0$ for other multi-indices α, we finally obtain the matrix coefficient problem of (26)-type:

$$(42)\qquad \begin{cases} \dfrac{d}{dt}w(t) &= \displaystyle\sum_{|\alpha|\le 2} A_\alpha \circ D^\alpha w(t) + \tilde{f}(t) \quad \text{for a.e. } t \in J \\ w(t_0) &= u^0 \triangle u^1. \end{cases}$$

Since

$$(43)\qquad \sum_{|\alpha|\le 2} (i\xi)^\alpha A_\alpha = A_0 - \mu A_0^T |\xi|^2 - (\lambda+\mu) \sum_{j,k=1}^{3} \xi_j \xi_k A_{jk}.$$

and

$$(\nu^2 + \mu|\xi|^2)^2(\nu^2 + (2\mu+\lambda)|\xi|^2) = 0,$$

is the characteristic equation of (43), the eigenvalues of the matrix are double: $i\sqrt{\mu}|\xi|$, $-i\sqrt{\mu}|\xi|$ and single: $i\sqrt{2\mu+\lambda}|\xi|$, $-i\sqrt{2\mu+\lambda}|\xi|$. Thus $\operatorname{Re}\nu = 0$ for every $\nu \in \sigma\left(\sum_\alpha (i\xi)^\alpha A_\alpha\right)$, and according to Theorem 22, the solution of problem (42) is given by (27).

4.6. *Biparabolic equation.*

Let us consider the operator

$$P^2 := \left(\Delta - \frac{\partial}{\partial t}\right)\left(\Delta - \frac{\partial}{\partial t}\right) = \Delta^2 - 2\frac{\partial}{\partial t}\Delta + \frac{\partial^2}{\partial t^2}.$$

Assume that $u_0,\ u_1 \in \mathcal{D}'_{temp}(\mathbb{R}^n;\mathbb{C})$ and $f : [t_0, +\infty[\to \mathcal{D}'_{temp}(\mathbb{R}^n;\mathbb{C})$ is locally summable. Let us write the Cauchy problem for P^2:

$$(44)\qquad \begin{cases} \dfrac{d^2}{dt^2}u(t) &= 2\dfrac{d}{dt}\Delta u(t) - \Delta^2 u(t) + f(t) \quad \text{for a.e. } t \in [t_0, \infty[\\ u(t_0) &= u_0 \\ u'(t_0) &= u_1. \end{cases}$$

Setting

$$w(= w_1 \triangle w_2) : [t_0, \infty[\ni t \overset{\text{def}}{\longmapsto} u(t) \triangle u'(t) \in \mathcal{D}'_{temp}(\mathbb{R}^n;\mathbb{C}^2),$$

(44) can be written as

$$(45)\qquad \begin{cases} w'(t) &= \displaystyle\sum_{|\alpha|\le 4} A_\alpha \circ D^\alpha w(t) + \tilde{f}(t) \quad \text{for a.e. } t \in [t_0, \infty[\\ w(t_0) &= u_0 \triangle u_1, \end{cases}$$

where $\tilde{f} = 0 \triangle f : [t_0, \infty[\to \mathcal{D}'_{temp}(\mathbb{R}^n;\mathbb{C}^2)$, $A_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_{2e_j} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_{2e_j+2e_k} := 2A_{jk}$ for $j \neq k$, $A_{4e_j} := A_{jj}$, where $A_{jk} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $A_\alpha = 0$

for other multi-indices α. The matrix

$$\sum_{|\alpha|\leq 4} (i\xi)^\alpha A_\alpha = \begin{pmatrix} 0 & 1 \\ -|\xi|^4 & -2|\xi|^2 \end{pmatrix}$$

has non-positive eigenvalues $\lambda = -|\xi|^2$, and according to Theorem 22, we obtain the unique solution of (45) given by (27).

References

1. Ejdelman S.D., *Paraboličeskie sistemy,* Moskva, 1964.
2. Gelfand I.M., Šilov G., *Nekotorye voprosy teorii differencjalnych uravnenij,* Moskva 1958.
3. Holly K., *Absolutely continuous distribution valued curves,* manuscript.
4. Holly K., *Linear evolution equation in* $\mathbb{R}^n$*,* manuscript.
5. Rudin W., *Functional analysis,* McGraw-Hill, New York, 1973.
6. Wiciak M., *Product integral in a Fréchet space,* Univ. Iagel. Acta Math., **39** (2001), 281–298.

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