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TEMPERED GROUPS

by Josef Teichmann

Abstract. Tempered groups are defined to possess a subalgebra D of the continuous bounded complex valued functions $BC(G, \mathbb{C})$, such that D detects the topology on the group in a specified sense and such that some Lipschitz-properties are satisfied. With this subalgebra D it is possible to derive – by classical Banach space methods – some results on approximation of the exponential map on infinite dimensional Lie groups. It is remarkable that all up to now known Fréchet–Lie groups, finite or infinite dimensional, are tempered.

1. Introduction. Up to Banach spaces there is a powerful theory to solve ordinary differential equations. Already on Fréchet spaces one has to investigate the circumstances much more carefully to obtain results on solvability of differential equations [**4**]. Nevertheless Fréchet spaces appear naturally by modelling C^∞-diffeomorphism groups [**3**]. There are two possible ways how to approach this problem: Either one tries to translate the given initial value problem into a Banach space setting, which normally leads to a loss of differentiability properties, or one tries to find some rudiments of theory on convenient spaces, so differentiability is preserved, but there is a lack of powerful theorems. Tempered groups are defined by the perspective of the first method. We shall prove that on tempered groups smooth one parameter subgroups can be well approximated by simple product integrals. This is by the way the origin of the notion of a "Tempered Group" since the growth of the multiplication is controlled. Banach Lie groups and ILB-Lie groups shall be proved to be tempered if the model spaces admit C_b^2-bump functions.

The most important class of regular Fréchet–Lie groups was given by Hideki Omori et al. (see [**7**]) with the concept of strong ILB-groups, nevertheless this

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concept is rather complicated in application. For the concepts of convenient calculus and Frölicher spaces (smooth spaces) see [**3**].

Definition 1. *A Lie group G is a smooth manifold modeled on c^∞-open subsets of a convenient vector space with smooth multiplication $\mu\colon G\times G\to G$, where $\mu(x,y)=xy$, and smooth inversion $\nu\colon G\to G$, where $\nu(x)=x^{-1}$, for $x,y\in G$. We shall denote by $\mu_x\colon G\to G$ and $\mu^y\colon G\to G$ the smooth left and right translation by an element of G, i.e. $\mu_x(y)=\mu^y(x)=\mu(x,y)$ for $x,y\in G$.*

Remark that Lie groups are not topological groups in general, because the identity $c^\infty(E\times E)\to c^\infty E\times c^\infty E$ need not be a homeomorphism (see [**3**], ch.1). If the Lie group G is a topological group, then the underlying topological space is regular (since any Hausdorff topological group is regular, see [**6**]), but not necessarily smoothly regular, and we can assume, that a chosen chart (u,U) has the property that inverse images of closed bounded sets in the convenient vector space are closed in the group, not only relatively closed in $U\subset G$. We shall need this property to be able to lift functions from the convenient vector space to the group.

The classical basics of Lie theory can be carried over to this general setting without any problems (see [**3**], ch. 8): Via left or right translation one can trivialize the kinematic tangent bundle $TG=G\times\mathfrak{g}$, where $\mathfrak{g}$ denotes the tangent space at the identity e. This is done by left invariant vector fields ($X\in C^\infty(G\leftarrow TG)$ is called left invariant if $T\mu_x X=X\circ\mu_x$ or equivalently $\mu_x^* X=X$ for $x\in G$) or right invariant vector fields. $\mathfrak{g}$ becomes a Lie algebra, isomorphic to the Lie algebra of left invariant vector fields and antiisomorphic to the Lie algebra of right invariant vector fields on G ($X\in C^\infty(G\leftarrow TG)$ is called right invariant if $\mu^{x*}X=X$ for $x\in G$). We denote the (anti-) isomorphism by L (respectively R). We have the following formulas for $x\in G$:

$$L(X)_x=\frac{d}{dt}|_{t=0}\,xc(t)\text{ and }R(X)_x=\frac{d}{dt}|_{t=0}\,c(t)x$$

for $X\in\mathfrak{g}$ and a curve $c\colon\mathbb{R}\to G$ with $c(0)=e$ and $c'(0)=X$.

An exponential mapping is a map $\exp\colon\mathfrak{g}\to G$, so that $Fl^{L(X)}(t,x)=x\exp(tX)$ is the global flow of the left invariant vector field $L(X)$. An exponential mapping is unique if it exists, remark that the global flow of $R(X)$ is given through $Fl^{R(X)}(t,x)=\exp(tX)x$. Furthermore we obtain for a smooth group homomorphism ϕ of groups, which admit an exponential mapping, the formula $\exp(\phi'(X))=\phi(\exp(X))$ for $X\in\mathfrak{g}$.

Definition 2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The conjugation by an element $x\in G$ defined through $conj_x(y)=xyx^{-1}$ for $y\in G$ is a smooth group automorphism. $Ad_x:={conj_x}'$ defines a smooth representation $Ad\colon G\to GL(\mathfrak{g})$. The adjoint representation Ad of the group maps into the subspace of* *smooth Lie algebra automorphisms. The derivative of Ad (even in the sense of smooth groups) is* $ad\colon \mathfrak{g} \to L(\mathfrak{g})$. *The adjoint representation ad of the Lie algebra maps in the subspace of derivations of the Lie algebra* $\mathfrak{g}$.

A smooth group or Frölicher–Lie-group is a group G with the structure of a smooth space (for this concept see [**3**], ch. 5), such that the multiplication and the inversion are smooth. All smoothly regular Lie groups are smooth groups in a canonical way. The group $GL(E)$ of invertible linear bounded maps on a convenient vector space E is a standard example, and not a Lie group in general, with the following smooth structure:

$$\mathcal{C}_{GL(E)} := \{c\colon \mathbb{R} \to GL(E) | c\colon \mathbb{R} \to L(E) \text{ and } inv \circ c\colon \mathbb{R} \to L(E) \text{ are smooth}\},$$
$$\mathcal{F}_{GL(E)} := \{f\colon GL(E) \to \mathbb{R} | f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in C_{GL(E)}\}.$$

Notice that the restrictions of all linear functionals on $L(E)$ lie in $\mathcal{F}_{GL(E)}$, furthermore the restriction of the composition of a linear functional with the inversion lies in $\mathcal{F}_{GL(E)}$, so $GL(E)$ becomes a smooth space. Multiplication and inversion are smooth. A smooth map $f\colon G \to H$ between Frölicher–Lie-groups is called initial if for any curve $c\colon \mathbb{R} \to G$ with $f \circ c$ smooth smoothness of c is implied. The following example might explain the interest in smooth groups:

Example 3. *The* ∞*-Torus* $\prod_{k\in\mathbb{N}} S^1$ *is a smooth group with smooth curves being the componentwise smooth ones. The* ∞*-Torus is a product in the category of locally compact topological groups and a product in the category of smooth groups. Furthermore the* ∞*-Torus can never be made to a manifold, because it should be infinite dimensional and locally compact.*

Remark 4. *Let* G *be a convenient smootly regular Lie group and denote by* $Ad\colon G \to GL(\mathfrak{g})$ *the adjoint representation. If* Ad *is initial, i.e.* $Ad \circ c$ *smooth for a curve* $c\colon \mathbb{R} \to G$ *implies smoothness of the curve, then the inner automorphisms* $\mathrm{Inn}(\mathfrak{g}) := Ad(G)$ *constitute a Frölicher–Lie-group diffeomorphic to* G *in the category of Frölicher–Lie groups. This is due to the fact that the Frölicher–Lie structure is induced by the general linear group* $GL(\mathfrak{g})$ *and well-defined, the diffeomorphism is given by* Ad.

Proposition 5. *Let* M *denote a compact finite-dimensional manifold and* $\mathrm{Diff}(M)$ *the Fréchet–Lie group of diffeomorphisms of* M*, then* $Ad\colon \mathrm{Diff}(M) \to GL(\mathfrak{X}(M))$ *is an initial map, so* $\mathrm{Diff}(M)$ *is canonically isomorphic to the Frölicher–Lie-group of inner automorphisms.*

Proof. The proof is done in several steps: First we prove that a curve c in G with $Ad \circ c$ smooth has the following property: For any $t_0 \in \mathbb{R}$, $x_0 \in M$ and any open neighborhood V of $c^{-1}(t_0, x_0)$ there is $\delta > 0$ and an open neighborhood U of x_0 with $c^{-1}(t, x) \in V$ for $|t - t_0| < \delta$ and $x \in U$. Otherwise there would exist $t_0 \in \mathbb{R}$, $x_0 \in M$, an open neighborhood V of $c^{-1}(t_0, x_0)$ and sequences $t_n \to t_0$ and $x_n \to x_0$ with $c^{-1}(t_n, x_n) \notin V$. Now take a vector field X on M having support in V with $X(c^{-1}(t_0, x_0)) \neq 0$. The formula for the action of Ad on vector fields is

$$Ad_{c_t}(X)(x) = T_{c^{-1}(t,x)}c_t(X(c^{-1}(t,x)).$$

Consequently $Ad_{c_{t_n}}(X)(x_n) = 0$ for all n, but $Ad_{c_{t_0}}(X)(x_0) \neq 0$. However, $Ad_{c_t}(X)$ is smooth, so a smooth curve of smooth sections in the tangent bundle, a contradiction.

Second step: it is sufficient to prove the following fact: Let c be a curve passing at $t = 0$ through e with $Ad \circ c$ smooth, then there is a neighborhood of 0, where the curve is smooth. From this we conclude easily the general case by looking at the smoothness of the composition and the curve $c_{t_0}^{-1}c$ around t_0.

Third step: we apply the first observation to prove the assertion of the second step: Let c be a curve passing at $t = 0$ through e with $Ad \circ c$ smooth, then there is a chart domain $V \subset M$, open around $x_0 = c^{-1}(0, x_0)$ mapped to a ball in $\mathbb{R}^n$, furthermore $\delta > 0$ and an open neighborhood $U \subset V$ of x_0 with $c^{-1}(t, x) \in V$ for $|t| < \delta$ and $x \in U$. Smoothness of $Ad \circ c$ locally can be read as follows

$$(T_x c_t^{-1})^{-1}(\frac{\partial}{\partial x^i}) \text{ is smooth for } i = 1, ..., n.$$

This means, by smoothness of the inversion of matrices, that $(\frac{\partial (c_t^{-1})^j)}{\partial x^i}(x))$ is smooth locally around x_0 and 0. Consequently by compactness of the manifold we conclude that there is a small interval around zero where $c_t^{-1}$is a smooth curve of diffeomorphisms, so c_t is smooth. □

Since we do neither possess any information on the initiality of Ad in general nor on some good other linear representation, which induces a strongly continuous Banach space representation, we shall try to construct a subspace of the bounded continuous functions, on which G acts faithfully, to be able to apply some Banach space methods to obtain the existence of the exponential map.

2. Tempered groups. We shall apply a remarkable theorem of Paul Chernoff on the approximation of C_0-contraction semigroups on a Banach space X (see [**1**]):

THEOREM 6. *Let X be a Banach space, and $c\colon \mathbb{R}_{\geq 0} \to L(X)$ a curve of uniformly power-bounded operators, i.e. there is $s_0 > 0$ and $M \geq 1$ such that $||c(t)^n|| \leq M$ for $t \in [0, s_0]$ and $n \in \mathbb{N}$. If there is a dense subset $D \subset X$ such* *that*

$$\lim_{t\downarrow 0} \frac{c(t)x - x}{t} = Ax \text{ for } x \in D$$

and there is $\lambda_0 > 0$ *with* $(\lambda_0 - A)D$ *dense in* X*, then* $\overline{A}$ *is the infinitesimal generator of a strongly continuous semigroup* T *and*

$$s\text{-}lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

uniformly on compact subsets of $\mathbb{R}_{\geq 0}$.

The following Theorem is needed for approximation on convenient vector spaces as stated in the following corollary (see [**8**] for the proofs):

Theorem 7. *Let* E *be a convenient vector space and* $c\colon \mathbb{R}_{\geq 0} \to L(E)$ *a smooth curve of bounded linear mappings with* $c(0) = id$ *such that there exists* $s > 0$ *with* $\{c(\frac{t}{n})^n \mid n > 0, 0 \leq t \leq s\}$ *a bounded set of bounded linear mappings, then there is a smooth semigroup* $T\colon \mathbb{R}_{\geq 0} \to L(E)$ *with infinitesimal generator* $c'(0)$ *and*

$$lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

in $C^\infty(\mathbb{R}_{\geq 0}, L(E))$.

Corollary 8. *Let* G *be a smoothly regular, convenient Lie group (with* $c^\infty G$ *being a topological group) and* $c\colon \mathbb{R}_{\geq 0} \to G$ *a smooth curve with* $c(0) = e$ *such that there exists* $s > 0$ *with* $\{c(\frac{t}{n})^n \mid n > 0, 0 \leq t \leq s\}$ *relatively compact in the smooth topology, then there are a smooth group* $T_t = \lim_{n\to\infty} c(\frac{t}{n})^n$ *of* G *and a chart* (u, U) *around* e *with the following properties: There is small* $\epsilon > 0$ *such that* $\{c(\frac{t}{n})^n \mid n > 0, 0 \leq t \leq \epsilon\} \subset U$ *and* $u(c(\frac{t}{n})^n)$ *converges in all derivatives to* $u(T_t)$ *for* $0 \leq t \leq \epsilon$.

We shall work in the Banach space $BC(G)$ of continuous complex valued bounded functions on G normed by the supremum norm. Given $X \in \mathfrak{g}$, there is a smooth one-parameter subgroup $\exp(tX)$ of G, if G admits an exponential map. We shall investigate the group T of linear operators on $BC(G)$ given by

$$T_t(f)(x) := f(\exp(tX)x) = (f \circ \mu_{\exp(tX)})(x) \text{ for all } x \in G,\ f \in BC(G)$$

for $t \in \mathbb{R}$. For a given smooth curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ and $c'(0) = X$ we can proceed in the same manner, so we obtain a curve C of isometries on $BC(G)$. To be able to apply Chernoff's theorem, we need an appropriate domain D in $BC(G)$, such that D detects the topology of G in a certain sense and satisfies certain properties concerning translations. This leads us to the concept of tempered groups. It is clear that the above one-parameter subgroup has some dense domain of definiton for the infinitesimal generator. That the intersection of all these maximal domains is again dense in the case of all strong ILB-groups is the result of this section.

DEFINITION 9 (tempered groups). *A smooth group G is said to be tempered if it is a smooth space and if a unital subalgebra $D \subset BC(G)$ is given, such that the following conditions are satisfied.*

1. *D is invariant under left translations, that means $f \circ \mu_a \in D$ for all $f \in D$ and $a \in G$.*
2. *D detects the converging sequences on G:*
$$\forall \{x_n\}_{n\in\mathbb{N}},\ x \in G : \sup_{y\in G} |f(x_n y) - f(xy)| \to 0 \text{ for all } f \in D \Rightarrow$$
$$x_n \text{ converges to } x \text{ in } G.$$
3. *For every smooth curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ the curve $C\colon \mathbb{R} \to L(BC(G))$ of left translations by $c(t)$ for $t \in \mathbb{R}$ is differentiable at $f \in D$ for $t = 0$ in the supremum-norm topology of $BC(G)$.*

LEMMA 10. *Let G be a tempered group and $c\colon \mathbb{R} \to G$ a smooth curve, then c is continuous and $C\colon \mathbb{R} \to L(BC(G))$ is differentiable at $f \in D$ for all $t \in \mathbb{R}$.*

PROOF. Let $c\colon \mathbb{R} \to G$ be a smooth curve, then $b(t) := c(t)c(0)^{-1}$ for $t \in \mathbb{R}$ is a smooth curve with $b(0) = e$, so $B\colon \mathbb{R} \to L(BC(G))$ is differentiable at $f \in D$ for $t = 0$. So for any $f \in D$ there exists $g \in BC(G)$, such that
$$\sup_{x\in G} |\frac{f(c(t)c(0)^{-1}x) - f(x)}{t} - g(x)| \overset{t\to 0}{\to} 0,$$
consequently by left translation we obtain
$$\sup_{y\in G} |\frac{f(c(t)y) - f(c(0)y)}{t} - g(c(0)y)| \overset{t\to 0}{\to} 0,$$
which is the desired assertion. The rest follows by property 2. □

LEMMA 11. *Let G be a topological group, $U \subset G$ an open neighborhood of e. Then there is a neighborhood $V \subset G$ of e, such that for any continuous curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ one can find a small open interval J around zero with $\cup_{t\in J} c(t)^{-1}V \subset U$.*

PROOF. The mapping $G \times G \to G$, $(g, h) \to g^{-1}h$ is continuous, so the conclusion follows immediately. □

The following proposition asserts that on tempered topological groups smooth one parameter groups can be well approximated:

PROPOSITION 12. *Let G be a tempered group (with $c^\infty G$ a topological group). Let $c\colon \mathbb{R} \to G$ be a smooth curve with $c(0) = e$ touching a smooth one-parameter group S at $t = 0$, more precisely*
$$\forall f \in D,\ x \in G :\ (f \circ \mu^x(c))'(0) = (f \circ \mu^x(S))'(0).$$

Then we obtain

$$\lim_{n\to\infty} c(\frac{t}{n})^n = S_t$$

uniformly on compact subsets of $\mathbb{R}$*, i.e.* $c(\frac{t}{n})^n S_{-t}$ *converges to* e *as* $n \to \infty$ *uniformly on compact subsets of* $\mathbb{R}$*. If* G *is a smoothly regular tempered Lie group, then the convergence of* $c(\frac{t}{n})^n$ *to* S_t *is uniform in all derivatives.*

Proof. The first part of the proof is a simple application of the core theorem, which asserts that the closure of the restriction of an infinitesimal generator of a strongly continuous semigroup to an invariant and dense subspace is the infinitesimal generator (see [**1**]). We shall denote the closure of the subspace $D \subset BC(G)$ by X. C and T denote the curves of contractions on $BC(G)$ given by left translation with $c(t)$ and $S(t)$, respectively. By property 1 D is invariant under the action of $C(t)$ and $T(t)$ for $t \in \mathbb{R}$, by property 3 the first derivatives of C and T at $t = 0$ exist pointwise for $f \in D$ and they coincide. So $T|_X$ defines a C_0-group on X, $C|_X$ is a curve of contractions on X. D is a dense, $T|_X$-invariant subspace of the domain of the infinitesimal generator of T, consequently the closure of the restriction of the infinitesimal generator to D is the infinitesimal generator. The application of Chernoff's theorem leads to

$$\lim_{n\to\infty} (C|_X(\frac{t}{n}))^n (f) = (T|_X)_t(f)$$

for all $f \in X$ uniformly on compact subsets of $\mathbb{R}$. In fact we have to apply the theorem twice to obtain the assertion for the whole real line. By property 2 we are lead to the existence of the limit in the group G uniformly on compact subsets of $\mathbb{R}$. Uniformity is due to our specified detection of the topology of G. Suppose that the sequence does not converge uniformly on a given compact interval K to the limit e, so there is an open neighborhood U of e and sequences $\{n_k\}_{k\in\mathbb{N}}$, a monotone, diverging sequence of natural numbers, and $\{t_k\}_{k\in\mathbb{N}}$ in K, so that $c(\frac{t_k}{n_k})^{n_k} S_{-t_k} \notin U$ for $k \in \mathbb{N}$, but

$$\sup_{x\in G} |f(c(\frac{t_k}{n_k})^{n_k} x) - f(S_{t_k} x)| \overset{k\to\infty}{\to} 0$$

by the convergence theorem. Consequently

$$\sup_{y\in G} |f(c(\frac{t_k}{n_k})^{n_k} S_{-t_k} y) - f(y)| \overset{k\to\infty}{\to} 0$$

which leads by 2 to a contradiction. So the limit exists uniformly on compact subsets of $\mathbb{R}$. By Theorem 7 and its Corollary 8 the smoothly regular case is already proved, we give another perspective in the general case, too.

Let G be additionally a Lie group, then the adjoint representation Ad maps sequentially smoothly compact sets to bounded ones, since it is continuous with respect to the smooth topologies (see [**3**]).

We denote by $Y \in \mathfrak{g}$ the generator of $S_t = \exp(Yt)$. As established above we know that $\lim_{n\to\infty} c_n(t) = \exp(Yt)$ uniformly on compact subsets of $\mathbb{R}$. The rest of the proof is devoted to the uniform convergence on compact intervals of $\delta^r c_n(t)$ to Y as $n \to \infty$. In fact it is an easy consequence of calculations with right logarithmic derivatives: For smooth curves $c, d\colon \mathbb{R} \to G$ we have

$$\delta^r(cd)(t) = \delta^r c(t) + Ad_{c(t)}\delta^r d(t)$$

for $t \in \mathbb{R}$. Consequently we obtain

$$\delta^r c_n(t) = \frac{1}{n}\left(\sum_{i=0}^{n-1} Ad^i_{c(\frac{t}{n})}\right)\delta^r c(\frac{t}{n})$$

for all $t \in \mathbb{R}$. The adjoint action of G maps sequentially compact to bounded sets in $L(\mathfrak{g})$, so there is a bounded absolutely convex subset $B \subset L(\mathfrak{g})$ so that $Ad_{c(\frac{t}{n})^n S_{-t}} \in B$ for t in a closed zero neighborhood. By the general approximation theorem on convenient algebras we obtain the following Mackey-limit:

$$\lim_{n\to\infty} Ad^n_{c(\frac{t}{n})} = Ad_{S_t}$$

uniformly on compact subsets of $\mathbb{R}$. The sequence measuring Mackey-convergence is given by $\{\frac{t^2}{n}\}_{n\in\mathbb{N}_+}$ on the interval $[0,t]$. To conclude we look at the above sum as an approximation of the integral $\frac{1}{t}\int_0^t Ad_{S_s}ds$ for $t \neq 0$ in the convenient algebra $L(\mathfrak{g})$. In fact we can choose n_0 big enough, so that for $n \geq n_0$ the approximation of the limit Ad_{S_t} by $Ad_{c_n(t)}$ is good enough. By uniformity of the respective limits, we obtain that

$$\frac{t}{n}(\sum_{i=0}^{n-1} Ad^i_{c(\frac{t}{n})}) = \frac{t}{n}(\sum_{i=0}^{n_0} Ad^i_{c(\frac{t}{n})} + \sum_{i=n_0}^{n-1}(Ad^i_{c(\frac{ti}{in})} - Ad_{S_{\frac{ti}{n}}}) + \sum_{i=n_0}^{n-1} Ad_{S_{\frac{ti}{n}}}$$

converges Mackey to the integral uniformly on compact subsets. Consequently

$$\lim_{n\to\infty}\delta^r c_n(t) = \frac{1}{t}\int_0^t Ad_{S_s}ds\, Y = Y$$

uniformly on compact subsets of $\mathbb{R}$. So the assertions are proved. On the way we obtain naturally $c'(0) = Y$. □

There is a general simple method to detect differentiability on Banach spaces, which is in fact valid for much more general situations (see [**2**] for details).

LEMMA 13. *Let E be a Banach space, $S \subset E'$ a norming subspace of the dual space, i.e. $\|x\| = \sup\{|l(x)| \,|\, l \in S$ and $\|l\| \leq 1\}$. Let $I \subset \mathbb{R}$ be an open bounded interval, then a curve $c\colon I \to E$ is Lip^n for a given $n \in \mathbb{N}$ if there* *are curves* $c^i\colon I \to E$ *for* $1 \le i \le n+1$ *with* $(l \circ c)^{(i)} = l \circ c^i$ *for* $l \in S$ *and* $1 \le i \le n+1$ *and* c^{n+1} *is bounded on* I. *In this case* $c^{(i)} = c^i$ *for* $0 \le i \le n$.

Theorem 14. *Let* E *be a Banach space, which admits* C_b^2*-bump functions, then any Lie group modeled on* E *is tempered.*

Proof. We denote by $B(0,r)$ for $r > 0$ the open ball around zero in E. The linear space of C_b^2-functions with values in $\mathbb{C}$ and support in $B(0,r)$ having bounded first and second derivative is denoted by $C_b^2(r)(E)$. If the Banach space admits C_b^2-bump functions, $C_b^2(r)(E)$ detects the converging sequences:

$$x_n \overset{n\to\infty}{\to} x \iff \phi(x_n) - \phi(x) \overset{n\to\infty}{\to} 0 \text{ for all } \phi \in C_b^2(r)(E)$$

for all sequences $\{x_n\}_{n\in\mathbb{N}}$ and $x \in B(0,r)$. Now we take two charts (u_1, U_1), (u_2, U_2) mapping 0 to e around e of the Banach–Lie group, so that the C^∞-function $\mu^0 := u_2 \circ \mu \circ (id \times {u_1}^{-1})\colon U_3 \times B(0,1)) \to B(0,1)$, where $U_3 \subset G$ is an open chart domain of E, has the property that $\check{\mu}^0\colon B(0,1) \to C^\infty(U_3, E)$ is globally Lipschitz and consequently bounded. This is possible by applying Theorem 12.7 in [**3**] and the fact that Lip^0-functions on a Banach space with values in a convenient vector space are locally Lipschitz around any point in the domain of definition.

Let $\phi \in C_b^2(1)(E)$ be a bump function and $c\colon \mathbb{R} \to E$ with $c(0) = 0$ a smooth curve, then $C\colon I \subset \mathbb{R} \to BC(B(0,1))$, given by $C_t(f)(x) = f(\mu^0(c(t),x))$ for $x \in B(0,1)$, $t \in I$ and $f \in BC(B(0,1))$, where I is a sufficiently small open interval around zero (so that $c(t) \in U_3$ for $t \in \overline{I}$), is differentiable at ϕ on I. This will be detected by point evaluations ev_x for $x \in B(0,1)$, which span a norming subspace for the supremum norm on $BC(B(0,1))$:

$$\begin{aligned}
\frac{d}{dt} ev_x(C_t(\phi)) &= d\phi(\mu^0(c(t),x))(d_1{\mu^0}_{(c(t),x)}(c'(t))) \\
\frac{d^2}{dt^2} ev_x(C_t(\phi)) &= d^2\phi(\mu^0(c(t),x))(d_1{\mu^0}_{(c(t),x)}(c'(t)), d_1{\mu^0}_{(c(t),x)}(c'(t))) \\
&\quad + d\phi(\mu^0(c(t),x))(d_1^2{\mu^0}_{(c(t),x)}(c'(t), c'(t))) \\
&\quad + d\phi(\mu^0(c(t),x))(d_1{\mu^0}_{(c(t),x)}(c''(t)))
\end{aligned}$$

for $t \in I$ and $x \in B(0,1)$. The right hand side of the respective derivative is the evaluation of a curve to $BC(B(0,1))$, because the linear parts are bounded in E. Consequently $ev_\phi \circ C\colon I \to BC(B(0,1))$ is Lip^1 on I for all $\phi \in C_b^2(1)(E)$ by the previous lemma. Now we lift $C_b^2(r)$ to the group G with the chart map u_2 given around e, where $0 < r < 1$ is chosen sufficiently small: We chose $0 < r_1 < 1$ so that ${u_2}^{-1}(B(0,r_1)) \subset U_1$. Applying the topological lemma leads to $0 < r < r_1$, so that for every continuous curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ there is a small interval J around zero with $\cup_{t\in J}\, c(t)^{-1}[{u_2}^{-1}(B(0,r))] \subset {u_2}^{-1}(B(0,r_1))$.

To prove differentiability we take $\phi \in C_b^2(r)(E)$, the lifting $\psi = \phi \circ u_2 \in BC(G)$ has support in ${u_2}^{-1}(B(0,r))$. Let $c\colon \mathbb{R} \to G$ be smooth with $c(0) = e$, then there is J, open around zero with the above property, let g denote the lifting of the first derivative along the curve c of ϕ at $t = 0$, then

$$\sup_{x\in G} |\frac{\psi(c(t)x) - \psi(x)}{t} - g(x)| = \sup_{x\in U_1} |\frac{\psi(c(t)x) - \psi(x)}{t} - g(x)|$$
$$= \sup_{x\in B(0,1)} |\frac{\phi(\check{\mu}(c(t),x)) - \phi(x)}{t} - d\phi(\mu(e,x))(d_1\check{\mu}_{(e,x)}(c'(0)))|$$

for $t \in J$. Hence we obtain differentiability of $ev_\psi \circ C\colon \mathbb{R} \to BC(G)$ at $t = 0$ for $\psi = \phi \circ u_2$ with $\phi \in C_b^2(r)$ by Taylor's remainder theorem. We obtain that for every smooth curve the associated left translations are everywhere differentiable at the lifted functions. $C_b^2(r)$ is an algebra, by lifting, moving the elements via left translation and associating the unit we can generate a unital subalgebra of $BC(G)$, which will be denoted by D.

We have to prove the assertions of Definition 9: 1 is clear by definition, 2 is clear by the structure of $C_b^2(r)$ as the translated functions detect every converging sequence, 3 is clear up to the following consideration. Let $c\colon \mathbb{R} \to BC(G)$ be a smooth curve with $c(0) = e$, $y \in G$. Let ϕ be in the lifting of $C_b^2(r)$ to the group, then there is $g \in BC(G)$, such that

$$\sup_{x\in G} |\frac{\phi(yc(t)x) - \phi(yx)}{t} - g(x)| \overset{t\to 0}{\to} 0.$$

Consequently the curve C is differentiable on the left translation by y of ϕ on G, because it is differentiable along the curve $yc(.)$ as remarked before. So all the properties are proved and the Banach–Lie group is tempered. □

Theorem 15. *Let G be a smooth group structure, where $c^\infty G$ is a topological group, $G = \operatorname{proj\,lim}_{\alpha\in\Omega} G_\alpha$, where the G_α are topological groups, the limit is given in the category of topological groups.*

If G_α is a Banach manifold modeled on a Banach space E_α, which admits C_b^2-bump functions for $\alpha \in \Omega$, and if the canonically given multiplication $\mu_\alpha\colon G \times G_\alpha \to G_\alpha$ is smooth, then G is a tempered group.

If furthermore the C^∞-vector fields

$$X_x^\alpha := \frac{d}{dt}|_{t=0}\mu_\alpha(c(t), x)$$

for $x \in G_\alpha$ are globally integrable on G_α for every smooth curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ and $\alpha \in \Omega$, then for every smooth curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ there is a continuous group T and

$$\lim_{n\to\infty} c(\frac{t}{n})^n = T_t$$

uniformly on compact intervals in $\mathbb{R}$.

Proof. We proceed by the same method as in the preceding examples, but we have to look carefully at the multiplication μ_α for a given $\alpha \in \Omega$. There are charts u_1, u_2 for the group G_α around e and a open chart domain $U_3 \subset G$ around e, so that $\mu_\alpha^0 = u_2 \circ \mu_\alpha \circ (id \times {u_1}^{-1})\colon U_3 \times B_\alpha(0,1) \to B_\alpha(0,1)$. By shrinking the chart (u_1, U_1) we may assume that $\check{u}_\alpha\colon B_\alpha(0,1) \to C^\infty(U_3, E_\alpha)$ is globally Lipschitz. Now we can apply the same method as before: We take the algebra $C_b^2(r)(E_\alpha)$, where $0 < r < 1$ is chosen sufficiently small due to the above consideration. We calculate the derivatives under evaluations and prove differentiability in $BC(B_\alpha(0,1))$ of curves of left translations by a smooth curve at functions from $C_b^2(1)(E_\alpha)$. We lift the algebra $C_b^2(r)$ on the group G_α and – via the canonically given smooth projections – from G_α to G. Finally we arrive at the differentiability properties. Redoing the program for all $\alpha \in \Omega$, moving around the functions concentrated at the identity and using the properties of the projections and the limit leads to a subalgebra $D \subset BC(G)$, which proves temperedness of G.

Assume now that the described C^∞-vector fields are globally integrable on G_α. Redoing the first part of the proof we can find a unital subalgebra $D_\alpha \subset BC(G_\alpha)$, which satisfies 1 and 2 of Definition 9, the third property is satisfied only for smooth curves $c\colon \mathbb{R} \to G$ with $c(0) = e$, which are projected to G_α (The resulting curve is denoted by c_α). It is worth mentioning that we are not given the structure of a tempered group, because the multiplication on G_α is not smooth.

Fix now a smooth curve $c\colon \mathbb{R} \to G$ with $c(0) = e$ to construct X^α. We denote the global flow associated to X^α by T^α. By inserting in functions of D_α we have the problem that the T^α are not translations, but we can argue directly as $\frac{d}{dt}T_t^\alpha(x) = \frac{d}{ds}|_{s=0}\mu_\alpha(c(s), T_t^\alpha)$. The exact formulation of the above construction leads to curves $C\colon \mathbb{R} \to BC(G_\alpha)$ which are two times differentiable under the evaluations ev_x for $x \in G_\alpha$ on a small interval around zero on functions $\phi \in D_\alpha$, where the derivatives lie in $BC(G_\alpha)$, the second one is bounded on the interval. Given $\phi \in D_\alpha$ and $x \in G_\alpha$ we obtain

$$\frac{d}{dt}\phi(T_t^\alpha(x)) = (C'(0)\phi)(T_t^\alpha(x))$$
$$\frac{d^2}{dt^2}\phi(T_t^\alpha(x)) = (C'(0)^2\phi)(T_t^\alpha(x))$$

for t in the interval. The right hand sides are bounded on the interval, so we conclude that the given curve is differentiable in $BC(G_\alpha)$. We shall look at the closure of D_α in $BC(G_\alpha)$, an algebra of differentiable functions concentrated at a small neighborhood, so we are able to detect convergence in a small neighborhood of the identity. By a slight generalization of Chernoff's theorem (we leave away the condition $\overline{A|_D} = A$, but obtain only convergence on the closure of D) we arrive at uniform convergence on compact subsets of a small neighborhood of the identity of the following limit:

$$\lim_{n\to\infty} c_\alpha(\frac{t}{n})^n = T_t^\alpha(e).$$

This means that the limit exists uniformly on compact subsets of $\mathbb{R}$ due to continuity of the multiplication. Therefore we have proved that the right hand side of the limit is a continuous group in G_α. The above procedure can be done for any $\alpha \in \Omega$, consequently we have proved the assertion by the properties of the limit, more precisely: There exists a continuous group T^α on G_α with the property $\lim_{n\to\infty} c_\alpha(\frac{t}{n})^n = T_t^\alpha$, satisfying the limit conditions. So we can lift it to G and there we obtain the desired equation. □

Corollary 16. *All strong ILB-Lie groups, where the respective Banach spaces in the chain admit C_b^2-bump functions, are tempered.*

Proof. See [**7**] for the properties of strong ILB-groups. In particular we get into the assumptions of Theorem 15, since a strong ILB-Lie group is the projective limit $\operatorname{proj\,lim}_n G_n = G$ of topological groups G_n, which are Banach manifolds, and $\mu_n \colon G \times G_n \to G_n$ are smooth for $n \geq 0$. □

On tempered Lie groups we can easily characterize the existence of an exponential mapping.

Theorem 17. *Let G be a tempered Lie group (in particular smoothly Hausdorff). Let $D \subset BC(G)$ be the given unital subalgebra. Let G satisfy the following completeness condition: If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with $f \circ \mu_{x_n}$ a Cauchy sequence in $BC(G)$ for $f \in D$, then there is $x \in G$ with $x_n \to x$ as $n \to \infty$.*

Then the Lie group admits a smooth group in each direction if and only if for all smooth curves $c\colon \mathbb{R} \to G$ with $c(0) = e$ the mapping $id - C'(0)\colon D \to X$ has dense image in the closure of D, denoted by X, where $C\colon \mathbb{R} \to L(BC(G))$ denotes the curve of left translations by $c(t)$.

Proof. We have to apply several results of classical theory of C_0-semigroups. Suppose first that G admits an exponential mapping, then the respective generators of the C_0-groups on G, given through $C'(0)$ for $c\colon \mathbb{R} \to G$, obey the condition $\mathbb{R} \setminus \{0\} \subset \rho(C'(0))$, consequently $id - C'(0)\colon D \to X$ is closable and the closure is invertible on X, so the image of D is dense.

Suppose the density condition is satisfied, then we can apply Chernoff's theorem. By approximation we obtain that the following limit exists uniformly on compact intervals of $\mathbb{R}$: $\lim_{n\to\infty} C(\frac{t}{n})^n = T_t$. By the completeness assumption we obtain the existence of a continuous group S in G with $\lim_{n\to\infty} c(\frac{t}{n})^n = S_t$ uniformly on compact subsets of $\mathbb{R}$. By [**8**] we obtain smoothness. □

REMARK 18. *The existence of such differentiable functions with respect to the uniform topology gave the idea to look for some differentiable right invariant semi-metrics on the given Lie group, which all together generate the sequential topology. This idea was worked out in* [**9**].

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Technical University of Vienna
E105, Wiedener Hauptstrasse 8–10
A-1040 Vienna, Austria
e-mail: `josef.teichmann@fam.tuwien.ac.at`