UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLII
2004

ON FIBRATIONS WITH THE GRASSMANN MANIFOLD OF TWO-PLANES AS FIBER

by Július Korbaš

Abstract. Let $p : E \to B$ be a Serre fibration with E compact, B a connected finite CW-complex, and fiber either the real Grassmann manifold $O(n)/O(2) \times O(n-2)$ or the complex Grassmann manifold $U(n)/U(2) \times U(n-2)$, where $n \geq 4$. We prove that if n is odd, then the fiber is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.

1. Introduction and statement of a theorem. Let $FG_{n,k}$ be the Grassmann manifold of all k-dimensional vector subspaces in F^n, where F is either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers. In the sequel, we shall suppose that $2k \leq n$ (the manifolds $FG_{n,k}$ and $FG_{n,n-k}$ can naturally be identified with each other). Let ξ_k and γ_k be the canonical k-plane bundles over $\mathbb{R}G_{n,k}$ and $\mathbb{C}G_{n,k}$, respectively. The i-th Stiefel–Whitney class of a real vector bundle α will be denoted by $w_i(\alpha)$, and the i-th Chern class of a complex vector bundle β by $c_i(\beta)$.

It is known (cf. Hiller [**3**]) that the mod 2 cohomology algebra of $\mathbb{R}G_{n,k}$ is

$$H^*(\mathbb{R}G_{n,k}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\xi_k), \dots, w_k(\xi_k)]/J(k, n-k),$$

where the ideal $J(k, n-k)$ is generated by the homogeneous elements

$$f_{1,n-k}, \dots, f_{k,n-k}$$

2000 *Mathematics Subject Classification.* Primary: 55R20; Secondary: 57R19, 57R20.

Key words and phrases. Grassmann manifold, Serre fibration, totally non-homologous to zero, derivation, cohomology spectral sequence, Stiefel–Whitney class, Chern class.

The author was supported in part by two grants of VEGA (Slovakia). The paper is in final form and will not be published elsewhere.

given by

$$\begin{pmatrix} f_{1,n-k} \\ f_{2,n-k} \\ \vdots \\ f_{k,n-k} \end{pmatrix} = \begin{pmatrix} w_1(\xi_k) & 1 & 0 & \dots & 0 \\ w_2(\xi_k) & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & 1 \\ w_k(\xi_k) & 0 & 0 & \dots & 0 \end{pmatrix}^{n-k+1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Borel [**1**], there is an isomorphism of the cohomology algebras,

$$\varphi : H^*(\mathbb{R}G_{n,k}; \mathbb{Z}_2) \to H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2),$$

$$\varphi(w_i(\xi_k)) = w_{2i}(\gamma_k),$$

where $w_{2i}(\gamma_k)$ is the $2i$-th Stiefel–Whitney class of the realification of the complex vector bundle γ_k.

Now we consider the special case of $k = 2$. Our aim is to prove the following generalization of Theorem B(2) of Korbaš [**5**].

THEOREM 1.1. *Let $p : E \to B$ be a Serre fibration with E compact, B a connected finite CW–complex, and fiber either the real Grassmann manifold $\mathbb{R}G_{n,2}$ ($n \geq 4$) or the complex Grassmann manifold $\mathbb{C}G_{n,2}$ ($n \geq 4$). If n is odd, then the fiber is totally non-homologous to zero in E with respect to $\mathbb{Z}_2$.*

In [**5**], where we proved a particular case of this result for n of the form $1 + 2^s$, one can find other interpretations of 1.1, comments on its applications, and some related results and considerations.

2. Proof of Theorem 1.1. We shall abbreviate the Stiefel–Whitney class $w_j(\xi_k) \in H^j(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$ to w_j. In the proof of Theorem 1.1, we shall need the following auxiliary result.

LEMMA 2.1. *Let $n \geq 4$. Then*

$$H^*(\mathbb{R}G_{n,2}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\xi_2), w_2(\xi_2)]/J(2, n-2),$$

where the ideal $J(2, n-2)$ is generated by the two homogeneous elements

$$f_{1,n-2} = \sum_{i=0}^{\infty} \binom{n-i-1}{i} w_1^{n-2i-1}(\xi_2) w_2^i(\xi_2)$$

(in dimension $n-1$), and

$$f_{2,n-2} = \sum_{i=0}^{\infty} \binom{n-i-1}{i-1} w_1^{n-2i}(\xi_2) w_2^i(\xi_2)$$

(in dimension n). Here $\binom{u}{v}$ *is the binomial coefficient reduced mod* 2 *if* $u \geq v$*;* $\binom{u}{v} = 1$ *if* $v = 0$*, and* $\binom{u}{v} = 0$ *if* $u < v$*.*

Lemma 2.1 can readily be proved using (for instance) the Hiller description; we shall omit the details.

Proof of Theorem 1.1. We proved in [**5**, Proposition 3] that the fibrations considered in the theorem under proof are $\mathbb{Z}_2$-orientable. Hence to prove the theorem it is enough to verify that the graded $\mathbb{Z}_2$-vector space $\mathrm{Der}_{<0}(H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2))$ resp. $\mathrm{Der}_{<0}(H^*(\mathbb{C}G_{n,2};\mathbb{Z}_2))$ of all derivations (in the graded $\mathbb{Z}_2$-algebras $H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ resp. $H^*(\mathbb{C}G_{n,2};\mathbb{Z}_2)$) of negative degrees is trivial if n is odd. Indeed, this enables us to conclude (change the coefficient field $\mathbb{Q}$ to $\mathbb{Z}_2$ in Meier [**6**, Lemma 2.5]) that the corresponding Leray–Serre spectral sequence collapses, and the fiber is therefore totally non-homologous to zero with respect to $\mathbb{Z}_2$.

We shall show that $\mathrm{Der}_{<0}(H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2)) = 0$ if n is odd; the complex case can be analysed analogously, when one uses the above mentioned isomorphism

$$\varphi : H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2) \to H^*(\mathbb{C}G_{n,2};\mathbb{Z}_2).$$

In the rest of the proof, the number n will be odd.

Since the algebra $H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2)$ is generated by the Stiefel–Whitney classes w_1 and w_2, it is clear that an element in $\mathrm{Der}_{<0}(H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2))$ will be trivial if it vanishes at w_1 and w_2.

If an element θ of $\mathrm{Der}_{<0}(H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2))$ has a nontrivial value at w_1, then θ must be of degree -1, so $\theta(w_1) = 1$ in $H^0(\mathbb{R}G_{n,2};\mathbb{Z}_2) \cong \mathbb{Z}_2$. It is known (Stong [**7**]) that if s is the unique integer such that $2^s < n \leq 2^{s+1}$, then $w_1^{2^{s+1}-2} \neq 0$, but $w_1^{2^{s+1}-1} = 0$. We see that $\theta(w_1) = 1$ implies

$$0 = \theta(w_1^{2^{s+1}-1}) = \theta(w_1)w_1^{2^{s+1}-2} + w_1\theta(w_1^{2^{s+1}-2}) = 1\cdot w_1^{2^{s+1}-2} + w_1\cdot 0 = w_1^{2^{s+1}-2},$$

which is a contradiction. Hence, for any $\theta \in \mathrm{Der}_{<0}(H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2))$, $\theta(w_1) = 0$.

Now, an element of $\mathrm{Der}_{<0}(H^*(\mathbb{R}G_{n,2};\mathbb{Z}_2))$ having a nonzero value at w_2 must be of degree -1 or -2. Suppose that σ is a derivation of degree -1 such that $\sigma(w_2) \neq 0$, and that τ is a derivation of degree -2 such that $\tau(w_2) \neq 0$. Then $\sigma(w_2) = w_1$, because

$$H^1(\mathbb{R}G_{n,2};\mathbb{Z}_2) \cong \mathbb{Z}_2 \cong \{0, w_1\},$$

and we conclude

$$\tau(w_2) = 1 \in H^0(\mathbb{R}G_{n,2};\mathbb{Z}_2).$$

Further, we know from Lemma 2.1 that $\sum_{i=0}^{\infty}\binom{n-i-1}{i}w_1^{n-2i-1}w_2^i = 0$. Using this, together with the fact that $\sigma(w_1) = 0$, we compute

$$\begin{aligned}
0 &= \sigma(\sum_{i=0}^{\infty}\binom{n-i-1}{i}w_1^{n-2i-1}w_2^i) \\
&= \sigma(w_1^{n-1} + w_1^{n-3}w_2 + \sum_{i=2}^{\infty}\binom{n-i-1}{i}w_1^{n-2i-1}w_2^i) \\
&= \sigma(w_1^{n-1}) + \sigma(w_1^{n-3}w_2) + \sum_{i=2}^{\infty}\binom{n-i-1}{i}\sigma(w_1^{n-2i-1}w_2^i) \\
&= 0 + w_1^{n-2} + \sum_{i\geq 3,\ i \text{ odd}}^{\infty}\binom{n-i-1}{i}\sigma(w_1^{n-2i-1}w_2^i),
\end{aligned}$$

because for even values of $i \geq 2$

$$\sigma(w_1^{n-2i-1}w_2^i) = \sigma((w_1^{\frac{n-2i-1}{2}}w_2^{\frac{i}{2}})^2) = 0.$$

In other words,

$$\begin{aligned}
0 &= w_1^{n-2} + \sum_{j=1}^{\infty}\binom{n-2j-2}{2j+1}\sigma(w_1^{n-4j-3}w_2^{2j+1}) \\
&= w_1^{n-2} + \sum_{j=1}^{\infty}\binom{n-2j-2}{2j+1}w_1^{n-4j-3}\sigma(w_2 \cdot w_2^{2j}) \\
&= w_1^{n-2} + \sum_{j=1}^{\infty}\binom{n-2j-2}{2j+1}w_1^{n-4j-2}w_2^{2j}.
\end{aligned}$$

But this is a contradiction, because (as is well known; one also can see it from the Hiller description) w_1 and w_2 satisfy no algebraic relations in dimensions less than or equal to $n-2$. In this way we have shown that $\sigma(w_2) = 0$.

Now in a similar way we show that $\tau(w_2) = 0$. Indeed, using Lemma 2.1 we obtain

$$\begin{aligned}
0 &= \tau(\sum_{i=0}^{\infty}\binom{n-i-1}{i}w_1^{n-2i-1}w_2^i) \\
&= \tau(w_1^{n-1}) + \tau(w_1^{n-3}w_2) + \sum_{i=2}^{\infty}\binom{n-i-1}{i}\tau(w_1^{n-2i-1}w_2^i) \\
&= 0 + w_1^{n-3} + \sum_{j=1}^{\infty}\binom{n-2j-2}{2j+1}\tau(w_1^{n-4j-3}w_2^{2j+1})
\end{aligned}$$

$$= w_1^{n-3} + \sum_{j=1}^{\infty} \binom{n-2j-2}{2j+1} w_1^{n-4j-3} \tau(w_2 \cdot w_2^{2j})$$
$$= w_1^{n-3} + \sum_{j=1}^{\infty} \binom{n-2j-2}{2j+1} w_1^{n-4j-3} w_2^{2j}.$$

This again is an impossible algebraic relation, and therefore $\tau(w_2) = 0$. This finishes the proof of Theorem 1.1. □

We have not found a way to prove it, but the following might be true.

CONJECTURE 2.2. Theorem 1.1 remains valid when the fiber is any $\mathbb{R}G_{n,k}$ $(k \leq n-k)$ with n odd or any $\mathbb{C}G_{n,k}$ $(k \leq n-k)$ with n odd.

Note that for smooth fiber bundles the conjecture was proved in Horanská, Korbaš [**4**] and Korbaš [**5**]. In attempts to prove the conjecture in general, one perhaps can use a combination of the "smooth" results with something similar to the Fiber Smoothing Theorems of Casson and Gottlieb [**2**].

References

1. Borel A., *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts,* Ann. of Math., **57** (1953), 115–207.
2. Casson A., Gottlieb D.H., *Fibrations with compact fibres,* Amer. J. Math., **99** (1977), 159–189.
3. Hiller H., *On the cohomology of real Grassmannians,* Trans. Amer. Math. Soc., **257** (1980), 521–533.
4. Horanská Ľ., Korbaš J., *On cup products in some manifolds,* Bull. Belgian Math. Soc., **7** (2000), 21–28.
5. Korbaš J., *On fibrations with Grassmannian fibers,* Bull. Belgian Math. Soc., **8** (2001), 119–130.
6. Meier W., *Rational universal fibrations and flag manifolds,* Math. Ann., **258** (1982), 329–340.
7. Stong R.E., *Cup products in Grassmannians,* Topology Appl., **13** (1982), 103–113.

Received December 3, 2002

Comenius University
Department of Algebra
Faculty of Mathematics, Physics,
and Informatics
Mlynská dolina
SK-842 48 Bratislava 4, Slovakia
e-mail: `korbas@fmph.uniba.sk`

Slovak Academy of Sciences
Mathematical Institute
Štefánikova 49
SK-814 73 Bratislava 1, Slovakia
e-mail: `matekorb@savba.sk`