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THE INITIAL BOUNDARY VALUE PROBLEM FOR THE HIGH ORDER PARABOLIC EQUATION IN UNBOUNDED DOMAIN

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Abstract. In this paper we consider the initial boundary value problem for the following equation $u_{tt} + A_1 u + A_2 u_t + g(u_t) = f(x,t)$ in unbounded domain, where A_1 is a linear elliptic operator of the fourth order and A_2 is an linear elliptic operator of the second order. We establish the theorem on the uniqueness of the weak solution for the problem (1)–(3).

In paper [**7**], the uniqueness of a weak solution in the class of functions which do not grow faster than the function $e^{a|x|}\ for\ |x| \to \infty$ has been shown by the method of introducing a parameter. In many papers authors have considered this problems in unbounded domains for the parabolic equations of the higher order with the first derivative with respect to time. There is much less papers which concern the problems for the parabolic equation with the second derivative with respect to time. In particular, in papers [**1**]–[**3**] and [**5**]–[**6**], authors have considered the Cauchy problem for the parabolic equation of the high order and the properties of its solutions. In the paper [**7**], authors have obtained some conditions for the uniqueness of the solution of the initial boundary value problem for the general linear parabolic systems in unbounded domains.

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain and $\partial\Omega \in C^1$, $\Omega \cap B_R = \Omega^R$ be a domain for all $R > 0$,
where $B_R = \{x \in R^n, |x| < R\}$ and $Q_T = \Omega \times (0,T)$, $Q_T^R = \Omega^R \times (0,T)$, $\Omega_\tau = Q_\tau \cap \{t = \tau\}$, $Q_{\tau_0,\tau_1} = \Omega \times (\tau_0, \tau_1)$.

We shall consider the equation of the form

$$
\begin{aligned}
(1)\qquad & u_{tt}(x,t) + \sum_{i,j,k,l=1}^{n} (a_{ij}^{kl}(x) u_{x_i x_j}(x,t))_{x_k x_l} - \\
& - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i}(x,t))_{x_j} - \sum_{i,j=1}^{n} (b_{ij}(x,t) u_{t x_i}(x,t))_{x_j} + \\
& + a(x)u(x) + g(x,t,u_t) = f(x,t)
\end{aligned}
$$

in the domain Q_T.
For this equation, we put the following boundary and initial conditions

$$(2)\qquad u|_{S_T} = 0, \ \ \frac{\partial u}{\partial \nu}|_{S_T} = 0,$$

$$(3)\qquad u|_{t=0} = u_0(x), \ \ u_t|_{t=0} = u_1(x),$$

where $S_T = \partial\Omega \times (0,T)$ and ν is a normal vector for S_T. Let us start with some notation.

$$H_{loc}^{0,k}(\overline{\Omega}) = \{u \in H^k(\Omega^R) : u|_{\partial\Omega \cap B_R} = 0, \frac{\partial^{k-1} u}{\partial \nu^{k-1}}|_{\partial\Omega\cap B_R} = 0, \forall R > 0\},\ k = 1, 2,$$

$$L_{loc}^2(\overline{\Omega}) = \{u \in L^2(\Omega^R), \forall R > 0\}.$$

For the equation (1), we adapt the following system of assumptions:

(A₁) $a_{ij}^{kl} \in L^\infty(\Omega)$; $\quad a_{ij}^{kl}(x) = a_{kl}^{ij}(x)$, for almost all $x \in \Omega$;
$\sum\limits_{i,j,k,l=1}^{n} a_{ij}^{kl}(x)\xi_{ij}\xi_{kl} \geq a_2 \sum\limits_{i,j=1}^{n} \xi_{ij}^2$ for almost all $x \in \Omega$
and for all $\xi \in R^{n(n-1)/2}$, where $a_2 > 0$ is a constant;

(A₂) $a_{ij}, a_{ijx_i} \in L^\infty(\Omega), i,j = 1, ..., n$; $\quad a_{ij}(x) = a_{ji}(x)$, for almost all $x \in \Omega$;
$\sum\limits_{ij=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0$, for almost all $x \in \Omega$ and for all $\xi \in R^n$;

(A₃) $a \in L^\infty(\Omega)$, $\quad a(x) \geq a_0 > 0$ for almost all $x \in \Omega$, where a_0 is a constant;

(B) $b_{ij} \in L^\infty(Q_\tau)$, $\quad \sum\limits_{ij=1}^{n} b_{ij}(x,t)\xi_i\xi_j \geq b_0 \sum\limits_{i=1}^{n} \xi_i^2$ for almost all $(x,t) \in Q_\tau$ and for all $\xi \in R^n$, where $b_0 > 0$ is a constant;

(G) The function $(x,t) \to g(x,t,\xi)$ is continuous for every $\xi \in R$
and the function $\xi \to g(x,t,\xi)$ is measurable for almost all $(x,t) \in Q_\tau$ and satisfies the following inequalities:
$(g(x,t,\xi) - g(x,t,\mu))(\xi - \mu) \geq g_0|\xi - \mu|^p$ for almost all $(x,t) \in Q_\tau$ and for all $\xi, \mu \in R$, $g_0 = const \geq 0$;
$|g(x,t,\xi)| \leq g_1|\xi|^{p-1}$, $\quad p \in (1, +\infty)$ for almost all $(x,t) \in Q_\tau$ and for all $\xi \in R$.

Under this assumptions, we will obtain the uniqueness of a weak solution of problem (1)–(3).

Definition. We call a function u a weak solution of problem (1)–(3) if

$$u \in L^2\Big((0,T); H^{0,2}_{loc}(\overline{\Omega})\Big), \quad u_{tt} \in L^2\Big((0,T); (H^{0,2}_{loc}(\overline{\Omega}))^*\Big),$$

$$u_t \in L^2\Big((0,T); H^{0,1}_{loc}(\overline{\Omega})\Big) \cap L^p\Big((0,T); L^p_{loc}(\overline{\Omega})\Big),$$

and u satisfies the following integral equation

$$\int\limits_{\Omega} u_t(x,T)v(x,T)dx + \int\limits_{Q_T} \Big[-u_t v_t + \sum_{i,j,k,l=1}^{n} a^{kl}_{ij}(x) u_{x_i x_j} v_{x_k x_l} +$$

$$+ \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + a(x)uv + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{x_i t} v_{x_j} + g(x,t,u_t)v\Big] dxdt$$

$$= \int\limits_{Q_T} f(x,t)v dxdt + \int\limits_{\Omega} u_1(x)v(x,0)dx$$

$$\forall v \in L^2\Big((0,T); H^{0,2}_{loc}(\overline{\Omega})\Big) \cap L^p\Big((0,T); L^p_{loc}(\overline{\Omega})\Big), \quad v_t \in L^2\Big((0,T); L^2_{loc}(\overline{\Omega})\Big),$$

where supp v is bounded.

Let

$$\phi_R(x) = \begin{cases} \frac{1}{R}(R^2 - |x|^2), & \text{for} \quad 0 \le |x| \le R, \\ 0, & \text{for} \quad |x| > R. \end{cases}$$

Now, we define the function Ψ by the formula

$$\Psi(x) = [\phi_R(x)]^\beta,\ \beta > 1. \tag{4}$$

From (4) there follows

$$\Psi_{x_i} = \beta(\phi_R)^{\beta-1}\phi_{Rx_i}, \phi_{Rx_i} = -\frac{2x_i}{R}, |\phi_{Rx_i}| \le 2$$

for $i = 1, 2, .., n$. Hence

$$\frac{\Psi^2_{x_i}}{\Psi} \le 4\beta^2 \frac{\phi_R^{2\beta-2}}{\phi_R^\beta} = C\phi_R^{\beta-2}, \tag{5}$$

where $C = 4\beta^2$, for $i = 1, 2, ..., n$. From (4) and (5) we obtain

$$\frac{\Psi^2_{x_i}}{\Psi} \le C(2R)^{\beta-2} \tag{6}$$

and

$$|\Psi_{x_i x_j}| \le \mu_0 (\phi_R)^{\beta-2}. \tag{7}$$

THEOREM. *If conditions* ($\mathbf{A_1}$)–($\mathbf{A_3}$), (**B**), (**G**) *hold, then problem* (1)–(3) *has at most one weak solution in the class of the functions u such that*

$$\int\limits_{Q^R} \left[|u|^2 + \sum_{i=1}^{n} |u_t|^2 + \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 \right] dxdt \le e^{aR^2} \ \ \forall R > 0,$$

where $a > 0$ is a constant.

PROOF. To obtain a contradiction, suppose that there exist two solutions u^1, u^2 of problem (1)–(3) such that $u^1 \ne u^2$.

Let $\tau_0, \tau_1 \in (0, \tau)$, $\tau_0 < \tau_1$ be fixed and Θ_m be a continuous function on $[0, T]$ such that

$$\Theta_m(t) = \begin{cases} \text{is a linear function} & \text{in } (\tau_0 + \frac{1}{m}, \tau_0 + \frac{2}{m}) \text{ and } (\tau_1 - \frac{2}{m}, \tau_1 - \frac{1}{m}), \\ 1, & \text{if } \quad \tau_0 + \frac{2}{m} \le t \le \tau_1 - \frac{2}{m}, \\ 0, & \text{if } \quad \tau_1 - \frac{1}{m} \le t \text{ or } t \le \tau_0 + \frac{1}{m}. \end{cases}$$

By $\{\rho_l\}$ we denote the sequence which satisfies the following conditions

$$\rho_l(t) = \rho_l(-t), \int\limits_{-\infty}^{\infty} \rho_l(t) dt = 1, \text{supp } \rho_l \subset \left[-\frac{1}{l}, \frac{1}{l} \right],$$

$l \in N$, $l > 2m$, [**4**]. Moreover, we put

$$v = \left((\Theta_m u_t \Psi e^{-\frac{\gamma t}{2}}) * \rho_l * \rho_l \right) \Theta_m e^{-\frac{\gamma t}{2}}, \ \gamma > 0,$$

where $*$ denotes the convolution with respect to t. Now, if we apply $*$ to functions u^1, u^2, then for $u = u^1 - u^2$ we obtain

$$\begin{aligned} \int\limits_{Q_T} \Bigg[-u_t v_t &+ \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x) u_{x_i x_j} v_{x_k x_l} + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + a(x) uv \\ &+ \sum_{i,j=1}^{n} b_{ij}(x,t) u_{t x_i} v_{x_j} + (g(x,t,u_t^2) - g(x,t,u_t^1)) v \Bigg] dxdt = 0. \end{aligned} \tag{8}$$

Let $< \cdot, \cdot >$ denote a scalar product in $L^2(\Omega^R)$. Taking into account the assumption of the theorem and form of function v we obtain the following equality

$$
\begin{aligned}
I_1(m,l) :=\ & -\int\limits_{Q_T} u_t v_t dxdt = \\
& = \frac{\gamma}{2}\int\limits_{Q_T} u_t\Theta_m \cdot \Big((\Theta_m u_t \Psi e^{-\frac{\gamma t}{2}}) * \rho_l * \rho_l\Big) e^{-\frac{\gamma t}{2}} dxdt \\
& - \int\limits_{Q_T} u_t\Theta_m' \Big((\Theta_m u_t \Psi e^{-\frac{\gamma t}{2}}) * \rho_l * \rho_l\Big) e^{-\frac{\gamma t}{2}} dxdt \\
& - \int\limits_{Q_T} u_t\Theta_m \Big((\Theta_m u_t \Psi e^{-\frac{\gamma t}{2}}) * \rho_l * \rho_l\Big)_t e^{-\frac{\gamma t}{2}} dxdt \\
& = \frac{\gamma}{2}\int\limits_0^T < (u_t\Theta_m e^{-\frac{\gamma t}{2}}) * \rho_l, (u_t\Theta_m e^{-\frac{\gamma t}{2}}) * \rho_l \Psi > dt \\
& - \int\limits_0^T < (u_t\Theta_m' e^{-\frac{\gamma t}{2}}) * \rho_l, (u_t\Theta_m e^{-\frac{\gamma t}{2}}) * \rho_l \Psi > dt \\
& - \int\limits_0^T < (u_t\Theta_m e^{-\frac{\gamma t}{2}})_t * \rho_l, (u_t\Theta_m e^{-\frac{\gamma t}{2}}) * \rho_l \Psi > dt \to \\
& \to \frac{\gamma}{2}\int\limits_{Q_T} u_t^2\Theta_m^2 e^{-\gamma t}\Psi dxdt - \int\limits_{Q_T} u_t^2\Theta_m\Theta_m' e^{-\gamma t}\Psi dxdt - 0 \to \\
& \to \int\limits_{Q_T} (\frac{\gamma}{2}\Theta_m^2 - \Theta_m\Theta_m') u_t^2 e^{-\gamma t}\Psi dxdt,
\end{aligned}
$$

when $l \to +\infty$. Next

$$
I_2(m,l) := \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_i x_j} v_{x_k x_l} dxdt = I_2^1 + I_2^2 + I_2^3.
$$

Here

$$I_2^1(m,l) := \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} \cdot \Big((\Theta_m u_{tx_kx_l} \Psi e^{-\frac{\gamma t}{2}}) * \rho_l * \rho_l \Big) \Theta_m e^{-\frac{\gamma t}{2}} dxdt$$

$$= \int_0^T \sum_{i,j,k,l=1}^{n} < a_{ij}^{kl} \Big(\Theta_m u_{x_ix_j} e^{-\frac{\gamma t}{2}} \Big) * \rho_l, \Big(\Theta_m u_{x_kx_l} e^{-\frac{\gamma t}{2}} \Big)_t * \rho_l \Psi > dt$$

$$+ \frac{\gamma}{2} \int_0^T \sum_{i,j,k,l=1}^{n} < a_{ij}^{kl} \Big(\Theta_m u_{x_ix_j} e^{-\frac{\gamma t}{2}} \Big) * \rho_l, \Big(\Theta_m u_{x_kx_l} e^{-\frac{\gamma t}{2}} \Big) * \rho_l \Psi > dt$$

$$- \int_0^T \sum_{i,j,k,l=1}^{n} < a_{ij}^{kl} \Big(\Theta_m u_{x_ix_j} e^{-\frac{\gamma t}{2}} \Big) * \rho_l, l \Big(\Theta'_m u_{x_kx_l} e^{-\frac{\gamma t}{2}} \Big) * \rho_l \Psi > dt$$

$$= \int_0^T \sum_{i,j,k,l=1}^{n} < a_{ij}^{kl} \Big(\Theta_m u_{x_ix_j} e^{-\frac{\gamma t}{2}} \Big) * \rho_l,$$

$$\Big(\frac{\gamma}{2} (\Theta_m u_{x_kx_l} e^{-\frac{\gamma t}{2}}) - (\Theta'_m u_{x_kx_l} e^{-\frac{\gamma t}{2}}) \Big) * \rho_l \Psi > dt$$

$$\to \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} \Theta_m (\frac{\gamma}{2} \Theta_m - \Theta'_m) u_{x_kx_l} e^{-\gamma t} \Psi dxdt,$$

when $l \to +\infty$,

$$I_2^2(m,l) := \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} \Theta_m \cdot \Big(\Theta_m e^{-\frac{\gamma t}{2}} (u_{tx_k} \Psi_{x_l} + u_{tx_l} \Psi_{x_k}) \Big) * \rho_l * \rho_l e^{-\frac{\gamma t}{2}} dxdt$$

$$\to \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} \Theta_m^2 (u_{tx_k} \Psi_{x_l} - u_{tx_l} \Psi_{x_k}) e^{-\gamma t} dxdt,$$

when $l \to +\infty$,

$$I_2^3(m,l) := \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} \Theta_m \cdot \Big(\Theta_m e^{-\frac{\gamma t}{2}} u_t \Psi_{x_lx_l} \Big) * \rho_l * \rho_l e^{-\frac{\gamma t}{2}} dxdt$$

$$\to \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_t \Theta_m^2 \Psi_{x_kx_l} e^{-\gamma t} dxdt,$$

when $l \to +\infty$. For the next integral we have

$$
\begin{aligned}
I_3(m,l) &:= \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} dxdt \\
&= \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij} u_{x_i} \Theta_m \cdot \left(\Theta_m e^{-\frac{\gamma t}{2}} u_{tx_j} \Psi \right) * \rho_l * \rho_l e^{-\frac{\gamma t}{2}} dxdt \\
&\quad + \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij} u_{x_i} \Theta_m \cdot \left(\Theta_m e^{-\frac{\gamma t}{2}} u_t \Psi_{x_j} \right) * \rho_l * \rho_l e^{-\frac{\gamma t}{2}} dxdt \\
&\to \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_{tx_j} \Theta_m^2 \Psi e^{-\gamma t} dxdt + \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_t \Theta_m^2 \Psi_{x_j} e^{-\gamma t} dxdt,
\end{aligned}
$$

when $l \to +\infty$. For the next integral we obtain

$$
\begin{aligned}
I_4(m,l) &:= \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} v_{x_j} dxdt \\
&= \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} \Theta_m \cdot \left(\Theta_m e^{-\frac{\gamma t}{2}} u_{tx_j} \Psi \right) * \rho_l * \rho_l e^{-\frac{\gamma t}{2}} dxdt \\
&\quad + \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} \Theta_m \cdot \left(\Theta_m e^{-\frac{\gamma t}{2}} u_t \Psi_{x_j} \right) * \rho_l * \rho_l e^{-\frac{\gamma t}{2}} dxdt \\
&\to \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} u_{tx_j} \Theta_m^2 \Psi e^{-\gamma t} dxdt + \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij} u_{x_i} u_t \Theta_m^2 \Psi_{x_j} e^{-\gamma t} dxdt,
\end{aligned}
$$

when $l \to +\infty$. And for next one there is

$$
\begin{aligned}
I_5(m,l) &:= \int\limits_{Q_T} (g(x,t,u_t^1) - g(x,t,u_t^2)) v dxdt \\
&\to \int\limits_{Q_T} (g(x,t,u_t^1) - g(x,t,u_t^2)) u_t \Theta_m^2 \Psi e^{-\gamma t} dxdt
\end{aligned}
$$

and

$$
I_6(m,l) := \int\limits_{Q_T} auvdxdt \to \int\limits_{Q_T} auu_t \Theta_m^2 \Psi e^{-\gamma t} dxdt.
$$

If we pass with m to infinity, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega_{\tau_1}} u_t^2 e^{-\gamma\tau_1}\Psi dx - \frac{1}{2}\int\limits_{\Omega_{\tau_0}} u_t^2 e^{-\gamma\tau_0}\Psi dx \\
&+\frac{1}{2}\int\limits_{\Omega_{\tau_1}} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_{x_kx_l} e^{-\gamma\tau_1}\Psi dx \\
&-\frac{1}{2}\int\limits_{\Omega_{\tau_0}} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_{x_kx_l} e^{-\gamma\tau_0}\Psi dx \\
&+\int\limits_{Q_{\tau_0,\tau_1}} \Big[\frac{\gamma}{2} u_t^2 \Psi(x) + \frac{\gamma}{2}\sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_{x_kx_l}\Psi \\
&\qquad + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j}(u_{tx_k}\Psi_{x_l} u_{tx_l}\Psi_{x_k}) + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_t \Psi_{x_kx_l} \\
&\qquad + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{tx_j}\Psi + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_t \Psi_{x_j} \\
&\qquad + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{tx_i} u_{tx_j}\Psi + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{tx_i} u_t \Psi_{x_j} \\
&\qquad + a(x) u u_t \Psi + (g(x,t,u_t^2) - g(x,t,u_t^1)) u_t \Psi(x)\Big] e^{-\gamma t} dx dt = 0,
\end{aligned} \tag{9}
$$

for almost all $\tau_0, \tau_1 \in (0,T]$. If we extend $u, a_{ij}, b_{ij}, f, a, g$ by zero for $t < 0$, then we can choose such τ_0 in (9) that

$$\frac{1}{2}\int\limits_{\Omega_{\tau_0}} u_t^2 e^{-\gamma\tau_0}\Psi dx = 0$$

and

$$\frac{1}{2}\int\limits_{\Omega_{\tau_0}} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_{x_kx_l} e^{-\gamma\tau_0}\Psi dx = 0.$$

From condition $(\mathbf{A_1})$ we infer

$$I_8 := \frac{1}{2}\int\limits_{\Omega_{\tau_1}} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_ix_j} u_{x_kx_l} e^{-\gamma\tau_1}\Psi dx \geq \frac{1}{2} a_2 \int\limits_{\Omega_{\tau_1}} \sum_{i,j=1}^{n} |u_{x_ix_j}| e^{-\gamma\tau_1}\Psi dx.$$

Next

$$I_{10} :=\frac{\gamma}{2}\int\limits_{Q_{\tau_0,\tau_1}}\sum_{i,j,k,l=1}^{n} a_{ij}^{kl}u_{x_ix_j}u_{x_kx_l}e^{-\gamma\tau_1}\Psi dx \geq \frac{\gamma}{2}a_2\int\limits_{Q_{\tau_0,\tau_1}}\sum_{i,j=1}^{n}|u_{x_ix_j}|e^{-\gamma\tau_1}\Psi dx,$$

$$\begin{aligned}
I_{11} :=& \int\limits_{Q_{\tau_0,\tau_1}}\sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x)u_{x_ix_j}\Big(u_{tx_k}\Psi_{x_l}+u_{tx_l}\Psi_{x_k}\Big)e^{-\gamma t}dxdt\\
=& \int\limits_{Q_\tau}\sum_{i,j,k,l=1}^{n}(a_{ij}^{kl})u_{x_ix_j}u_{tx_k}\Psi_{x_l}e^{-\gamma t}dxdt+\int\limits_{Q_\tau}\sum_{i,j,k,l=1}^{n}(a_{ij}^{kl})u_{x_ix_j}u_{tx_l}\Psi_{x_k}e^{-\gamma t}dxdt\\
\leq& \int\limits_{Q_\tau}\sum_{i,j,k,l=1}^{n}\left[\frac{a_2^0}{2\delta_3}|u_{x_ix_j}|^2\left(\frac{\Psi_{x_l}^2}{\Psi}+\frac{\Psi_{x_k}^2}{\Psi}\right)+\frac{\delta_3}{2}(|u_{tx_k}|^2+|u_{tx_l}|^2)\Psi\right]e^{-\gamma t}dxdt\\
\leq& n^3\delta_3\int\limits_{Q_\tau}\sum_{i=1}^{n}|u_{tx_i}|^2e^{-\gamma t}\Psi(x)dxdt+\frac{4\beta^2a_2^0}{\delta_3}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}|u_{x_ix_j}|^2(\phi_R)^{\beta-2}e^{-\gamma t}dxdt,
\end{aligned}$$

$$\begin{aligned}
I_{12} :=& \int\limits_{Q_\tau}\sum_{i,j,k,l=1}^{n}a_{ij}^{kl}u_{x_ix_j}u_te^{-\gamma t}\Psi_{x_kx_l}dxdt\\
\leq& \frac{1}{2}\int\limits_{Q_\tau}\sum_{i,j,k,l=1}^{n}\left[\delta_2(a_{ij}^{kl})^2|u_{x_ix_j}|^2\Psi(x)+\frac{1}{\delta_2}|u_t|^2\frac{(\Psi_{x_kx_l})^2}{\Psi}\right]e^{-\gamma t}dxdt\\
\leq& \frac{\delta_2a_2^0n^2}{2}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}|u_{x_ix_j}|^2e^{-\gamma t}\Psi(x)dxdt+\frac{n^4\mu_0}{2\delta_2}\int\limits_{Q_\tau}|u_t|^2(\phi_R)^{\beta-2}e^{-\gamma t}dxdt.
\end{aligned}$$

From assumption $(\mathbf{A_2})$ and the initial condition we obtain

$$\begin{aligned}
I_{13} :=& \int\limits_{Q_\tau}\sum_{i,j=1}^{n}a_{ij}u_{x_i},u_{tx_j}e^{-\gamma t}\Psi(x)dxdt\\
=& \frac{1}{2}\int\limits_{\Omega_{\tau_1}}\sum_{i,j=1}^{n}(a_{ij}u_{x_i}u_{x_j}e^{-\gamma t}\Psi(x))_tdx+\frac{\gamma}{2}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}e^{-\gamma t}\Psi(x)dxdt\\
=& \frac{1}{2}\int\limits_{\Omega_{\tau_1}}\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}e^{-\gamma\tau_1}\Psi(x)dx+\frac{\gamma}{2}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}e^{-\gamma t}\Psi(x)dxdt.
\end{aligned}$$

Next, we have

$$\begin{aligned}
I_{14} :=& \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u_{x_i} u_t e^{-\gamma t} \Psi_{x_j} dxdt \\
=& \int\limits_{0}^{\tau} \sum_{i,j=1}^{n} (a_{ij} u, u_t e^{-\gamma t} \Psi_{x_j})_{x_i} dt - \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ijx_i} u u_t e^{-\gamma t} \Psi_{x_j} dxdt \\
&- \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u u_{tx_i} e^{-\gamma t} \Psi_{x_j} dxdt - \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij} u u_t e^{-\gamma t} \Psi_{x_i x_j} dxdt \\
\le& \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[\delta_4 |u|^2 \Psi + \frac{a_{ijx_i}}{\delta_4} |u_t|^2 \frac{(\Psi_{x_j})^2}{\Psi} \right] e^{-\gamma t} dxdt \\
&+ \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[\delta_5 |u_{tx_i}|^2 \Psi + \frac{1}{\delta_5} (a_{ij})^2 |u|^2 \frac{(\Psi_{x_j})^2}{\Psi} \right] e^{-\gamma t} dxdt \\
&+ \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[(a_{ij})^2 |u_t|^2 |\Psi_{x_i x_j}| + |u|^2 |\Psi_{x_i x_j}|^2 \right] e^{-\gamma t} dxdt \le \\
\le& \frac{\beta^2 a_1^1 n^2}{\delta_4} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt + \frac{1}{2} n^2 \delta_4 \int\limits_{Q_\tau} |u|^2 \Psi e^{-\gamma t} dxdt \\
&+ \frac{\beta^2 a_1^0 n^2}{\delta_5} \int\limits_{Q_\tau} |u|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt + \frac{1}{2} n \delta_5 \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 \Psi e^{-\gamma t} dxdt \\
&+ \frac{a_1^0 n^2 \mu_0}{2} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt + \frac{n^2 \mu_0}{2} \int\limits_{Q_\tau} |u|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt.
\end{aligned}$$

From (**B**), there follows

$$\begin{aligned}
I_{15} =& \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} u_{tx_j} e^{-\gamma t} \Psi(x) dxdt \ge b_0 \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 e^{-\gamma t} \Psi(x) dx, \\
I_{16} =& \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij} u_{tx_i} u_t e^{-\gamma t} \Psi_{x_j} dxdt \\
\le& \frac{1}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} \left[\delta_0 (b_{ij})^2 |u_{tx_i}|^2 \Psi + \frac{1}{\delta_0} |u_t|^2 \frac{(\Psi_{x_j})^2}{\Psi} \right] e^{-\gamma t} dxdt \\
\le& \frac{\delta_0 b^0 n}{2} \int\limits_{Q_\tau} \sum_{i=1}^{n} |u_{tx_i}|^2 e^{-\gamma t} \Psi(x) dxdt + \frac{n \beta^2 2}{\delta_0} \int\limits_{Q_\tau} |u_t|^2 (\phi_R)^{\beta-2} e^{-\gamma t} dxdt.
\end{aligned}$$

Next, from ($\mathbf{A_3}$) and the initial condition, we infer

$$\begin{aligned} I_{17} &= \int\limits_{Q_\tau} a(x,t)uu_t e^{-\gamma t}\Psi dxdt \\ &= \frac{1}{2}\int\limits_{\Omega_{\tau_1}} (a|u|^2\Psi e^{-\gamma t})_t dx + \frac{\gamma}{2}\int\limits_{Q_\tau} a(x,t)|u|^2\Psi e^{-\gamma t}dxdt \\ &\geq \frac{1}{2}a_0 \int\limits_{\Omega_{\tau_1}} |u|^2\Psi e^{-\gamma\tau_1}dx + \frac{\gamma}{2}a_0\int\limits_{Q_\tau} |u|^2\Psi e^{-\gamma t}dxdt. \end{aligned}$$

Moreover, by condition ($\mathbf{G}$), we obtain

$$I_{18} = \int\limits_{Q_\tau} (g(x,t,u_t^2) - g(x,t,u_t^1))u_t^R\Psi e^{-\gamma t}dxdt \geq 0.$$

From the estimates of the integrals $I_8 - I_{18}$ and (9), we obtain

$$\begin{aligned} &\frac{1}{2}\int\limits_{\Omega_{\tau_1}} u_t^2\Psi e^{-\gamma\tau_1}dx + \frac{a_2}{2}\int\limits_{\Omega_{\tau_1}}\sum_{ij=1}^n |u_{x_ix_j}|^2\Psi e^{-\gamma\tau_1}dx \\ &+ \frac{a_0}{2}\int\limits_{\Omega_{\tau_1}} |u|^2\Psi e^{-\gamma\tau_1}dx + \frac{\gamma}{2}\int\limits_{Q_{\tau_0,\tau_1}} |u_t|^2\Psi e^{-\gamma t}dxdt \\ &+ \left[\frac{\gamma a_2}{2} - \frac{\delta_2 a_2^0 n^2}{2}\right]\int\limits_{Q_{\tau_0,\tau_1}}\sum_{i,j=1}^n |u_{x_ix_j}|^2\Psi e^{-\gamma t}dxdt \\ &+ \left[b_0 - n^3\delta_3 - \frac{b^0\delta_0 n}{2} - \frac{\delta_5 n}{2}\right]\int\limits_{Q_{\tau_0,\tau_1}}\sum_{i=1}^n |u_{tx_i}|^2\Psi e^{-\gamma t}dxdt \\ &+ \left[\frac{\gamma a_0}{2} - \frac{\delta_4 n^2}{2}\right]\int\limits_{Q_{\tau_0,\tau_1}} |u|^2\Psi e^{-\gamma t}dxdt \\ \leq &\left[\frac{n^4\mu_0}{2\delta_2} + \frac{2\beta^2 n}{\delta_0} + \frac{2na_1^1\beta^2}{\delta_4} + \frac{n^2 a_1^0\mu_0}{2}\right]\int\limits_{Q_{\tau_0,\tau_1}} |u_t|^2(\phi_R)^{\beta-2}e^{-\gamma t}dxdt \\ &+ \left[\frac{n^2\mu_0}{2} + \frac{2a_1^0 n\beta^2}{\delta_5}\right]\int\limits_{Q_{\tau_0,\tau_1}} |u|^2(\phi_R)^{\beta-2}e^{-\gamma t}dxdt \\ &+ \frac{4a_2^0\beta^2}{\delta_3}\int\limits_{Q_{\tau_0,\tau_1}}\sum_{ij=1}^n |u_{x_ix_j}|^2(\phi_R)^{\beta-2}e^{-\gamma t}dxdt. \end{aligned} \tag{10}$$

Let R_1 be fixed and $R_1 < R$.
Now, we choose $\gamma = \gamma_0 + \gamma_1$, $\delta_2 = \frac{2}{a_2^0 n^2}$, $\delta_4 = \frac{2}{n^2}$, $\gamma_1 = \max\{\frac{2}{a_0}, \frac{2}{a_2}\}$, $\delta_0 = \frac{2b_0}{3b^0 n}$, $\delta_3 = \frac{b_0}{3n^3}$, $\delta_5 = \frac{2b_0}{3n}$.
From (10), we obtain the following inequality

$$\gamma_0 \int\limits_{Q_\tau^{R_1}} \sum_{i,j=1}^{n} \Big[|u_t|^2 + |u_{x_i x_j}|^2 + |u|^2 \Big] \Psi e^{-\gamma_0 t} dxdt \leq \tag{11}$$

$$\leq K R^{\beta-2} \int\limits_{Q_\tau^{R}} \sum_{i,j=1}^{n} \Big[|u_t|^2 + |u_{x_i x_j}|^2 + |u|^2 \Big] e^{-\gamma_0 t} dxdt.$$

Then, from (11), we conclude:

$$\gamma_0 \int\limits_{Q_\tau^{R_1}} w e^{-\gamma_0 t} \Psi dxdt \leq K R^{\beta-2} \int\limits_{Q_\tau^{R}} w e^{-\gamma_0 t} dxdt, \tag{12}$$

where $w = \sum\limits_{ij=1}^{n} \Big[|u_t|^2 + |u_{x_i x_j}|^2 + |u|^2 \Big]$.
In Ω^{R_1},

$$\Psi(x) = [\phi_R(x)]^\beta = \left[\frac{1}{R}(R - |x|)(R + |x|) \right]^\beta \geq (R - R_1)^\beta. \tag{13}$$

Using (13) we obtain, from (12),

$$\gamma_0 e^{-\gamma_0 \tau} (R - R_1)^\beta \int\limits_{Q_\tau^{R_1}} w dxdt \leq K R^{\beta-2} \int\limits_{Q_\tau^{R}} w dxdt$$

or

$$\int\limits_{Q_\tau^{R_1}} w dxdt \leq \frac{1}{\gamma_0} \left(\frac{R}{R - R_1} \right)^\beta \frac{e^{\gamma_0 \tau}}{R^2} \int\limits_{Q_\tau^{R}} w dxdt. \tag{14}$$

Since $\lim\limits_{R \to \infty} \frac{R}{R - R_1} = 1$, we get, from (14),

$$\int\limits_{Q_\tau^{R_1}} w dxdt \leq \frac{K}{\gamma_0} \frac{1}{R^2} e^{\gamma_0 \tau} \int\limits_{Q_\tau^{R}} w dxdt. \tag{15}$$

Let $R = R_2$. Then

$$\int\limits_{Q_\tau^{R_1}} w dxdt \leq \frac{K}{\gamma_0} \frac{1}{(R_2 - R_1)^2} e^{\gamma_0 \tau} \int\limits_{Q_\tau^{R_2}} w dxdt. \tag{16}$$

Moreover, let $\rho_k = \frac{R_2-R_1}{k}$, $k \in N$, $R_1(s) = R_1 + s\rho_k$, $R_2(s) = R_1(s) + \rho_k$,$s = 0, 1, ..., k-1$. Then inequality (16) for $R_1(s)$, $R_2(s)$ and ρ_k will take the form

$$\int\limits_{Q_\tau^{R_1(s)}} wdxdt \leq \frac{K}{\gamma_0}\frac{1}{\rho_k^2}e^{\gamma_0\tau} \int\limits_{Q_\tau^{R_2(s)}} wdxdt. \tag{17}$$

From the form of $R_1(s)$, $R_2(s)$, ρ_k and (17), we infer

$$\int\limits_{Q_\tau^{R_1}} wdxdt \leq \left(\frac{K}{\gamma_0\rho_k^2}\right)^k e^{\gamma_0\tau} \int\limits_{Q_\tau^{R_2}} wdxdt. \tag{18}$$

Choosing K and γ_0 such that $\left(\frac{K}{\gamma_0\rho_k^2}\right)^k < e^{-1}$, putting the constants κ, λ, a, b_0 such that $R_1 = 2^m$, $R_2 = 2^{m+1}$, $m \in N$, $\kappa = \lambda 2^{m+1}$, $\lambda = 2 + [a]$, $\gamma = b_0\lambda^2 2^{m+1}$, $b_0 = 8K \cdot e$, and assuming that

$$\int\limits_{Q_\tau^{R_2}} wdxdt \leq e^{aR_2^2}$$

we obtain, from (15),

$$\int\limits_{Q_\tau^{R_1}} wdxdt \leq e^{(-\kappa+\gamma_0\tau_0)} \int\limits_{Q_\tau^{R_2}} wdxdt \leq e^{(-\kappa+\gamma_0\tau_0+aR_2^2)}.$$

Since

$$-\kappa + \gamma_0\tau_0 + aR_2^2 = (-2 + a - [a] + b_0\lambda^2\tau_0)2^{m+1} \leq (-1 + b_0\lambda^2\tau_0)2^{m+1},$$

it follows that for $\tau_0 = \min\{T, \frac{1}{2b_0}\lambda^2\}$

$$\int\limits_{Q_{\tau_0}^{R_1}} wdxdt \leq e^{-2^{m+1}}.$$

Hence for $m \to +\infty$, $w(x,t) = 0$ almost everywhere in Q_{τ_0}. If $\tau_0 < T$, then we can by analogy prove that $w(x,t) = 0$ almost everywhere in $Q_{\tau_0, 2\tau_0}$ etc.
Hence $u(x,t) = u^1(x,t) - u^2(x,t) = 0$. □

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