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ON THE TYPE SEQUENCES OF SOME ONE DIMENSIONAL RINGS

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Abstract. In this article in Section 2 we describe the holes and their positions of a numerical semigroup and use this description to compute the type sequence of the semigroup generated by an arithmetic sequence $m_0, m_1, \ldots, m_{p+1}$ explicitly (see 3.8 and 3.9).

Introduction. Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the $\mathfrak{m}$-adic completion $\hat{R}$ of R is a domain or, equivalently, the integral closure $\overline{R}$ of R in its quotient field $\mathrm{Q}(R)$ is a discrete valuation ring and a finite R-module. We further assume that R is residually rational, i.e., R and $\overline{R}$ have the same residue field. A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

Let $v : \mathrm{Q}(R) \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation of $\overline{R}$ and let $\mathfrak{C} := \mathrm{ann}_R(\overline{R}/R) = \{x \in R \mid x\overline{R} \subseteq R\}$ be the conductor ideal of R in $\overline{R}$. Then the value semigroup $v(R) = \{v(x) \mid x \in R, x \neq 0\}$ is a numerical semigroup, that is, $\mathbb{N} \setminus v(R)$ is finite and therefore $v(R) = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\} \cup \{z \in \mathbb{N} \mid z \geq c\}$, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ are elements of $v(R)$, $n := n(R) = \ell(R/\mathfrak{C})$ and the integer $c = c(R) := \ell_{\overline{R}}(\overline{R}/\mathfrak{C})$ is also determined by $\mathfrak{C} = \{x \in \mathrm{Q}(R) \mid v(x) \geq c\}$ or, equivalently $\mathfrak{C} = (\mathfrak{m}_{\overline{R}})^c$.

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In [**5**] Matsuoka have studied the degree of singularity $\delta = \delta(R) := \ell(\overline{R}/R) = \mathrm{card}(\mathbb{N} \setminus v(R))$ of R by introducing the saturated chain of fractionary ideals

$$\mathfrak{C} = \mathfrak{A}_n \subsetneq \cdots \subsetneq \mathfrak{A}_1 = \mathfrak{m} \subsetneq \mathfrak{A}_0 = R \subsetneq \mathfrak{A}_1^{-1} \subsetneq \cdots \subsetneq \mathfrak{A}_n^{-1} = \overline{R},$$

where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq \mathrm{v}_i\}$ and $\mathfrak{A}_i^{-1} = (R : \mathfrak{A}_i)$, $i = 0, 1, \ldots, n$. Moreover, each $\mathfrak{A}_i^{-1}, i = 0, \ldots, n$ is a overring of R which satisfies the assumptions that we assume for R. The sequence $\mathrm{t}_i = \mathrm{t}_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \ldots, n$, is called the type sequence of R.

The above numerical invariants of R carry information of the ring and hence to study various algebraic and geometric properties of the ring R; several authors (see e.g. [**1, 2, 3**]) have been studied the above numerical invariants. For example the first term t_1 is the Cohen–Macaulay type of R and the sum $\sum_{i=1}^{n} \mathrm{t}_i$ is the degree of singularity of R.

In Section 3 we give an algorithmic method (see 3.7) to compute the type sequence of the coordinate ring of an algebroid monomial curve defined by an arithmetic sequence $m_0, m_1, \ldots, m_{p+1}$. For this we make use of the explicit description of the standard basis of the numerical semigroup generated by arithmetic sequence which was done in [**7**]. We also give some illustrative examples.

1. Preliminaries – assumptions and notation. Throughout this article we make the following assumptions and notation.

NOTATION 1.1. Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of all natural numbers and all integers, respectively. Note that we assume $0 \in \mathbb{N}$. Further, for $a, b \in \mathbb{N}$, we denote $[a, b] := \{r \in \mathbb{N} \mid a \leq r \leq b\}$ and $\mathbb{N}_a := \{n \in \mathbb{N} \mid n \geq a\}$.

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the integral closure $\overline{R}$ of R in its quotient field $\mathrm{Q}(R)$ is a discrete valuation ring and is a finite R-module. We further assume that R is residually rational, i.e., the residue field $k_{\overline{R}}$ of $\overline{R}$ is equal to the residue field k_R of R. A particular important class of rings which satisfy these assumptions are semi-group rings which are coordinate rings of algebroid monomial curves.

We shall now recall the notions of *type sequences* and *almost Gorenstein rings.*

1.2. (Type sequences — almost Gorenstein rings) Let R be as in 1.1 and let $v(R)$ be its numerical semigroup, $c = c(v(R))$ be the conductor of $v(R)$, $n = n(R) = \ell(R/\mathfrak{C}) = \mathrm{card}(v(R) \setminus \mathbb{N}_c)$ and $\delta = \delta(R) = \ell(\overline{R}/R) = \mathrm{card}(\mathbb{N} \setminus v(R))$ be the degree of singularity of R (see [**5**]). Let $0 = \mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ be elements of $v(R)$ such that $v(R) \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$. Further as noted in [**5**], the degree of singularity $\delta(R)$ can be seen as the sum of n positive integers $t_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \ldots, n$, where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq \mathrm{v}_i\}$ and $\mathfrak{A}_i^{-1} := (R : \mathfrak{A}_i) := \{x \in \mathrm{Q}(R) \mid x\mathfrak{A}_i \subseteq R\}$. The first positive integer $t_1(R)=\ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type τ_R of R. The sequence $t_1(R), t_2(R), \ldots, t_n(R)$ is called the *type sequence* of R. Several authors have studied the properties of type sequences (see e.g. [**1, 4**]). The term "type sequence" is chosen since (as noted above) the first term $t_1(R)=\ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type of R. Further, we have $1\le t_i(R)\le \tau_R$ for every $i=1,\ldots,n$ (see [**5**, §3, Proposition 2 and Proposition 3]) and hence (see also [**4**, Proposition 2.1]) $\ell^*(R)\le(\tau_R-1)\,(\ell(R/\mathfrak{C})-1)$, where $\ell^*(R):=\tau_R\cdot\ell(R/\mathfrak{C})-\ell(\overline{R}/R)$. Moreover, the equality holds if and only if $\ell(\overline{R}/R)=\tau_R+\ell(R/\mathfrak{C})-1$, or, equivalently, $t_i(R)=1$ for $i=2,\ldots,n$. Type sequence of a numerical semi-group can also be defined analogously: Let Γ be a numerical semigroup, $c\in\mathbb{N}$ be its conductor and let $\Gamma\setminus\mathbb{N}_c=\{0=\mathrm{v}_0,\mathrm{v}_1,\ldots,\mathrm{v}_{n-1}\}$, where $0=\mathrm{v}_0<\mathrm{v}_1<\cdots<\mathrm{v}_{n-1}<\mathrm{v}_n:=c$ are elements of Γ. Further, for $i=0,\ldots,n$, let $\Gamma_i:=\{h\in\Gamma\mid h\ge \mathrm{v}_i\}$, $\Gamma(i):=\{x\in\mathbb{Z}\mid x+\Gamma_i\subseteq\Gamma\}$ and let $\mathrm{t}_i=\mathrm{card}\,(\Gamma(i)\setminus\Gamma(i-1))$. Then $\Gamma=\Gamma(0)\subseteq\Gamma(1)\subseteq\cdots\subseteq\Gamma(n-1)\subseteq\Gamma(n)=\mathbb{N}$ and the sequence t_i, $i=1,\ldots,n$ is called the *type sequence* of Γ. In particular, the cardinality t_1 of the set $\mathrm{T}(\Gamma):=\Gamma(1)\setminus\Gamma$ is called the *Cohen–Macaulay type* of the semigroup Γ.

The type sequence of a ring R need not be same as the type sequence of the numerical semi-group $v(R)$ of R (see e.g. [**4**]).

A ring R in (1.1) is called *almost Gorenstein* if the type sequence of R is $\{\tau_R,1,1,\ldots,1\}$, or, equivalently, $\ell^*(R)$ attains its upper bound, i.e., $\ell(\overline{R}/R)=\tau_R-1+\ell(R/\mathfrak{C})$. It is clear that Gorenstein rings are almost Gorenstein but not conversely (see [**8**], (1.2)–(1)).

Examples 1.3. Using the above definitions we shall compute the type sequences of the semigroups of the examples ([**8**], (1.2)).

(1) Let $e\in\mathbb{N}$, $a\in[0,e-1]$ with $e\ge 3$, $b:=\begin{cases}\ge 1, & \text{if } a=0,\\ \ge 2, & \text{if } a\ge 1,\end{cases}$ and put $c:=be-a$. Let Γ be the semi-group generated by the almost arithmetic sequence $e,c,c+1,\ldots,c+e-1$. Then:

(i) $c(\Gamma)=c(R)=\begin{cases}c, & \text{if } a\in[0,e-2]\,,\\ (b-1)e, & \text{if } a=e-1\end{cases}$ and

$\Gamma\backslash\mathbb{N}_c=\begin{cases}\{0,e,2e,\ldots,(b-1)e\}, & \text{if } a\in[0,e-2]\,,\\ \{0,e,2e,\ldots,(b-2)e\}, & \text{if } a=e-1\,.\end{cases}$ Therefore, $n=n(R)=\begin{cases}b, & \text{if } a\in[0,e-2]\,,\\ b-1, & \text{if } a=e-1\end{cases}$ and $\mathrm{v}_i=ie$ for $i=0,\ldots,n-1$.

(ii) For each $i = 1, \dots, n$, we have $\Gamma(i) \setminus \Gamma(i-1) =$

$$\begin{cases} [(b-i)e+1, (b-i+1)e-1], & \text{if } a = 0, \\ [(b-i)e-a, (b-i+1)e-a-1] \setminus \{(b-i)e\}, & \text{if } a \geq 1 \text{ and } \\ & \quad i \in [1, n-1], \\ [1, e-a-1], & \text{if } a \in [1, e-2] \text{ and } \\ & \quad i = n, \\ [1, e-1], & \text{if } a = e-1 \text{ and } \\ & \quad i = n. \end{cases}$$

In particular, $\mathrm{t}_i = \mathrm{t}_i(\Gamma) = \begin{cases} e-1, & \text{if } a \in \{0, e-1\} \text{ and } i \in [1,n], \\ e-1, & \text{if } a \in [1, e-2] \text{ and } i \in [1, n-1], \\ e-a-1, & \text{if } a \in [1, e-2] \text{ and } i = n. \end{cases}$

The type sequence of Γ is $\begin{cases} \underbrace{e-1, \dots, e-1}_{n\text{-times}}, & \text{if } a \in \{0, e-1\}, \\ \underbrace{e-1, \dots, e-1}_{(n-1)\text{-times}}, e-a-1, & \text{if } a \in [1, e-2]. \end{cases}$

In particular, R is almost Gorenstein if and only if (a, b) is one of the following three pairs $(0,1), (e-2, 2), (e-1, 2)$. Therefore, the semi-group ring $K[\![X^3, X^5, X^7]\!]$ (take $e = 3$, $a = 1$ and $b = 2$) is almost Gorenstein of type 2 and hence not Gorenstein.

(2) Let $e \in \mathbb{N}$ with $e \geq 4$ and $m := 3e+1$. Let Γ be the semi-group generated by the sequence $e, 2e-1, m, m+1, \dots, m+e-4$. Then:

(i) $c = c(\Gamma) = c(R) = 3e-1$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, 2e-1, 2e\}$. Therefore, $n = n(R) = 4$ and $\mathrm{v}_1 = e, \mathrm{v}_2 = 2e-1, \mathrm{v}_3 = 2e, \mathrm{v}_4 = c$.

(ii) $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [2e+1, 3e-2]$, $\Gamma(2) \setminus \Gamma(1) = [e+1, 2e-2]$, $\Gamma(3) \setminus \Gamma(2) = \{e-1\}$ and $\Gamma(4) \setminus \Gamma(3) = [1, e-2]$. Therefore, $\mathrm{t}_1 = \tau_R = e-2, \mathrm{t}_2 = e-2, \mathrm{t}_3 = 1, \mathrm{t}_4 = e-2$ and the type sequence of Γ is $e-2, e-2, 1, e-2$. Therefore, R is not almost Gorenstein, since $e \geq 4$.

(3) Let $e, r' \in \mathbb{N}$ with $e \geq 3$, $1 \leq r'$, $2r' \leq e-1$ and $c := 2e$. Let Γ be the semi-group generated by the sequence $e, e+r', c+1, c+2, \dots, c+e-1$. Then:

(i) $c = c(\Gamma) = c(R) = 2e$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e+r'\}$. Therefore, $n = n(\Gamma) = n(R) = 3$ and $\mathrm{v}_1 = e, \mathrm{v}_2 = e+r', \mathrm{v}_3 = c$.

(ii) $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [e+1, e+r'-1] \cup [e+r'+1, 2e-1]$, $\Gamma(2) \setminus \Gamma(1) = [e-r', e-1]$ and $\Gamma(3) \setminus \Gamma(2) = [1, e-r'-1]$. Therefore, $\mathrm{t}_1 = \tau_R = e-2, \mathrm{t}_2 = r', \mathrm{t}_3 = e-r'-1$ and the type sequence of Γ is $e-2, r', e-r'-1$. Therefore, R is almost Gorenstein if and only if $r' = 1$ and $e = 3 \iff R$ is Gorenstein. Hence, if $e \geq 4$ then R is not almost Gorenstein.

(4) Let $e, r, r' \in \mathbb{N}$ with $e \geq 3$, $1 \leq r$, $1 \leq r'$, $r + r' \leq e - 1$ and let Γ be the semi-group generated by the sequence $e, e + r, e + r + r', e + r + r' + 1, \ldots, 2e + r + r' - 1$.

We consider the four cases (i) $r' = r = 1$; (ii) $r' = 1$, $r \geq 2$; (iii) $1 < r' \leq r$; (iv) $r < r'$ separately.

Case (i): $(r', r) = (1, 1)$: This case is included in example (1) ($a = 0$ and $b = 1$).

Case (ii): $r' = 1$ and $r \geq 2$: In this case $c = e + r$ and $\Gamma \setminus \mathbb{N}_c = \{0, e\}$. Therefore, $n = 2$ and $\mathrm{v}_1 = e$. Further, $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [r, e-1] \cup [e+1, e+r-1]$ and $\Gamma(2) \setminus \Gamma(1) = [1, r-1]$. Therefore, $\mathrm{t}_1 = \tau_R = e-1, \mathrm{t}_2 = r-1$ and the type sequence of Γ is $e-1, r-1$. Therefore, R is almost Gorenstein if and only if $r = 2$.

Case (iii): $1 < r' \leq r$: In this case $c = e + r + r'$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e + r\}$. Therefore, $n = 3$ and $\mathrm{v}_1 = e$, $\mathrm{v}_2 = e + r$. Further, we have $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = \{r\} \cup [r+r', e+r+r'-1] \setminus \{e, e+r\}$, $\Gamma(2) \setminus \Gamma(1) = \begin{cases} [r+1, r+r'-1], & \text{if } r = r', \\ [r', r+r'-1] \setminus \{r\}, & \text{if } r' < r, \end{cases}$ and $\Gamma(3) \setminus \Gamma(2) = \begin{cases} [1, r-1], & \text{if } r' = r, \\ [1, r'-1], & \text{if } r' < r. \end{cases}$

Therefore, $\mathrm{t}_1 = \tau_R = e-1, \mathrm{t}_2 = \begin{cases} r' - 1, & \text{if } r' = r, \\ r - 1, & \text{if } r' < r, \end{cases}$ $\mathrm{t}_3 = \begin{cases} r - 1, & \text{if } r' = r, \\ r' - 1, & \text{if } r' < r, \end{cases}$

and the type sequence of Γ is $\begin{cases} e - 1, r' - 1, r - 1, & \text{if } r' = r, \\ e - 1, r - 1, r' - 1, & \text{if } r' < r. \end{cases}$ Therefore, R is almost Gorenstein if and only if $(r', r) = (2, 2)$.

Case (iv): $r < r'$: In this case $c = e + r + r'$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e + r\}$. Therefore, $n = 3$ and $\mathrm{v}_1 = e$, $\mathrm{v}_2 = e + r$. Further, we have $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [r + r', e + r + r' - 1] \setminus \{e, e + r\}$, $\Gamma(2) \setminus \Gamma(1) = [r', r + r' - 1]$ and $\Gamma(3) \setminus \Gamma(2) = [1, r' - 1]$. Therefore, $\mathrm{t}_1 = \tau_R = e - 2, \mathrm{t}_2 = r, \mathrm{t}_3 = r' - 1$ and the type sequence of Γ is $e - 2, r, r' - 1$. Therefore, R is almost Gorenstein if and only if $(r, r') = (1, 2)$.

2. Holes of first and second type. Let R be as in 1.1. In this section we describe the holes of first and second type of the numerical semigroup $v(R)$ of R. In addition to the Notations of § 1, we also fix the following:

Notation 2.1. Put $\Gamma := v(R)$ and let $\Gamma_i := v(\mathfrak{A}_i)$, $\Gamma(i)$ and t_i, $i = 1, \ldots, n$ be as in 1.2.

In order to compute some type sequences explicitly, we need to study the "*holes*" of Γ, i.e., elements of $\mathbb{N} \setminus \Gamma$. The positions of the holes will therefore determine the type sequence of Γ. To make these things more precise first let us make the following:

Definition 2.2. An element $z \in \mathbb{Z} \setminus \Gamma$ is called a *hole of first type* (respectively, *hole of second type*) of Γ if $c-1-z \in \Gamma$ (respectively, if $c-1-z \notin \Gamma$). Then $\Gamma' := \{z \in \mathbb{Z} \setminus \Gamma \mid c-1-z \in \Gamma\} = \{c-1-h \mid h \in \Gamma\}$ is the set of holes of first type of Γ and $\Gamma'' := \{z \in \mathbb{Z} \setminus \Gamma \mid c-1-z \notin \Gamma\}$ is the set of holes of second type of Γ. Therefore, $\mathbb{Z} = \Gamma \uplus \Gamma' \uplus \Gamma''$. Further, it is easy to see that:

$$\text{(2.2.a)} \qquad \begin{cases} \Gamma' \cap \mathbb{N} = \{c-1-\mathrm{v}_i \mid i \in [0,n-1]\}; |\Gamma' \cap \mathbb{N}| = n = c - \delta, \\ \Gamma'' \subseteq \mathbb{N} \setminus \Gamma, c-1 \notin \Gamma'' \quad \text{and} \quad \mathrm{T}(\Gamma) \subseteq \{c-1\} \cup \Gamma''. \end{cases}$$

In particular, Γ is symmetric if and only if $\Gamma'' = \emptyset$. For this reason the cardinality of Γ'' is called the *symmetry-defect* of Γ.

Lemma 2.3. $(\Gamma(i) \setminus \Gamma(i-1)) \cap \Gamma' = \{c-1-\mathrm{v}_{i-1}\}$ *for each* $i = 1, \ldots, n$.

Proof. First note that $(\Gamma(i) \setminus \Gamma(i-1)) \cap \Gamma' \subseteq \{c-1-\mathrm{v}_k \mid k = 0, \ldots, n-1\}$ and that $c-1-\mathrm{v}_{i-1}$ is the greatest element in $\Gamma(i) \setminus \Gamma(i-1)$ by [**5**], Proposition 2. Now suppose that $c-1-\mathrm{v}_k \in \Gamma(i) \setminus \Gamma(i-1)$ for some $k \neq i-1$. Then $c-1-\mathrm{v}_k < c-1-\mathrm{v}_{i-1}$ and so $k > i-1$. Therefore, $c-1-\mathrm{v}_k \in \Gamma(i) \subseteq \Gamma(k)$ and hence $c-1 = (c-1-\mathrm{v}_k) + v_k \in \Gamma$ a contradiction. □

Lemma 2.4. *Every element* $z \in \Gamma''$ *can be written in the form* $z = x - h$ *with* $x \in \Gamma(1) \setminus \Gamma$, $x \neq c-1$ *and* $h \in \Gamma$. *In particular, we have:*

$\Gamma'' \subseteq \{x - \mathrm{v}_i \mid x \in \Gamma(1) \setminus \Gamma, x \neq c-1$ *and* $i \in [0, n-1]\}$.

Proof. If $z \in \Gamma(1)$, then take $x = z$ and $h = 0$. In the case $z \notin \Gamma(1)$, i.e., $z + \Gamma_1 \nsubseteq \Gamma$, let $i := \max\{k \in [0, n-1] \mid z + \mathrm{v}_k \notin \Gamma\}$ and $x := z + \mathrm{v}_i$. Then $x \neq c-1$ (otherwise, $z = x - \mathrm{v}_i = c-1-\mathrm{v}_i \in \Gamma'$) and $x \in \Gamma(1) \setminus \Gamma$ by definition of i. Therefore, we can take $x := z + \mathrm{v}_i$ and $h = \mathrm{v}_i$. □

The following 2.5, 2.6 and 2.7 are used to determine the positions of the holes of second type.

Lemma-Definition 2.5. First let us recall that $m := \mathrm{v}_1$ is the multiplicity of R and the set $S_m(\Gamma) := \{z \in \Gamma \mid z - m \notin \Gamma\}$ is called the *standard basis* or the *Apéry set of* Γ with respect to m. We put $\mathrm{S} := \mathrm{S}_m(\Gamma)$ and write $\mathrm{S} = \{0 = s_0, s_1, \ldots, s_{m-1}\}$ with $0 = s_0 < s_1 < \cdots < s_{m-1}$. Note that every element $h \in \Gamma$ can be written in the unique form $h = \rho m + s$ with $\rho \in \mathbb{N}$ and $s \in \mathrm{S}$. Further, note that $s_{m-1} = c-1+m$. With these definitions, we have:

For each $z \in \Gamma''$ *and each* $s \in \mathrm{S}$, *the following minima exist:*

(1) $\kappa(z) := \mathrm{Min}\{k \in [0, m-1] \mid z + s_j \in \Gamma$ *for all* $k \leq j \leq m-1\}$.

(2) $\alpha_s(z) := \mathrm{Min}\{\alpha \in \mathbb{N} \mid z + s + \alpha m \in \Gamma\}$.

Proof. (1) Since $\Gamma'' \subseteq \mathbb{N}$ by (2.2.a), we have $z + s_{m-1} = z + c - 1 + m \geq c$ and hence $z + s_{m-1} \in \Gamma$. (2) For every $s \in \mathrm{S}$, $z + s + \alpha m \in \Gamma$ for large $\alpha >> 0$. □

Lemma 2.6. *For* $z \in \Gamma''$ *and for* $s \in \mathrm{S}$, *we have*

(1) $\kappa(z) = \mathrm{Min}\{k \in [0, m-1] \mid \alpha_{s_k}(z) = 0\}$.

(2) $z + s + \rho m \notin \Gamma$ *for all* $\rho \in [0, \alpha_s(z) - 1]$, $z + s + \alpha_s(z)m \in \Gamma$ *and* $\alpha_{s_k}(z) = 0$, *i.e.,* $z + s_k \in \Gamma$ *for all* $k \geq \kappa(z)$.

(3) If $z = x - \rho m$ *with* $x \in \Gamma(1) \setminus \Gamma$, *then* $\alpha_{s_0}(z) = \rho + 1$.

Proof. (1) and (2) are immediate from definitions and (3) follows from: $z + \rho m = x \notin \Gamma$ and $z + (\rho + 1)m = x + m \in \Gamma$. □

Definition 2.7. For $r \in \mathbb{N}$ and $z \in \Gamma''$, let

$(*_r(z))$ For each $j \in [r, n]$, we have $\mathrm{v}_j = s_k + \rho m$ with $s_k \in \mathrm{S}, \rho \in \mathbb{N}$ and either $k \geq \kappa(z)$, or $\rho \geq \alpha_{s_k}(z)$.

Proposition 2.8. *Let* $x \in \Gamma(1) \setminus \Gamma$, $i \in [0, n-1]$ *be such that* $z := x - \mathrm{v}_i \in \Gamma''$. *Further, let* r *be the least positive integer with* $r > i$ *and* $(*_r(z))$ *holds. Then* $z \in \Gamma(r) \setminus \Gamma(r-1)$.

Proof. First we prove that $z \in \Gamma(r)$, i.e., $z + \Gamma_r \subseteq \Gamma$. It is enough to prove that:

$$z + \mathrm{v}_j \in \Gamma \quad \text{for all} \quad j \in [r, n]. \tag{2.8.a}$$

Now, since $(*_r(z))$ holds, for each $j \in [r, n]$ we have $\mathrm{v}_j = s_k + \rho m$ with $s_k \in \mathrm{S}, \rho \in \mathbb{N}$ and either $k \geq \kappa(z)$, or $\rho \geq \alpha_{s_k}(z)$. We consider these two cases separately.

Case: $k \geq \kappa(z)$: In this case $z + s_k \in \Gamma$ by 2.5.(1) and so $z + \mathrm{v}_j = z + s_k + \rho m \in \Gamma$.

Case: $\rho \geq \alpha_{s_k}(z)$: In this case $\rho = \alpha_{s_k}(z) + \beta$ for some $\beta \in \mathbb{N}$ and so $z + \mathrm{v}_j = z + s_k + \alpha_{s_k}(z)m + \beta m \in \Gamma$. This proves (2.8.a).

Now we prove that $z \notin \Gamma(r-1)$, i.e., $z + \Gamma_{r-1} \nsubseteq \Gamma$. It is enough to prove that:

$$z + \mathrm{v}_j \notin \Gamma \quad \text{for some} \quad j \in [r-1, n]. \tag{2.8.b}$$

By definition of r, we have either $r - 1 \leq i$, or $(*_{r-1}(z))$ does not hold. In the case $r - 1 \leq i$, taking $j = i$, we have $z + \mathrm{v}_j = x \notin \Gamma$ by assumption, which proves (2.8.b). If $(*_{r-1}(z))$ does not hold, i.e., there exists $j \in [r-1, n]$ such that $\mathrm{v}_j = s_k + \rho m$ with $s_k \in \mathrm{S}, \rho \in \mathbb{N}$, $k < \kappa(z)$ and $\rho < \alpha_{s_k}(z)$. Therefore, $z + \mathrm{v}_j = z + s_k + \rho m \notin \Gamma$ by 2.6.(1). This proves (2.8.b). □

Corollary 2.9. *Let* $x \in \Gamma(1) \setminus \Gamma$ *and* $i \in [0, n-1]$ *be such that* $z := x - \mathrm{v}_i \in \Gamma''$. *Further, assume that*

$$\kappa(z) = \mathrm{Min}\{k \in [0, m-1] \mid s_k > \mathrm{v}_i\} \tag{2.9.a}$$

and that

$$\alpha_s(z) \in \{0, 1\} \quad \textit{for all} \quad s \in \mathrm{S} \quad \textit{with} \quad s \leq \mathrm{v}_i. \tag{2.9.b}$$

Then $z \in \Gamma(i+1) \setminus \Gamma(i)$.

Proof. In view of 2.8 it is enough to prove that $(*_{i+1}(z))$ holds. For this let $j \in [i+1, n]$ and $\mathrm{v}_j = s_k + \rho m$ with $s_k \in \mathrm{S}$ and $\rho \in \mathbb{N}$. To show that either $k \geq \kappa(z)$, or $k < \kappa(z)$ and $\rho \geq \alpha_{s_k}(z)$. If $s_k > \mathrm{v}_i$, then $k \geq \kappa(z)$ by the Assumption (2.9.a). If $s_k \leq \mathrm{v}_i$, then $\rho \geq 1$, since $\mathrm{v}_j > \mathrm{v}_i$ and hence $\alpha_{s_k}(z) \leq \rho$ by the Assumption (2.9.b). □

In Section 3, we shall consider a class of rings such that the Assumptions of 2.9 are satisfied by the holes of second type.

Corollary 2.10. *Let $x \in \Gamma(1) \setminus \Gamma$ and $i \in [0, n-1]$, $\beta \in \mathbb{N}^+$ be such that $\mathrm{v}_i = \beta m$ and $z := x - \mathrm{v}_i \in \Gamma''$. Further, assume that $\kappa(z) = 1$. Then $z \in \Gamma(i+1) \setminus \Gamma(i)$.*

Proof. In view of 2.8 it is enough to prove that $(*_{i+1}(z))$ holds. For this let $j \in [i+1, n]$ and $\mathrm{v}_j = s_k + \rho m$ with $s_k \in \mathrm{S}$ and $\rho \in \mathbb{N}$. Since $\kappa(z) = 1$ by assumption, it is enough to show that: if $k < 1$, i.e., if $k = 0$, then $\rho \geq \alpha_{s_0}(z) = \beta + 1$ (see 2.5.(2)). This is immediate from $\rho m = \mathrm{v}_j > \mathrm{v}_i = \beta m$. □

3. Numerical invariants of semigroups generated by arithmetic sequences. In this section we give an explicit description of the type sequence of a semigroup generated by an arithmetic sequence. In addition to the notation, definitions and results of 1.1 and 2.1, we further fix the following notation.

Notation 3.1. Let $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 1$ be such that $\gcd(m, d) = 1$ and let p be an integer $p \geq 1$ and put $m_i := m + id$ for $i = 0, 1, \ldots, p+1$. Let $\Gamma := \sum_{i=0}^{p+1} \mathbb{N} m_i$ be the semigroup generated by the arithmetic sequence $m_0, m_1, \ldots, m_{p+1}$.

For any positive natural number $k \in \mathbb{N}^+$, let $q_k \in \mathbb{N}$ and $r_k \in [1, p+1]$ be the unique integers defined by the equation $k = q_k(p+1) + r_k$. We put $q := q_{m-1}$ and $r := r_{m-1} - 1$. Therefore, $q \in \mathbb{N}$, $r \in [0, p]$ and $m - 2 = q(p+1) + r$.

Put $s_0 = 0$ and $s_k := m_{r_k} + q_k m_{p+1} = (1+q_k)m + (r_k + q_k(p+1))\, d$ for $k \in [1, m-1]$. Further, we put $\mathrm{S}_1 := \{m_i + j m_{p+1} \mid i \in [1, p+1] \text{ and } j \in [0, q-1]\}$ and $\mathrm{S}_2 := \{m_i + q m_{p+1} \mid i \in [1, r+1]\}$. Note that $\mathrm{S}_1 = \emptyset$, if $q = 0$.

Proposition 3.2. *With the notations as in 3.1 we have:*

(1) The standard basis $\mathrm{S} := S_m(\Gamma)$ with respect to the multiplicity $m = m_0$ of Γ is:

$$\mathrm{S} = \{s_k \mid k \in [0, m-1]\} = \{0\} \cup \mathrm{S}_1 \cup \mathrm{S}_2.$$

(2) The conductor $c := c(\Gamma)$ and the degree of singularity $\delta := \delta(\Gamma)$ of Γ are:

$$c = (m-1)(d+q) + q + 1 \quad \textit{and} \quad \delta = ((m-1)(d+q) + (r+1)(q+1))\,/2.$$

(3) The set $\mathrm{T} := T(\Gamma) = \Gamma(1) \setminus \Gamma = \{m_i + q m_{p+1} - m_0 \mid i \in [1, r+1]\} = \{c - 1 - (r - i + 1)d \mid i \in [1, r+1]\}$. In particular, the Cohen–Macaulay type of Γ is $\tau := \tau_\Gamma = r + 1$.

Proof. (1) and (3) are special cases of the general results proved in [**7**], (3.5) and [**6**], § 5. (2) is proved in [**9**], § 3, Supplement 6. □

Now we give an explicit description of the positions of the holes of second type of Γ.

Lemma 3.3. *With the notations as in 2.1 and 3.1, we have:* $\operatorname{card}(\Gamma'') = (q+1)r$. *Moreover,* $\Theta := \{x - jm_{p+1} \mid x \in \Gamma(1)\setminus\Gamma, x \neq c-1$ *and* $j \in [0,q]\} = \Gamma''$.

Proof. Note that $\operatorname{card}(\Gamma'') = \operatorname{card}((\mathbb{N}\setminus\Gamma)) - \operatorname{card}(\Gamma'\cap\mathbb{N}) = \delta - n = 2\delta - c = (q+1)r$ by 2.1 and 3.2.(2). Therefore, since $|\Theta| = |\Gamma''|$, it is enough to prove that $\Theta \subseteq \Gamma''$. For this, let $x \in \Gamma(1)\setminus\Gamma$, $x \neq c-1$ and $j \in [0,q]$. Then $x - jm_{p+1} \geq c - 1 - rd - qm_{p+1} = d > 0$ and $x - jm_{p+1} \notin \Gamma$, since $x \notin \Gamma$ and $jm_{p+1} \in \Gamma$. Therefore, further it is enough to prove that $x - jm_{p+1} \notin \Gamma'$. Note that by 3.2.(3) $x = c-1-(r-i+1)d$ for some $i \in [1,r]$. Therefore, if $x - jm_{p+1} \in \Gamma'$, then $x - jm_{p+1} = c-1-h$ for some $h \in \Gamma$ and so $(r-i+1)d + jm_{p+1} = h \in \Gamma$. Now, adding m on both sides we get $s_k = m_{r-i+1} + jm_{p+1} = m + h \notin \mathrm{S}$ a contradiction, since $k = j(p+1) + r - i + 1 \leq q(p+1) + r + 1 \leq m-1$. This proves that $x - jm_{p+1} \notin \Gamma'$. □

Lemma 3.4. *Let* $j \in [0,q]$. *Then* $jm_{p+1} = s_{j(p+1)} \in \mathrm{S}$ *and* $jm_{p+1} < c$. *In particular,* $\{k \in [0, m-1] \mid s_k > jm_{p+1}\} \neq \emptyset$ *and if* $s_k > jm_{p+1}$ *for* $k \in [0, m-1]$, *then* $k > j(p+1)$.

Proof. Using 3.1 and 3.2, it is easy to verify that $jm_{p+1} = s_{j(p+1)} \in \mathrm{S}$ and $jm_{p+1} < c$. Further, since $c = s_{m-1} - m + 1$, we have $jm_{p+1} < s_{m-1}$ and hence the last assertion is clear. □

Proposition 3.5. *Let* $j \in [0,q]$, $x \in \Gamma(1)\setminus\Gamma$, $x \neq c-1$ *and let* $z = x - jm_{p+1}$. *Then* $\kappa(z) = \operatorname{Min}\{k \in [0,m-1] \mid s_k > jm_{p+1}\}$ *and* $\alpha_s(z) \in \{0,1\}$ *for all* $s \in \mathrm{S}$.

Proof. First note that, by 3.3, $z \in \Gamma''$ and, by 3.2.(3), $x = m_i + qm_{p+1} - m$ for some $i \in [1, r+1]$ and so $z + m = m_i + (q-j)m_{p+1} \in \Gamma$, since $j \leq q$ and so $z + m + s \in \Gamma$ for every $s \in \mathrm{S}$. In particular, $\alpha_s(z) \in \{0,1\}$.

If $j = 0$, then $z = x \in \Gamma(1)\setminus\Gamma$ and so $z + s \in \Gamma$ for every $s \in \mathrm{S}$, $s \neq 0$ and $z \notin \Gamma$. Therefore, $\kappa(z) = 1 = \operatorname{Min}\{k \in [0,m-1] \mid s_k > 0\}$.

Now assume that $j > 0$. Let $s_k \in \mathrm{S}$ be such that $s_k > jm_{p+1}$. Then $z + s_k =$

$$x + s_k - jm_{p+1} = \begin{cases} x + (q_k + 1 - j)m_{p+1}, & \text{if } k = q_k(p+1) + (p+1), \\ x + m_{r_k} + (q_k - j)m_{p+1}, & \text{if } k = q_k(p+1) + r_k,\ r_k \neq p+1. \end{cases}$$

Further, since $k > j(p+1)$ by 3.4, we have

$$\begin{cases} q_k + 1 > j, & \text{if } \ k = q_k(p+1) + (p+1), \\ q_k \geq j, & \text{if } \ k = q_k(p+1) + r_k, \ \ r_k \neq p+1. \end{cases}$$

Therefore, it follows that $z + s_k \in \Gamma$, since $x \in \Gamma(1)$. This proves that

$$\text{(3.5.a)} \qquad \alpha_{s_k}(z) = 0 \quad \text{for every} \quad s_k \in \mathrm{S} \quad \text{with} \quad s_k > jm_{p+1}.$$

Further, $\alpha_{jm_{p+1}}(z) \geq 1$, since $z + jm_{p+1} = x \notin \Gamma$. Therefore, by (3.5.a), we have:

$$\kappa(z) = \mathrm{Min}\{k \in [0, m-1] \mid \alpha_{s_k}(z) = 0\} = \mathrm{Min}\{k \in [0, m-1] \mid s_k > jm_{p+1}\}. \qquad \square$$

Definition 3.6. Let $0 = \mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ be elements of Γ such that $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$. For $i \in [0, n]$, the element $\mathrm{v}_i \in \Gamma$ is called the i-th element of Γ. Note that, by 3.4, for every $j \in [0, q]$, there exists a unique integer $\mathrm{i}(j) \in [0, n-1]$ such that $jm_{p+1} = \mathrm{v}_{\mathrm{i}(j)}$ is the $\mathrm{i}(j)$-th element of Γ.

Corollary 3.7. *Let $j \in [0, q]$ and let $\mathrm{i}(j) \in [0, n-1]$ be as in the definition 3.6. Then*

$$\Gamma\,(\mathrm{i}(j)+1) \setminus \Gamma\,(\mathrm{i}(j)) = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma\}.$$

In particular, $\mathrm{card}\,(\Gamma\,(\mathrm{i}(j)+1) \setminus \Gamma\,(\mathrm{i}(j))) = \tau_\Gamma = r + 1$.

Proof. First let $x \in \Gamma(1) \setminus \Gamma$, $x \neq c - 1$ and let $z = x - jm_{p+1}$. Then $z \in \Gamma''$ by 3.3. Further, since $jm_{p+1} = \mathrm{v}_{\mathrm{i}(j)}$ is the $\mathrm{i}(j)$-th element (see 3.6) of Γ, by 3.5 and 2.9, we have $z \in \Gamma\,(\mathrm{i}(j)+1) \setminus \Gamma\,(\mathrm{i}(j))$. Further, note that $c - 1 - jm_{p+1} = c - 1 - \mathrm{v}_{\mathrm{i}(j)}$ is the unique element of Γ' which belongs to $\Gamma\,(\mathrm{i}(j)+1) \setminus \Gamma\,(\mathrm{i}(j))$ by (2.3). Therefore, it follows from 3.3 that $\{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma\} = \Gamma\,(\mathrm{i}(j)+1) \setminus \Gamma\,(\mathrm{i}(j))$. Now the last assertion follows from 3.2.(3). $\square$

Theorem 3.8. *Let $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 1$ be such that $\gcd(m, d) = 1$ and let p be an integer with $1 \leq p \leq m - 2$. Let $\Gamma := \sum_{k=0}^{p+1} \mathbb{N} m_k$ be the semigroup generated by the arithmetic sequence $m_k := m + kd$, $k = 0, 1, \ldots, p+1$. Let $q \in \mathbb{N}$ and $r \in [0, p]$ be the unique integers defined by the equation $m - 2 = q(p+1) + r$. Further, let $c \in \Gamma$ be the conductor of Γ, $\mathbb{N}_c = \{z \in \mathbb{N} \mid z \geq c\}$ and let $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$ with $\mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$. Then the i-th term $\mathrm{t}_i = \mathrm{t}_i(\Gamma)$ of the type sequence $(\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n)$ of Γ is*

$$t_i = \begin{cases} 1, & \textit{if} \ \ \mathrm{v}_{i-1} \neq jm_{p+1} \ \ \textit{for every } j \in [0, q], \\ r + 1, & \textit{if} \ \ \mathrm{v}_{i-1} = jm_{p+1} \ \ \textit{for some } j \in [0, q]. \end{cases}$$

Proof. If $\mathrm{v}_{i-1} \neq jm_{p+1}$ for every $j \in [0,q]$, then $\Gamma(i) \setminus \Gamma(i-1) = \{c - 1 - v_{i-1}\}$ by 2.4 and 2.3, 3.3, 3.7 and hence $\operatorname{card}(\Gamma(i) \setminus \Gamma(i-1)) = 1$. If $\mathrm{v}_{i-1} = jm_{p+1}$ for some $j \in [0,q]$, then $\operatorname{card}(\Gamma(i) \setminus \Gamma(i-1)) = r+1$ by 3.7. □

Corollary 3.9. *In addition to the notations and assumptions as in* (3.8), *further assume that* $d = 1$. *Then the* i*-the term* t_i *of the type sequence* $(\mathrm{t}_1, \mathrm{t}_2, \dots, \mathrm{t}_n)$ *of* Γ *is*

$$t_i = \begin{cases} r+1, & \text{if } \ i = \binom{j+1}{2}(p+1) + j + 1 \ \text{ for some } j \in [0,q], \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Note that since $m_0, \dots, m_{p+1}$ is an arithmetic sequence, every element of Γ can be written uniquely in the form $am_0 + m_k + bm_{p+1}$ with $a, b \in \mathbb{N}$ and $k \in [0, p+1]$. Therefore, we have $\Gamma = \cup_{j \geq 0} \Gamma^{(j)}$, where $\Gamma^{(0)} := \{0\}$ and $\Gamma^{(j)} := \{am_0 + m_k + bm_{p+1} \mid (a,b) \in \mathbb{N}^2, k \in [0, p+1] \text{ and } a + b = j - 1\}$ for $j \geq 1$. Further, since $d = 1$, for every $j \geq 0$, elements of $\Gamma^{(j)}$ are consequtive positive integers, $\operatorname{Min}(\Gamma^{(j)}) = jm_0$, $\operatorname{Max}(\Gamma^{(j)}) = jm_{p+1}$ and $\operatorname{card}(\Gamma^{(j)}) = j(p+1)+1$. Furthermore, $\Gamma^{(j)} \cap \Gamma^{(j+1)} \neq \emptyset$ if and only if $j \geq q+1$. Therefore, for every $j \in [0,q]$, jm_{p+1} is the $(\mathrm{i}(j)-1)$-th element $\mathrm{v}_{\mathrm{i}(j)-1}$ in Γ, where $\mathrm{i}(j) := \operatorname{card}\left(\biguplus_{t=0}^{j} \Gamma^{(t)}\right) = \sum_{t=0}^{j} (t(p+1)+1) = \binom{j+1}{2}(p+1)+j+1$. Now the assertion is clear from 3.8. □

Corollary 3.10. *Let* m, d, p, q, r *and* Γ *be as in* 3.8 *and let* $R := K[\![\Gamma]\!]$ *be the semigroup ring of* Γ *over a field* K. *Then*

(1) R *is Gorenstein if and only if* $r = 0$.
(2) Assume that R *is not Gorenstein. Then* R *is almost Gorenstein if and only if* $m = p+2$. *Moreover, in this case we have* $\tau_R = m - 1$.

Proof. (1) R is Gorenstein if and only if $\tau_R = r + 1 = 1$, i.e. $r = 0$.
(2) R is almost Gorenstein if and only if the type sequence of R is $\tau_R = r+1, 1, \dots, 1$ or equivalently (by 3.8) $q = 0$, i.e. $m - 2 = r$. Now, since $m \geq p+2$ and $r \leq p$, we have $m - 2 = r$ if and only if $m - 2 = p$. □

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