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ON THE NAGATA AUTOMORPHISM

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Abstract. Let R be a commutative ring with unity of arbitrary characteristic. We give a direct proof that the Nagata automorphism σ of the ring $R[x,y,z]$: $\sigma(x) = x - 2y(xz+y^2) - z(xz+y^2)^2$, $\sigma(y) = y + z(xz+y^2)$, $\sigma(z) = z$ is a composition of six elementary automorphisms after the extension of it to the automorphism of $R[x,y,z,w]$ by setting $\sigma(w) = w$. We obtain an analogous result for the Anick automorphism of the free associative algebra $R\langle x,y,z\rangle$.

Introduction. Let R be a commutative ring with unity, of arbitrary characteristic, and $R[x_1,\dots,x_n]$ be the polynomials ring over R in the variables $x_1,\dots,x_n$. An automorphism φ of $R[x_1,\dots,x_n]$ is called *elementary* if it has the form

$$\begin{array}{rcll} \varphi(x_j) & = & x_j & \text{for} \quad j \neq i \\ \varphi(x_j) & = & x_i + f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) & \text{for} \quad j = i, \end{array}$$

where $i \in \{1,\dots,n\}$ and $f \in R[x_1,\dots,x_{i-1},x_{i+1},\dots,x_n]$. An automorphism ψ of $R[x_1,\dots,x_n]$ is called *linear* if

$$[\psi(x_1),\dots,\psi(x_n)] = [x_1,\dots,x_n]A$$

for some invertible matrix $A \in GL(n,R)$, where the right hand side is to be understood as matrix product. An automorphism φ of $R[x_1,\dots,x_n]$ is called *tame* if it is a composition of a finite number of elementary and linear automorphisms of $R[x_1,\dots,x_n]$.

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In 1972 (see [**6**], (1.1), Part 2) Masayoshi Nagata gave the following example of automorphism σ of the ring $R[x,y,z]$:

$$\begin{aligned}\sigma(x) &= x - 2y(xz+y^2) - z(xz+y^2)^2\\ \sigma(y) &= y + z(xz+y^2)\\ \sigma(z) &= z\end{aligned}$$

and he proved that σ is not tame as an automorphism of $D[x,y]$, where $D = R[z]$ ([**6**], Theorem 1.4, Part 2). Simultaneously, Nagata has conjectured that σ is not a tame automorphism of $R[x,y,z]$ ([**6**], Conjecture 3.1, Part 2). This conjecture was proved to be true by I. P. Shestakov and U. U. Umirbaev in [**7**], Corollary 9. Earlier in [**1**] H. Bass raised the question of whether Nagata's automorphism of the polynomial ring in three variables is *stably tame*, i.e., if it becomes tame after adding an additional variable w. M. Smith in [**8**] (also D. L. Wright in an unpublished note [MR1001475 (90f:13005)], see also [**5**], Corollary 6.1.5) proved that σ is indeed stably tame. In the proof of the above fact, M. Smith has used the exponential form of σ, i.e., $\sigma(r) = (\exp\Delta)(r) = \sum(1/i!)\Delta^i(r)$, where Δ is some locally nilpotent derivation of $R[x,y,z,w]$. Hence the proof remains valid only in the case of rings of characteristic zero. In the case of arbitrary characteristic, the M. Smith Theorem follows from Theorem 7 in E. Edo's paper [**4**].

In the paper, we give a direct proof of the M. Smith Theorem for a ring of arbitrary characteristic (Theorem 1, cf. [**9**], Theorem 1). At the end of this note, we prove an analogous result for the well-known Anick automorphism (see [**2**], p. 343) of the free associative algebra $R\langle x,y,z\rangle$ (Theorem 2).

1. Decomposition of the Nagata authomorphism.

THEOREM 1. *Let R be a commutative ring with unity, of arbitrary characteristic. Then the automorphism $\overline{\sigma}$ of $R[x,y,z,w]$ defined by*

$$\overline{\sigma}(x) = \sigma(x), \qquad \overline{\sigma}(y) = \sigma(y), \qquad \overline{\sigma}(z) = \sigma(z), \qquad \overline{\sigma}(w) = w$$

(under the obvious convention for characteristic 2) is a tame automorphism. Moreover, $\overline{\sigma}$ is a composition of six elementary automorphisms.

PROOF. Throughout the proof, one should substitute the number 0 for the number 2, when the ring R has characteristic 2. Let τ_1, τ_2, τ_3 be automorphisms of $R[x,y,z,w]$ of the following respective forms:

$$\begin{aligned}&\tau_1(x) = x, && \tau_2(x) = x + 2yw - zw^2, && \tau_3(x) = x,\\ &\tau_1(y) = y, && \tau_2(y) = y, && \tau_3(y) = y - zw,\\ &\tau_1(z) = z, && \tau_2(z) = z, && \tau_3(z) = z,\\ &\tau_1(w) = w + xz + y^2, && \tau_2(w) = w, && \tau_3(w) = w,\end{aligned}$$

Obviously, they are elementary automorphisms of $R[x,y,z,w]$ (for any ring R). Moreover, $\tau_1(w-(xz+y^2))=w$, so

$$\tau_1^{-1}(w)=w-(xz+y^2),$$

and obviously, $\tau_1^{-1}(x)=x$, $\tau_1^{-1}(y)=y$, $\tau_1^{-1}(z)=z$. Analogously, we obtain,

$$\tau_2^{-1}(x)=x-2yw+zw^2 \qquad \text{and} \qquad \tau_3^{-1}(y)=y+zw.$$

To complete the proof, it is sufficient to show that

$$\overline{\sigma}=\tau_2\circ\tau_3\circ\tau_1\circ\tau_3^{-1}\circ\tau_2^{-1}\circ\tau_1^{-1}.$$

To this end, we only need to prove that

$$(1) \qquad \overline{\sigma}\circ\tau_1\circ\tau_2\circ\tau_3\circ\tau_1^{-1}\circ\tau_3^{-1}\circ\tau_2^{-1}(a)=a \qquad \text{for} \qquad a\in\{x,y,z,w\}.$$

Since $\sigma(xz+y^2)=xz+y^2$ (see (1.2) in [**6**]), then

$$\begin{aligned}
\overline{\sigma}\circ\tau_1(x)&=\overline{\sigma}(x),\\
\overline{\sigma}\circ\tau_1(y)&=\overline{\sigma}(y),\\
\overline{\sigma}\circ\tau_1(z)&=z,\\
\overline{\sigma}\circ\tau_1(w)&=\tau_1(w)\ ,
\end{aligned}$$

thus

$$\begin{aligned}
\overline{\sigma}\circ\tau_1\circ\tau_2(x)&=x-2y(xz+y^2)-z(xz+y^2)^2\\
&\qquad +\ 2[y+z(xz+y^2)][w+(xz+y^2)]-z[w+(xz+y^2)]^2\\
&=x+2yw-zw^2=\tau_2(x),\\
\overline{\sigma}\circ\tau_1\circ\tau_2(y)&=\overline{\sigma}(y),\\
\overline{\sigma}\circ\tau_1\circ\tau_2(z)&=z,\\
\overline{\sigma}\circ\tau_1\circ\tau_2(w)&=\tau_1(w).
\end{aligned}$$

Hence

$$\begin{aligned}
\overline{\sigma}\circ\tau_1\circ\tau_2\circ\tau_3(x)&=\tau_2(x),\\
\overline{\sigma}\circ\tau_1\circ\tau_2\circ\tau_3(y)&=y+z(xz+y^2)-z[w+(xz+y^2)]=y-zw=\tau_3(y),\\
\overline{\sigma}\circ\tau_1\circ\tau_2\circ\tau_3(z)&=z,\\
\overline{\sigma}\circ\tau_1\circ\tau_2\circ\tau_3(w)&=\tau_1(w),
\end{aligned}$$

and in consequence

$$\overline{\sigma} \circ \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_1^{-1}(x) = \tau_2(x),$$
$$\overline{\sigma} \circ \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_1^{-1}(y) = \tau_3(y),$$
$$\overline{\sigma} \circ \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_1^{-1}(z) = z,$$
$$\overline{\sigma} \circ \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_1^{-1}(w) = w + (xz+y^2) - [(x+2yw-zw)z + (y-zw)^2] = w.$$

From the above it follows that $\overline{\sigma} \circ \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_1^{-1} \circ \tau_3^{-1} = \tau_2$, and so (1) is proved. □

REMARK 1. Let us consider the problem of the decomposition of an automorphism into elementary automorphisms, admitting triangular endomorphisms of $R[x, y, z]$ of the form

$$\alpha(x) = xf(y, z), \qquad \alpha(y) = yg(z), \qquad \alpha(z) = z,$$

where $f \in R[y, z]$, $g \in R[z]$. Then the Nagata automorphism σ has a decomposition into a finite number of elementary automorphisms and triangular endomorphisms of $R[x, y, z]$ of the following form:

$$\alpha \circ \sigma = \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1 \circ \alpha\ ,$$

where α is an endomorphism of $R[x, y, z]$ defined by

$$\alpha(x) = xz, \qquad \alpha(y) = yz, \qquad \alpha(z) = z,$$

and γ_1, γ_2 are elementary automorphisms of $R[z, y, z]$ of the following respective form:

$$\gamma_1(x) = x - y^2, \qquad \gamma_2(x) = x,$$
$$\gamma_1(y) = y, \qquad \gamma_2(y) = y + xz^2,$$
$$\gamma_1(z) = z, \qquad \gamma_2(z) = z.$$

Indeed, let φ be an endomorphism of $R[x, y, z]$ of the form

$$\varphi(x) = x - 2yz^2(x + y^2) - z^4(x + y^2)^2,$$
$$\varphi(y) = y + z^2(x + y^2),$$
$$\varphi(z) = z.$$

Then, it is easy to show that

$$\sigma \circ \alpha = \alpha \circ \varphi$$

and that $\varphi = \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$. In particular, φ is a tame automorphism of $R[x, y, z]$.

2. Decomposition of the Anick authomorphism. Let R be a commutative ring with unity, of arbitrary characteristic, and $R\langle x_1, \ldots, x_n\rangle$ be the free assiociative algebra over R with free generators $x_1, \ldots, x_n$. An automorphism φ of $R\langle x_1, \ldots, x_n\rangle$ is called *elementary* if it is of the form

$$\begin{array}{rcll} \varphi(x_j) & = & x_j & \text{for} \quad j \neq i \\ \varphi(x_j) & = & x_i + f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{for} \quad j = i, \end{array}$$

where $i \in \{1, \ldots, n\}$ and $f \in R\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\rangle$. An automorphism ψ of $R\langle x_1, \ldots, x_n\rangle$ is called *linear* if

$$[\psi(x_1), \ldots, \psi(x_n)] = [x_1, \ldots, x_n]A$$

for some invertible matrix $A \in GL(n, R)$, where the right hand side is to be understood as matrix product. An automorphism φ of $R\langle x_1, \ldots, x_n\rangle$ is called *tame* if it is a composition of finite number of elementary and linear automorphisms of $R\langle x_1, \ldots, x_n\rangle$.

Let us consider the Anick automorphism δ of the algebra $R\langle x, y, z\rangle$:

$$\begin{array}{rcl} \delta(x) & = & x + z(xz - zy) \\ \delta(y) & = & y + (xz - zy)z \\ \delta(z) & = & z. \end{array}$$

It is easy to observe that δ as an automorphism of $R[x, y, z]$, is tame. In the case of free associative algebras $R\langle x, y, z\rangle$ over field R of characteristic zero, the automorphism δ is not tame (see [**10, 3**]) and it is stably tame (see [**8**]). We prove that δ is stably tame for a ring R with unity, of arbitrary characteristic.

Theorem 2. *Let R be a commutative ring with unity, of arbitrary characteristic. Then the automorphism $\overline{\delta}$ of $R\langle x, y, z, w\rangle$ defined by*

$$\overline{\delta}(x) = \delta(x), \qquad \overline{\delta}(y) = \delta(y), \qquad \overline{\delta}(z) = \delta(z), \qquad \overline{\delta}(w) = w$$

is a tame automorphism. Moreover, $\overline{\delta}$ is a composition of six elementary automorphisms.

Proof. Let δ_1, δ_2, δ_3 be elementary automorphisms of $R\langle x, y, z, w\rangle$ of the following respective forms:

$$\begin{array}{lll} \delta_1(x) = x, & \delta_2(x) = x, & \delta_3(x) = x + zw, \\ \delta_1(y) = y, & \delta_2(y) = y + wz, & \delta_3(y) = y, \\ \delta_1(z) = z, & \delta_2(z) = z, & \delta_3(z) = z, \\ \delta_1(w) = w - xz + zy, & \delta_2(w) = w, & \delta_3(w) = w, \end{array}$$

Then we easily deduce that $\overline{\delta} = \delta_3^{-1} \circ \delta_2^{-1} \circ \delta_1^{-1} \circ \delta_3 \circ \delta_2 \circ \delta_1$. □

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