UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLV
2007

ON SOME APPLICATION OF THE BOCHENEK THEOREM

by Ludwik Byszewski

Dedicated to Professor Jan Bochenek on the occasion of his 80th birthday.

Abstract. The aim of this paper is to prove theorems about the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution Cauchy problem. The method of semigroups, Banach fixed-point theorem and Bochenek theorem about the existence and uniqueness of the classical solution of the linear first-order differential evolution problem in a not necessarily reflexive Banach space are used to prove the existence and uniqueness of the solutions of the problem considered. The results obtained are based on results by Balachandran and Ilamaran [**1**], Bochenek [**2**], Kato [**7**], Pazy [**8**], Winiarska [**11**] and the author [**3, 4**].

1. Introduction. In this paper, we prove two theorems on the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution Cauchy problem using the method of semigroups, Banach fixed-point theorem and Bochenek theorem (see [**2**]) about the existence and uniqueness of the classical solution of the linear first-order differential evolution problem in a not necessarily reflexive Banach space.

Let E be a real Banach space with norm $\|\cdot\|$ and let $A : E \to E$ be a closed densly defined linear operator. For an operator $A, D(A), \varrho(A)$ and A^* will denote its domain, resolvent set and adjoint, respectively.

For a Banach space $E, C(E)$ will denote the set of closed linear operators from E into itself.

We will need the class $G(\tilde{M}, \beta)$ of operators A satisfying the conditions:

There exist constants $\tilde{M} > 0$ and $\beta \in \mathbb{R}$ such that

(C_1) $\quad A \in C(E),\ \overline{D(A)} = E$ and $(\beta, +\infty) \subset \varrho(-A)$,

(C_2) $\quad \|(A+\varsigma)^{-k}\| \le \tilde{M}(\varsigma - \beta)^{-k}$ for each $\varsigma > \beta$ and $k = 1, 2, \ldots$

It is known (see Kato [**7**]) that for $A \in G(\tilde{M}, \beta)$ there exists exactly one strongly continuous semigroup $T(t) : E \to E$ for $t \geq 0$ such that $-A$ is its infinitesimal generator and

$$\|T(t)\| \leq \tilde{M}\, e^{\beta t} \ \text{ for } \ t \geq 0.$$

Throughout the paper we shall use the notation:

$$I = [0, a], \ \text{ where } \ a > 0,$$

$$M = \sup \{\|T(t)\|,\ t \in [0, a]\}$$

and

$$X = C(I, E).$$

The fuctional-diffrential evolution semilinear problem considered here is of the form

$$u'(t) + Au(t) = f(t, u(t), u(\sigma(t))),\ t \in (0, a], \tag{1.1}$$

$$u(0) + g(u) = u_0, \tag{1.2}$$

where $f : I \times E^2 \to E,\ g : X \to E,\ \sigma : I \to I$ are given functions satisfying some assumptions and u_0 is an element of E.

The results obtained are based on those by Balachandran and Ilamaran [**1**], Bochenek [**2**], Kato [**7**], Pazy [**8**], Winiarska [**11**] and the author [**3, 4**].

Some ordinary functional-differential equations were considered by Corduneanu [**5**], Hale [**6**], Pelczar and Szarski [**9**], and Przeworska-Rolewicz [**10**].

2. The Bochenek theorem. The results of this section were obtained by Professor Jan Bochenek (see [**2**]).

Let us consider the Cauchy problem

$$u'(t) + Au(t) = h(t), \ \ t \in I \setminus \{0\}, \tag{2.1}$$

$$u(0) = x. \tag{2.2}$$

A function $u : I \to E$ is said to be a *classical solution* of problem (2.1), (2.2) if:

(i) u is continuous on I and continuously differentiable on $I \setminus \{0\}$,
(ii) $u'(t) + Au(t) = h(t)$ for $t \in I \setminus \{0\}$,
(iii) $u(0) = x$.

Assumption (Z). The adjoint operator A^* is densly defined in E^*, i.e., $\overline{D(A^*)} = E^*$.

THEOREM 2.1. *Let conditions* (C_1), (C_2) *and Assumption* (Z) *be satisfied. Moreover, let* $h : I \to E$ *be Lipschitz continuous on* I *and* $x \in D(A)$.

Then Cauchy problem (2.1), (2.2) *has exactly one classical solution* u *given by the formula*

$$u(t) = T(t)x + \int_0^t T(t-s)h(s)ds, \ \ t \in I.$$

3. Theorem about a mild solution. A function $u \in X$ and satisfying the integral equation

$$u(t) = T(t)u_0 - T(t)g(u) + \int_0^t T(t-s)f(s, u(s), u(\sigma(s)))ds, \ t \in I,$$

is said to be a *mild solution* of semilinear Cauchy problem (1.1), (1.2).

THEOREM 3.1. *Assume that:*

(*i*) *the operator* $A : E \to E$ *satisfies conditions* (C_1) *and* (C_2),

(*ii*) $f : I \times E^2 \to E$ *is continuous with respect to the first variable on* $I, g : X \to E$, $\sigma : I \to I$ *is continuous on* I *and there exist positive constants* L *and* K *such that*

$$\|f(s, z_1, z_2) - f(s, \tilde{z}_1, \tilde{z}_2)\| \le L \sum_{i=1}^{2} \|z_i - \tilde{z}_i\| \quad \text{for} \quad s \in I, \ \ z_i, \tilde{z}_i \in E \quad (i = 1, 2) \tag{3.1}$$

and

$$\|g(w) - g(\tilde{w})\| \le K\|w - \tilde{w}\|_X \ \ \text{for} \ \ w, \tilde{w} \in X, \tag{3.2}$$

(*iii*) $M(2aL + K) < 1$,

(*iv*) $u_0 \in E$.

Then semilinear Cauchy problem (1.1), (1.2) *has a unique mild solution.*

PROOF. Introduce the operator F given by the formula

$$(Fw)(t) := T(t)u_0 - T(t)g(w) + \int_0^t T(t-s)f(s, w(s), w(\sigma(s)))ds, \ \ t \in I,$$

on the Banach space X.

It is easy to see that

$$F : X \to X. \tag{3.3}$$

Now we shall show that F is a contraction on X. For this purpose, consider the difference

$$(3.4)\qquad \begin{aligned}(Fw)(t)-(F\tilde{w})(t) = \int_0^t T(t-s)[f(s,w(s),w(\sigma(s)))\\ - f(s,\tilde{w}(s),\tilde{w}(\sigma(s)))]ds - T(t)[g(w)-g(\tilde{w})]\end{aligned}$$

for $w, \tilde{w} \in X$ and $t \in I$.

Then, from (3.4), (3.1) and (3.2),

$$(3.5)\qquad \begin{aligned}&\|(Fw)(t)-(F\tilde{w})(t)\| \le \|T(t)\|\|g(w)-g(\tilde{w})\|\\ &\quad + \int_0^t \|T(t-s)\|\|f(s,w(s),w(\sigma(s))) - f(s,\tilde{w}(s),\tilde{w}(\sigma(s)))\|ds\\ &\le ML\int_0^t (\|w(s)-\tilde{w}(s)\| + \|w(\sigma(s))-\tilde{w}(\sigma(s))\|)ds\\ &\quad + MK\|w-\tilde{w}\|_X \;\le\; M(2aL+K)\|w-\tilde{w}\|_X\end{aligned}$$

for $w, \tilde{w} \in X$ and $t \in I$.

If we define $q = M(2aL+K)$, then, by (3.5) and assumption (iii),

$$(3.6)\qquad \|Fw - F\tilde{w}\|_X \;\le\; q\|w-\tilde{w}\|_X \text{ for } w, \tilde{w} \in X$$

with $0 < q < 1$. This shows that the operator F is a contraction on X.

Consequently, by (3.3) and (3.6), operator F satisfies all the assumptions of the Banach theorem. Therefore, in space X there is the only one fixed point of F and this point is the mild solution of semilinear Cauchy problem (1.1), (1.2). So, the proof of Theorem 3.1 is complete. □

4. Theorem about a classical solution.

A function $u : I \to E$ is said be a *classical solution* of semilinear Cauchy problem (1.1), (1.2) if:

(i) u is continuous on I and continuously differentiable on $I\backslash\{0\}$,
(ii) $u'(t) + Au(t) = f(t, u(t), u(\sigma(t)))$ for $t \in I\backslash\{0\}$,
(iii) $u(0) + g(u) = u_0$.

THEOREM 4.1. *Suppose that:*

(i) *the operator* $A : E \to E$ *satisfies conditions* (C_1) *and* (C_2), *and Assumption* (Z),
(ii) $f : I \times E^2 \to E$, $g : X \to E$, $\sigma : I \to I$ *is continuous on* I *and there exist positive constants* C *and* K *such that*

$$(4.1)\qquad \begin{aligned}&\|f(s,z_1,z_2) - f(\tilde{s},\tilde{z}_1,\tilde{z}_2)\| \;\le\; C(|\, s-\tilde{s}\,| + \sum_{i=1}^{2}\|z_i-\tilde{z}_i\|)\\ &\textit{for}\quad s,\tilde{s} \in I,\; z_i,\tilde{z}_i \in E \quad (i=1,2)\end{aligned}$$

and

$$\|g(w) - g(\tilde{w})\| \le K\|w - \tilde{w}\|_X \quad \textit{for} \quad w, \tilde{w} \in X.$$

(iii) $M(2aC + K) < 1$.

Then semilinear Cauchy problem (1.1), (1.2) *has a unique mild solution* u. *Moreover, if* $u_0 \in D(A)$ *and* $g(u) \in D(A)$, *and if there exists a positive constant* κ *such that*

$$\|u(\sigma(s)) - u(\sigma(\tilde{s}))\| \le \kappa\|u(s) - u(\tilde{s})\| \textit{ for } s, \tilde{s} \in I \tag{4.2}$$

then u *is the unique classical solution of problem* (1.1), (1.2).

Proof. Since all the assumptions of Theorem 3.1 are satisfied, it is easy to see that semilinear Cauchy problem (1.1), (1.2) possesses a unique mild solution, which, in line with the last assumption, we denote by u.

Now, we shall show that u is the classical solution of problem (1.1), (1.2). To this end, introduce

$$N := \max_{s \in I} \|f(s, u(s),\ u(\sigma(s)))\| \tag{4.3}$$

and observe that

$$\begin{aligned} u(t+h) - u(t) &= [T(t+h)u_0 - T(t)u_0] - [T(t+h)g(u) - T(t)g(u)] \\ &\quad + \int_0^h T(t+h-s)f(s,u(s),u(\sigma(s)))ds \\ &\quad + \int_h^{t+h} T(t+h-s)f(s,u(s),u(\sigma(s)))ds \\ &\quad - \int_0^t T(t-s)f(s,u(s),u(\sigma(s)))ds \\ &= T(t)[T(h)-J]u_0 - T(t)[T(h)-J]g(u) \\ &\quad + \int_0^h T(t+h-s)f(s,u(s),u(\sigma(s)))ds \\ &\quad + \int_0^t T(t-s)[f(s+h,u(s+h),u(\sigma(s+h))) \\ &\quad - f(s,u(s),u(\sigma(s)))ds \end{aligned} \tag{4.4}$$

for $t \in [0,a), h > 0$ and $t + h \in (0,a]$.

Consequently, by (4.4), (4.3), (4.1) and (4.2),

$$\begin{aligned} \|u(t+h) - u(t)\| &\le hM\|Au_0\| + hM\|Ag(u)\| + hMN + MCah \\ &\quad + MC\int_0^t (\|u(s+h) - u(s)\| + \|u(\sigma(s+h)) - u(\sigma(s))\|)ds \\ &= C_*h + MC(1+\kappa)\int_0^t \|u(s+h) - u(s)\|ds \end{aligned} \tag{4.5}$$

for $t \in [0,a), h > 0$ and $t+h \in (0,a]$, where $C_* = M[\|Au_0\| + \|Ag(u)\| + N + aC]$.

From (4.5) and Gronwall's inequality, there follows

$$\|u(t+h) - u(t)\| \leq C_* e^{aMC(1+\kappa)} h$$

for $t \in [0,a), h > 0$ and $t+h \in (0,a]$. Hence u is Lipschitz continuous on I.

The Lipschitz continuity of u on I combined with the Lipschitz continuity of f on $I \times E^2$ implies that $t \to f(t, u(t), u(\sigma(t)))$ is Lipschitz continuous on I. This property of f together with the assumptions of Theorem 4.1 imply, by Bochenek Theorem 2.1 and by Theorem 3.1, that the linear Cauchy problem

$$\begin{aligned} &v'(t) + Av(t) = f(t, u(t), u(\sigma(t))), \quad t \in I \backslash \{0\}, \\ &v(0) = u_0 - g(u), \end{aligned}$$

which can be written in the equivalent form as:

$$\begin{aligned} &v'(t) + Av(t) = f(t, u(t), u(\sigma(t))), \quad t \in I \backslash \{0\}, \\ &v(0) + g(u) = u_0, \end{aligned}$$

has a unique classical solution v such that

$$v(t) = T(t)u_0 - T(t)g(u) + \int_0^t T(t-s) f(s, u(s), u(\sigma(s))) ds = u(t), \quad t \in I.$$

Consequently, u is the unique classical solution of semilinear Cauchy problem (1.1), (1.2) and therefore the proof of Theorem 4.1 is complete. □

Remark. If equation (1.1) does not depend on the functional argument and $g \equiv 0$, then Theorem 4.1 is a particular case of the Bochenek theorem (see [**2**], Theorem 2).

References

1. Blachandran K., Ilamaran S., *Existence and uniqueness of mild and strong solutions of a semilinear evolution equation with nonlocal conditions,* Indian J. Pure Appl. Math., **25.4** (1994), 411–418.
2. Bochenek J., *The existence of a solution of a semilinear first-order differential equation in a Banach space,* Univ. Iagel. Acta Math., **31** (1994), 61–68.
3. Byszewski L., *Differential and Functional-Differential Problems Together with Nonlocal Conditions,* Cracow University of Technology, Monograph 184, Cracow, 1995 (in Polish).
4. Byszewski L., *Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem,* Selected Problems of Mathematics, Anniversary Issue 6, Cracow, 1995, 25–33.
5. Corduneanu C., *Integral Equations and Applications,* Cambridge University Press, Cambridge, 1991.
6. Hale J., *Theory of Functional Diferential Equations,* Springer-Verlag, New York, Berlin, Heidelberg, 1977.

7. Kato T., *Perturbation Theory of Linear Operators,* Springer-Verlag, New York, Berlin, Heidelberg, 1966.
8. Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
9. Pelczar A., Szarski J., *Introduction to the Theory of Differential Equations,* Vol. I, Polish Scientific Publishers, Warsaw, 1987 (in Polish).
10. Przeworska-Rolewicz D., *Equations with Transformed Argument – an Algebraic Approach,* Polish Scientific Publishers, Warsaw, 1973.
11. Winiarska. T., *Differential Equations with Parameter,* Monograph 68, Technical University of Cracow, Cracow, 1988.

Received March 9, 2007

Institute of Mathematics
Cracow University of Technology
ul. Warszawska 24
31-155 Kraków,
Poland
e-mail: lbyszews@usk.pk.edu.pl