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BETA-REGRESSION MODEL FOR PERIODIC DATA WITH A TREND

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Abstract. In this paper, we prove that there exists exactly one maximum likelihood estimator for the beta-regression model, where beta distributed dependent variable is periodic with a trend. This is an important generalization of the result obtained by Dawidowicz ([**3**]). The model is useful when the dependent variable is continuous and restricted to a bounded interval. In such a model the classical regression should not be applied. The parameters are obtained by maximum likelihood estimation. We test a hypothesis of periodicity against the trend. An AIC is used to decide whether the hypothesis should be rejected or not. We analyze the goodness-of-fit sensitivity. We consider diagnostic techniques that can be used to identify departures from the postulated model and to identify influential observations.

1. Introduction. The linear regression model is widely used in applications to analyze data that is considered to be related to other variables. It should, however, not be used, in models, where dependent data is restricted to the interval $[0, 1]$. The dependence on time might be described as a combination of a cyclic and linear function. The term "beta regression" was defined by Dawidowicz, Stanuch and Kawalec at the ISCB conference in Stockholm in 2001 [**4**]. The Generalized Linear Model applied to beta regression is widely discussed in Ferrari, Cribari-Neto [**6**]. The application of small sample bias adjustments to the maximum likelihood estimators of parameters is discussed by Cribari-Neto and Vasconcellos [**2**].

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The aim of this article is to present a beta-regression model for periodic data with a trend and to prove that there exists exactly one maximum likelihood estimator for the beta-regression model for periodic data with a linear trend. This is an important generalization of the result obtained by Dawidowicz [**3**]. The paper is organised as follows. In Section 2, we present the beta-regression model. In Section 3, we discuss maximum likelihood estimation. Diagnostics measures are presented in Section 4.

2. Statement of the problem. The proposed model is based on the assumption that the dependent data is beta distributed. The beta density is given by

$$f(x,\varphi,r)=\frac{\Gamma(r)}{\Gamma(r\varphi)\Gamma((1-\varphi)r)}x^{r\varphi-1}(1-x)^{(1-\varphi)r-1}, 0\le x\le 1,$$

where $0<\varphi<1$, $r>0$ and $\Gamma(\cdot)$ is the gamma function. There is $E(x)=\varphi$ and $Var(x)=\frac{\varphi(1-\varphi)}{1+r}$.

Let $x_1,x_2,\dots,x_n$ be independent, beta distributed random variables. In the model it is assumed that the mean of the dependant variable has the form

$$E(x_j)=\varphi(t_j),\quad j=1,2,\dots,n,$$

where φ is the sum of cyclic function of period T and the monotonic function θ. r is an unknown precision parameter. The t_j's may be interpreted as time points.

We can restrict data to the interval $[0,1]$, so we consider the model, where

$$0\le x_j\le 1\quad j=1,2,\dots,n,$$

and

$$E(x_j)=\varphi(t_j)=\theta(t_j)+\beta_0+\sum_{k=1}^{p}\left(\alpha_k\sin\frac{2\pi k}{T}t_j+\beta_k\cos\frac{2\pi k}{T}t_j\right),$$

where $0\le\varphi(t_j)\le 1$.

The $\theta(\cdot)$ function is a strictly monotonic and differentiable function that maps $\mathbf{R}$ into $[0,1]$. Moreover, the θ function is twice continuously differentiable with respect to parameters. The $\theta(\cdot)$ function is responsible for modelling the trend. There are several possible choices of $\theta(\cdot)$ function. For instance, we can use the inverse logit function $\theta(t)=\exp(at)/(C+\exp(at))$, the inverse probit function $\theta(t)=\Phi(at+C)$, where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable, the inverse log-log functions $\theta(t)=\exp(-\exp(at+C))$ and $\theta(t)=1-\exp(-\exp(at+C))$, where $a>0$ and $C\in\mathbf{R}$.

In the paper we assume that a trend function is simpler, i.e., a linear function that maps a bounded interval which contains all t_j's from the model into $[0,1]$, that is $\theta(t) = at$.

Let $\bar{b} = (A, A_1, A_2, \ldots, A_p, B_0, B_1, \ldots, B_p)$ and let

$$T(\bar{b}, t) = At + B_0 + \sum_{k=1}^{p} \left(A_k \sin kt + B_k \cos kt\right).$$

The likelihood function in the beta regression model has the form

$$\begin{aligned} &L(t_1, t_2, \ldots, t_n, x_1, x_2, \ldots, x_n, \bar{b}, r) \\ &\qquad = \prod_{j=1}^{n} \frac{1}{B(T(\bar{b}, t_j), r - T(\bar{b}, t_j))} x_j^{T(\bar{b}, t_j)-1} (1 - x_j)^{r - T(\bar{b}, t_j) - 1}, \end{aligned} \tag{1}$$

where $B(\cdot, \cdot)$ denotes the beta function. The log-likelihood function in the beta regression model has the form

$$\begin{aligned} \ln L = \sum_{j=1}^{n} \Big(&- \ln B(T(\bar{b}, t_j), r - T(\bar{b}, t_j)) \\ &+ (T(\bar{b}, t_j) - 1) \ln x_j + (r - T(\bar{b}, t_j) - 1) \ln(1 - x_j) \Big). \end{aligned}$$

We shall rewrite discussed parameters. Let $b = (a, \alpha_1, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_p) = (b_a, b_1, \ldots, b_{2p+1})$, where $a = \frac{A}{r}$, $\alpha_k = \frac{A_k}{r}$ and $\beta_k = \frac{B_k}{r}$. Let

$$\varphi(b, t_j) = \frac{T(\bar{b}, t_j)}{r} = at_j + \beta_0 + \sum_{k=1}^{n} (\alpha_k \sin kt_j + \beta_k \cos kt_j).$$

Now

$$l = \ln L = \sum_{j=1}^{n} \bar{l}_j,$$

where

$$\begin{aligned} \bar{l}_j = &- \ln B(r\varphi(b, t_j), r(1 - \varphi(b, t_j))) \\ &+ (r\varphi(b, t_j) - 1) \ln x_j + (r(1 - \varphi(b, t_j)) - 1) \ln(1 - x_j). \end{aligned}$$

3. The maximum likelihood estimation.

Lemma 3.1. *The function* $\ln B(x, y)$ *is a convex function in* x *and* y.

LEMMA 3.2. *Let $[c,d]$ be a closed and bounded interval. The set $\overline{\boldsymbol{A}}$ of all such $(A, A_1, A_2, \dots, A_p, B_0, B_1, \dots, B_p, r) \in \boldsymbol{R}^{2p+3}$ that for every $x \in [c,d]$,*

$$0 \le Ax + B_0 + \sum_{k=1}^{p} (A_k \sin kx + B_k \cos kx) \le r$$

is closed and convex in $\boldsymbol{R}^{2p+3}$.

Proof of Lemma 3.1 is in Dawidowicz [**3**]. Proof of Lemma 3.2 is analogous to the proof in Dawidowicz [**3**].

LEMMA 3.3. *The set $\boldsymbol{A}$ of all $b = (a, \alpha_1, \alpha_2, \dots, \alpha_p, \beta_0, \beta_1, \dots, \beta_p) \in \boldsymbol{R}^{2p+2}$ satisfying the condition*

$$0 \le ax + \beta_0 + \sum_{k=1}^{p} (\alpha_k \sin kx + \beta_k \cos kx) \le 1 \tag{2}$$

for every $x \in \boldsymbol{R}$ is compact in $\boldsymbol{R}^{2p+2}$.

PROOF. Let

$$f_b(x) = ax + \beta_0 + \sum_{k=1}^{p} (\alpha_k \sin kx + \beta_k \cos kx)\,.$$

From inequality (2) it follows that for every $x \in [-\pi, \pi]$

$$-1 \le f_b(x) \sin kx \le 1 \qquad k = 1, 2, \dots, p$$

and

$$-1 \le f_b(x) \cos kx \le 1 \qquad k = 0, 1, \dots, p.$$

Integrating all these inequalities on the interval $[-\pi, \pi]$, we obtain

$$-1 \le \beta_0 \le 1 \qquad -2 \le \beta_k \le 2 \qquad k = 1, 2, \dots, p \tag{3}$$

and

$$-2(1+|a|) \le -2\left(1+\frac{|a|}{k}\right) \le \alpha_k \le 2\left(1+\frac{|a|}{k}\right) \le 2(1+|a|) \qquad k = 1, 2, \dots, p. \tag{4}$$

Substituting $x = \pi$ into (2) and using (3), we obtain

$$-\frac{2p+2}{\pi} \le a \le \frac{2p+2}{\pi}. \tag{5}$$

From inequalities (3), (4) and (5) there follows that the set $\mathbf{A}$ is bounded. The closedness is a natural consequence of it being defined by weak inequalities. $\square$

Lemma 3.4. *Exactly one of the following two conditions holds true*

1. For all $j = 1, 2, \ldots, n$

$$x_j = at_j + \beta_0 + \sum_{k=1}^{p} \left(\alpha_k \sin kt_j + \beta_k \cos kt_j\right).$$

2.

$$\lim_{r\to\infty} \frac{d}{dr} \sum_{j=1}^{n} \Big(-\ln B\big(r\varphi(b,t_j), r(1-\varphi(b,t_j))\big) + (r\varphi(b,t_j)-1)\ln x_j + (r(1-\varphi(b,t_j))-1)\ln(1-x_j)\Big) = -\infty.$$

Lemma 3.5. *The function* L *as a function in* $(\bar{b}, r)$ *is concave.*

Theorem 3.1. *For given* $t_1, t_2, \ldots, t_n \in [c,d]$ *and* $x_1, x_2, \ldots, x_n$, *exactly one of the following two conditions holds true*

1. There exist such $a, \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_p$ *that for all* $j = 1, 2, \ldots, n$

$$x_j = at_j + \beta_0 + \sum_{k=1}^{p} \left(\alpha_k \sin kt_j + \beta_k \cos kt_j\right).$$

2. There exists exactly one $(\widehat{\bar{b}}, \widehat{r}) \in \overline{\boldsymbol{A}}$ *such that*

$$L(\widehat{\bar{b}}, \widehat{r}) = \max_{(\bar{b},r)\in\overline{\boldsymbol{A}}} L(\bar{b}, r),$$

where L *is a likelihood function defined in 1.*

Proofs of Lemmas 3.4 and 3.5, as well as a proof of Theorem 3.1 are in Dawidowicz [**3**]. To prove Theorem 3.1, we need Lemmas 3.1–3.5.
We shall then obtain an expression for Fisher's information matrix.

Theorem 3.2. *Let* M *denote Fisher's information matrix. Then*

$$M = \begin{bmatrix} M_{b,b} & M_{b,r} \\ M_{r,b} & M_{r,r} \end{bmatrix},$$

where

$$M_{r,r} = -\frac{\partial^2 l}{\partial r^2}(b,r) = \sum_{j=1}^{n} \big(-\Psi'(r) + \varphi^2(b,t_j)\Psi'(\varphi(b,rt_j)) + (1-\varphi(b,t_j))^2\Psi'((1-\varphi(b,t_j))r)\big),$$

$$M_{b,r}^T = \left[\sum_{j=1}^{n} \frac{\partial\varphi(b,t_j)}{\partial b_a} Z, \sum_{j=1}^{n} \frac{\partial\varphi(b,t_j)}{\partial b_1} Z, \ldots, \sum_{j=1}^{n} \frac{\partial\varphi(b,t_j)}{\partial b_{2p+1}} Z\right],$$

where

$Z = Z(\varphi(b,t_j),r) = r\left[\varphi(b,t_j)\Psi'(\varphi(b,t_j)r) + (1-\varphi(b,t_j))\Psi'((1-\varphi(b,t_j))r)\right]$

and

$$W(\varphi(b,t_j),r) = \Psi((1-\varphi(b,t_j))r) - \Psi(\varphi(b,t_j)r) + \ln\frac{x(t_j)}{1-x(t_j)}.$$

$M_{b,b}$ *matrix elements* $m_{u,w},\ u,w = a,1,\dots,2p+1,$ *are of the form*

$$m_{u,w} = -r^2\sum_{j=1}^{n}\frac{\partial\varphi(b,t_j)}{\partial b_u}\frac{\partial\varphi(b,t_j)}{\partial b_w}G(\varphi(b,t_j),r),$$

where $G(\varphi(b,t_j),r) = -\Psi'(\varphi(b,t_j)r) - \Psi'((1-\varphi(b,t_j))r)$ *and* $\Psi(x) = \frac{d\ln\Gamma(x)}{dx}$.

Proof. Each $\varphi(b,t_j)$ is twice continuously differentiable with respect to parameter b. Since

$$E\left(\frac{\partial\bar{l}_j}{\partial b_u}\right) = rE\left(W(\varphi(b,t_j),r)\right) = 0,\quad u = a,1,2,\dots,2p+1,$$

and

$$E\left(\frac{\partial^2\bar{l}_j}{\partial b_u\partial b_w}\right) = rE\left(\frac{\partial^2\varphi(b,t_j)}{\partial b_u\partial b_w}W(\varphi(b,t_j),r)\right) + r^2E\left(\frac{\partial\varphi(b,t_j)}{\partial b_u}\frac{\partial\varphi(b,t_j)}{\partial b_w}G(\varphi(b,t_j),r)\right),$$

then under our assumptions, we obtain

$$E\left(\frac{\partial^2\bar{l}_j}{\partial b_u\partial b_w}\right) = r^2\frac{\partial\varphi(b,t_j)}{\partial b_u}\frac{\partial\varphi(b,t_j)}{\partial b_w}G(\varphi(b,t_j),r).$$

Hence

$$m_{u,w} = -r^2\sum_{j=1}^{n}\frac{\partial\varphi(b,t_j)}{\partial b_u}\frac{\partial\varphi(b,t_j)}{\partial b_w}G(\varphi(b,t_j),r).$$

Under our regularity assumptions, the matrix M is symmetric and

$$M_{r,b} = M_{b,r}^T.$$

After simple computations, we obtain the formulas for $M_{r,r}$ and $M_{b,r}^T$. □

Theorem 3.3. *The inverse of Fisher's information matrix is of the form*

$$M^{-1} = \begin{bmatrix} M_{b,b}^{-1} + \left(M_{b,b}^{-1}M_{b,r}\right)E^{-1}\left(M_{b,b}^{-1}M_{b,r}\right)^T & -E^{-1}M_{b,b}^{-1}M_{b,r} \\ -E^{-1}\left(M_{b,b}^{-1}M_{b,r}\right)^T & E^{-1}\end{bmatrix},$$

where $E = M_{r,r} - M_{b,r}^TM_{b,b}^{-1}M_{b,r} \in \boldsymbol{R}$.

One can prove the theorem using known facts of algebra, to be found, e.g., in Rao [**8**].

Under the same assumptions, we can find the matrix $M_{b,b}^{-1}$ using well-known algebraic recurrence formulas.

THEOREM 3.4. *The $MLE(b)$ and $MLE(r)$ are (assumed to be unique) maximum likelihood estimators of b and r, respectively. Their asymptotical distribution is*

$$\begin{pmatrix} MLE(b) \\ MLE(r) \end{pmatrix} \sim N_{2p+3}\left(\begin{pmatrix} b \\ r \end{pmatrix}, \quad M^{-1} \right),$$

where $2p+3$ is the number of estimated parameters.

Proof of this well known result can be found, e.g., in Stuart, Ord and Arnold [**9**]. The assumed uniqueness is a consequence of Theorem 3.1.

4. Analysis and diagnostics for the model. It is well known that, under some regularity assumptions, maximum likelihood estimators are consistent and asymptotically efficient. Fitting of the model should be followed by diagnostic analysis, which would check the goodness-of-fit of the evaluated model. Ferrari and Cribari-Neto [**6**] considered the correlation between the observed and predicted values as a basis for a measure of goodness-of-fit. Unfortunately, the statistic does not take into account the effect of dispersion covariates.

DEFINITION 4.1. Akaike's information criterion is

$$AIC = -2\left(l(b,r) - (2p+3)\right),$$

where $2p+3$ is the number of estimated parameters.

The model with minimum AIC is studied more thoroughly than other models (Akaike, [**1**]). Thus we obtain the set of AIC-optimal parameters. Subsequently, we obtain the number of harmonics. Evaluating AIC is a method of determining the best model when several models fit to the same data. When we use AIC, we do not require the models compared to be nested.

Let φ^b denote the mean of beta-distributed random variables parametrized with b and let b_m, b_n denote the set b with m and n parameters, respectively. We want to test the hypothesis

$$H_0: \quad \varphi^{b_m} = \varphi^{b_n}$$

versus

$$H_1: \quad \varphi^{b_m} \neq \varphi^{b_n}.$$

The AIC is used to verify the hypothesis.

Lemma 4.1. *For every $m > n$, under the assumption of hypothesis H_0, the statistics*

$$\chi^2_{AIC} = |AIC_m - AIC_n|$$

has asymptotically the chi-square distribution with $2(m-n)$ degrees of freedom.

A proof can be found in Akaike [**1**].

χ^2 is a measure of the goodness-of-fit of the model. It measures the relative deviations between the observed and the fitted values. Large individual components indicate observations not well accounted for by the model.
The discrepancy of fit can also be computed by residuals.

Definition 4.2. With the above notations, the residuals are

$$r_j = x_j - \varphi(MLE(b), t_j), \qquad j = 1, \dots, n.$$

The observation with a large absolute value of r_j may be considered discrepant. We can also define the standardized residuals.

Definition 4.3. With the above notations, the standardized residuals are

$$r_j^s = \frac{x_j - \varphi(MLE(b), t_j)}{\sqrt{Var(x_j)}},$$

where

$$Var(x_j) = \frac{\varphi(MLE(b), t_j)\,(1 - \varphi(MLE(b), t_j))}{1 + MLE(r)}, \qquad j = 1, \dots, n.$$

Generalized leverage can be used as a measure for assessing the importance of individual observations. We will use the generalized leverage proposed by Wei, Hu and Fung [**10**]. Let $x = (x_1, \dots, x_n)^T$ be a vector of observable responses. The expectation of x is $m = E(x)$ and can be expressed as $m = m(\alpha)$. Let $M(\alpha) = M(\alpha(x))$ denote an estimator of α. Then $M(x) = m(M(\alpha))$ is the predicted response vector.

Definition 4.4. With the above notations, the generalized leverage of estimator $M(\alpha)$ is defined as

$$GL(M(\alpha)) = \frac{\partial M(x)}{\partial x^T}.$$

By the definition, the (i, j) element of the matrix $GL(M(\alpha))$ is the instantaneous rate of change of the i-th predicted value with respect to the j-th response value. In other words, it measures the influence of observations on the fit of the model under the estimator $M(\alpha)$. The observations with large

$$GL_{(i,i)} = \frac{\partial M(x_i)}{\partial x_i^T}$$

are called leverage points.

Theorem 4.1. *If $l(b,x)$ has second order continuous derivatives with respect to b and x and $MLE(b)$ exists uniquely, then the generalized leverage of maximum likelihood estimator of b in the beta regression model with known r is*

$$GL(b) = -\frac{\partial \varphi}{\partial b^T} M_{b,b}^{-1} \frac{\partial^2 l(b,r)}{\partial b \partial x^T},$$

where $\varphi = (\varphi(b,t_1), \ldots, \varphi(b,t_n))$ and (u,v)th element of matrix $\frac{\partial^2 l(b,r)}{\partial b \partial x^T}$ is

$$\left(\frac{\partial^2 l(b,r)}{\partial b \partial x^T}\right)_{(u,v)} = r \sum_{j=1}^{n} \frac{\frac{\partial \varphi(b,t_j)}{\partial b_u}}{x_v(1-x_v)}, \quad u = a, 1, 2, \ldots, 2p+1 \text{ and } v = 1, 2, \ldots, n.$$

Let now r be unknown. The generalized leverage of maximum likelihood estimator of b in the beta regression model is

$$GL(b,r) = -\frac{\partial \varphi}{\partial (b,r)^T} M^{-1} \frac{\partial^2 l(b,r)}{\partial (b,r) \partial x^T},$$

and the elements of the last row of the matrix $\frac{\partial^2 l(b,r)}{\partial b \partial x^T}$ are

$$\left(\frac{\partial^2 l(b,r)}{\partial b \partial x^T}\right)_{(2p+3,v)} = \sum_{j=1}^{n} \frac{\varphi(b,t_j) - x_v}{x_v(1-x_v)}, \quad v = 1, 2, \ldots, n.$$

A proof is a consequence of a result obtained by Wei, Hu and Fung [**10**]. Let the null hypothesis for a given b^{m0} be $H_0 : b^m = b^{m0}$ and the alternative hypothesis be $H_1 : b^m \neq b^{m0}$, where b^m and b^{m0} are m-vectors and $m < 2p+3$. In order to check the asymptotic inference, we can perform Rao's score test. Let $S_m(b,r)$ denote the vector containing m out of the first $2p+3$ coefficients of score function $S(b,r)$, and let $M_{m,b,b}^{-1}$ be the matrix formed of the corresponding m rows and m columns of the matrix $M_{b,b}^{-1}$.

Definition 4.5. Rao's score statistic is

$$T_R = \left(S_m(MLE_0(b), MLE_0(r))\right)^T M_{m,b,b}^{-1} S_m(MLE_0(b), MLE_0(r)),$$

where $MLE_0(b)$ and $MLE_0(r)$ are restricted maximum likelihood estimators, computed under H_0.

It is well known (see e.g. Stuart, Ord and Arnold, [**9**]) that under the regularity conditions and the assumption of hypothesis H_0, the statistics asymptotically has the chi-square distribution with m degrees of freedom.

The hypothesis can also be tested with Wald's test.

Definition 4.6. Wald's statistic takes the form of

$$T_W = \left(MLE(b^m) - b^{m0}\right)^T M_{m,b,b}^{-1}(MLE(b), MLE(r)) \left(MLE(b^m) - b^{m0}\right).$$

Similarly, under the regularity conditions and the assumption of hypothesis H_0, the statistics asymptotically has the chi-square distribution with m degrees of freedom (see e.g. Stuart, Ord and Arnold, [**9**]).

For testing the significance of the single parameter b_u, $u = a, 1, \ldots, 2p+1$, we may use the statistic $T_W = (MLE(b_u))^2 m_{u,u}^{-1}$, where $m_{u,u}^{-1}$ is the (u,u)-th element of the matrix $M^{-1}(MLE(b), MLE(r))$. The square root of T_W asymptotically has standard normal distribution (see e.g. Stuart, Ord and Arnold, [**9**]).

We can determine the appopriate confidence intervals (see e.g. Stuart, Ord and Arnold, [**9**]).

Lemma 4.2. *The* $(1-\alpha)100\%$ *confidence interval for the single parameter* b_u*, where* $u = a, 1, \ldots, 2p+1$*, is*

$$\left(MLE(b_u) - \Phi^{-1}(1-\frac{\alpha}{2})\sqrt{m_{u,u}^{-1}}, MLE(b_u) + \Phi^{-1}(1-\frac{\alpha}{2})\sqrt{m_{u,u}^{-1}}\right).$$

The asymptotic $(1-\alpha)100\%$ *confidence interval for parameter* r *is*

$$\left(MLE(r) - \Phi^{-1}(1-\frac{\alpha}{2})\sqrt{m_{2p+3,2p+3}^{-1}}, MLE(r) + \Phi^{-1}(1-\frac{\alpha}{2})\sqrt{m_{2p+3,2p+3}^{-1}}\right),$$

where $m_{2p+3,2p+3}^{-1}$ *is equal to* $(2p+3, 2p+3)$*-th element of the inverse of Fisher information matrix calculated at the maximum likelihood estimates of all parameters.*

Similarly, we can evaluate approximate confidence regions for sets of parameters.

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