

STABILITY OF NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS

BY ZDZISŁAW KAMONT

Abstract. We give a theorem on the error estimate of approximate solutions for difference functional equations of the Volterra type with an unknown function in several variables. The error is estimated by a solution of an initial problem for nonlinear differential functional equation. We apply this general result to the investigation of the convergence of difference schemes generated by mixed problems for evolution functional differential equations. We assume nonlinear estimates of the Perron type with respect to the functional variable for given operators.

1. INTRODUCTION

Nonlinear parabolic differential functional equations and first order partial functional differential equations have the following property: difference methods for suitable initial or initial boundary value problems consist in replacing partial derivatives with difference operators. Moreover, because differential equations contain a functional variable which is an element of the space of continuous functions defined on a subset of a finite dimensional space, we need some interpolating operators. This leads to nonlinear difference functional problems which satisfy consistency conditions on all sufficiently regular solutions of functional differential equations. The main task in these considerations is to find a finite difference approximation of an original problem which is stable. The method of difference inequalities or simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference functional problems.

1991 *Mathematics Subject Classification.* 35R10, 39A11, 65N10.

Key words and phrases. Difference functional equations, stability and convergence, nonlinear estimates of the Perron type.

These considerations as a rule require a lot of calculations to reach the convergence result so the main property of the corresponding operators was not easy to be seen. The aim of the present paper is to show that the results mentioned above as well as many other theorems are consequences of a result concerning an abstract nonlinear difference functional equation with an unknown function in several variables.

Our results are based on a comparison technique. It is important in the paper that we have assumed nonlinear estimates of the Perron type for a given function with respect to the functional variable and that we use ordinary differential functional equations as comparison problems. It is easy to see that conditions indicated above are identical with the assumptions that guarantee the uniqueness of solutions of initial boundary value problems.

Now we formulate our functional differential problems. For any metric spaces X and Y , by $C(X, Y)$ we denote the class of all continuous functions from X into Y . We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E = [0, a] \times (-b, b), \quad D = [-d_0, 0] \times [-d, d],$$

where $a > 0$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $b_i > 0$ for $1 \leq i \leq n$, and $d_0 \in \mathbb{R}_+$, $d = (d_1, \dots, d_n) \in \mathbb{R}_+^n$, $\mathbb{R}_+ = [0, +\infty)$. Let $c = b + d$ and $\Omega = E \cup E_0 \cup \partial_0 E$, where

$$E_0 = [-d_0, 0] \times [-c, c], \quad \partial_0 E = [0, a] \times ([-c, c] \setminus (-b, b)).$$

For a function $z : \Omega \rightarrow \mathbb{R}$ and a point $(t, x) \in \bar{E}$, where \bar{E} is the closure of E , we define a function $z_{(t,x)} : D \rightarrow \mathbb{R}$ by $z_{(t,x)}(\tau, y) = z(t + \tau, x + y)$, $(\tau, y) \in D$. Then $z_{(t,x)}$ is the restriction of z to the set $[t - d_0, t] \times [x - d, x + d]$ and this restriction is shifted to the set D . For $\xi : E \rightarrow \mathbb{R}^{1+n}$ we put $\xi = (\xi_0, \xi')$ and $\xi' = (\xi_1, \dots, \xi_n)$. Write $\Sigma = E \times C(D, \mathbb{R}) \times \mathbb{R}^n$ and suppose that $f : \Sigma \rightarrow \mathbb{R}$, $\varphi : E_0 \cup \partial_0 E \rightarrow \mathbb{R}$, $\alpha : E \rightarrow \mathbb{R}^{1+n}$ are given functions. The require α satisfy $\alpha_0(t, x) \leq t$ and $\alpha(t, x) \in \bar{E}$ for $(t, x) \in E$. We consider the functional differential equation

$$(1) \quad \partial_t z(t, x) = f(t, x, z_{\alpha(t,x)}, \partial_x z(t, x))$$

with the initial boundary condition

$$(2) \quad z(t, x) = \varphi(t, x) \quad \text{on } E_0 \cup \partial_0 E,$$

where $x = (x_1, \dots, x_n)$ and $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$. We will consider classical solutions of (1), (2). We give examples of equations which can be derived from (1) by specializing f and α .

EXAMPLE 1.1. Assume that $d_0 = 0$, $d = \mathbf{0}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ and $\tilde{f} : E \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given function. We define f as follows:

$$f(t, x, w, q) = \tilde{f}(t, x, w(0, \mathbf{0}), q) \quad \text{on } \Sigma.$$

Then (1) reduces to the differential equation with deviated variables

$$(3) \quad \partial_t z(t, x) = \tilde{f}(t, x, z(\alpha(t, x)), \partial_x z(t, x)).$$

EXAMPLE 1.2. Suppose that $\tilde{f} : E \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given function and $\alpha : E \rightarrow \mathbb{R}^{1+n}$ satisfies the conditions: $-d_0 \leq \alpha_0(t, x) - t \leq 0$ and $-d \leq \alpha'(t, x) - x \leq d$ on E . Write

$$f(t, x, w, q) = \tilde{f}(t, x, w(\alpha(t, x) - (t, x)), w(0, \mathbf{0}), q) \text{ on } \Sigma.$$

Then (1) is equivalent to the differential equation with deviated variables

$$(4) \quad \partial_t z(t, x) = \tilde{f}(t, x, z(t, x), z(\alpha(t, x)), \partial_x z(t, x)).$$

Note that initial boundary sets corresponding to (3) and (4) are different.

EXAMPLE 1.3. Suppose that $\beta, \gamma : E \rightarrow \mathbb{R}^{1+n}$ and

$$\begin{aligned} -d_0 &\leq (\beta_0 - \alpha_0)(t, x) \leq 0, & -d_0 &\leq (\gamma_0 - \alpha_0)(t, x) \leq 0, \\ -d &\leq (\beta' - \alpha')(t, x) \leq d, & -d &\leq (\gamma' - \alpha')(t, x) \leq d, \end{aligned}$$

where $(t, x) \in E$. For a given function $\tilde{f} : E \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, we put

$$f(t, x, w, q) = \tilde{f}\left(t, x, \int_{(\beta-\alpha)(t,x)}^{(\gamma-\alpha)(t,x)} w(\tau, y) dy d\tau, q\right) \text{ on } \Sigma.$$

Then (1) reduces to the differential integral equation

$$(5) \quad \partial_t z(t, x) = \tilde{f}\left(t, x, \int_{\beta(t,x)}^{\gamma(t,x)} z(\tau, y) dy d\tau, \partial_x z(t, x)\right).$$

EXAMPLE 1.4. Suppose that $\tilde{f} : E \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given function. Put $\alpha(t, x) = (t, x)$ on E and

$$f(t, x, w, q) = \tilde{f}\left(t, x, w(0, \mathbf{0}), \int_D w(\tau, y) dy d\tau, q\right) \text{ on } \Sigma.$$

Then (1) is equivalent to the differential integral equation

$$(6) \quad \partial_t z(t, x) = \tilde{f}\left(t, x, z(t, x), \int_D z(t + \tau, x + y) dy d\tau, \partial_x z(t, x)\right).$$

It is clear that more complicated differential equations with deviated variables and differential integral problems can be obtained from (1). Note also that equations (3) and (5) cannot be obtained as particular cases of differential functional equations considered in [8, 9].

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of (1), (2) can be found in [7], see also [1, 2]. Difference approximations of classical solutions to first order partial differential functional equations were investigated in [3, 8] and [9], Chapter 5. Initial problems on

the Haar pyramid and initial boundary value problems were considered. The monograph [9] contains an exposition of recent developments in the field of first order partial functional differential equations.

Now we formulate initial boundary value problems for nonlinear parabolic functional differential equations. Let us denote by $M_{n \times n}$ the class of all $n \times n$ matrices with real elements. Write $\Xi = E \times C(D, \mathbb{R}) \times \mathbb{R}^n \times M_{n \times n}$ and suppose that $F : \Xi \rightarrow \mathbb{R}$, $\varphi : E_0 \cup \partial_0 E \rightarrow \mathbb{R}$, $\alpha : E \rightarrow \mathbb{R}^{1+n}$ are given functions. We assume that $\alpha_0(t, x) \leq t$ and $\alpha(t, x) \in \bar{E}$ for $(t, x) \in E$. We consider the functional differential equation

$$(7) \quad \partial_t z(t, x) = F(t, x, z_{\alpha(t, x)}, \partial_x z(t, x), \partial_{xx} z(t, x))$$

with the initial boundary condition

$$(8) \quad z(t, x) = \varphi(t, x) \text{ on } E_0 \cup \partial_0 E,$$

where

$$\partial_{xx} z = [\partial_{x_i x_j} z]_{i, j=1, \dots, n}.$$

We look for classical solutions to problem (7), (8). Differential equations with deviated variables and differential integral equations are particular case of (7). Examples analogous to (3)–(6) can be formulated for parabolic equations.

Difference approximations of nonlinear equations with initial boundary conditions of the Dirichlet type were studied in [10, 12]. The convergence of a general class of difference schemes for parabolic equations and solutions considered on unbounded domains were investigated in [13, 23]. Monotone iterative methods and finite difference schemes for computing approximate solutions of parabolic equations with time delay were studied in [15–17]. Numerical treatment of initial boundary value problems of the Neumann - Robin type can be found in [16]. Approximate projection difference schemes were developed in [18]. The numerical method of lines was considered in [11].

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of parabolic functional differential problems can be found in [4–6, 14, 19, 20, 22]. The monographs [4, 25] give an extensive survey of the theory of parabolic functional differential equations.

It should be noted that all problems considered in the paper have the following property: the unknown functions are the functional variables in differential equations. The partial derivatives appear in a classical sense.

The paper is divided into two parts. In the first part (Section 2) we propose a general method for the investigation of the stability of difference schemes generated by initial boundary value problems for evolution functional differential equations. We prove a theorem on error estimates for approximate solutions to functional difference equations of the Volterra type with the unknown function

in several variables. The error of an approximate solution is estimated by a solution of an initial problem for a nonlinear differential functional equation. In the second part of the paper (Section 3) we apply the above general idea to the investigation of the convergence of difference methods for evolution functional differential equations. We give sufficient conditions for the convergence of the Lax difference schemes and the Euler difference methods for (1), (2). We also deal with numerical methods for parabolic functional differential problems. We prove that there is a general class of difference schemes for (7), (8) which are convergent and the convergence follows from a theorem presented in Section 2. We give error estimates for all numerical methods considered in the paper.

Our approach assumes that the differential functional equations considered in [8, 9] and [10] are particular cases of (1) and (2), respectively. On the other hand, there are differential integral problems and differential equations with deviated variables covered by our theorems and the results presented in the above papers are not applicable to those equations. In the paper, we use general ideas for finite difference equations, as such ideas were introduced in [9, 21].

2. STABILITY OF FUNCTIONAL DIFFERENCE EQUATIONS

For any two sets U and W , by $\mathbb{F}(U, W)$, we denote the class of all functions defined on U and taking values in W . If $A \subset U$ and $f \in \mathbb{F}(U, W)$, then $f|_A$ denotes the restriction of f to the set A . Let \mathbb{N} and \mathbb{Z} be the sets of natural numbers and integers, respectively. We define a mesh on Ω in the following way. Suppose that $(h_0, h') = h$, $h' = (h_1, \dots, h_n)$, stand for steps of the mesh. For $(r, m) \in \mathbb{Z}^{1+n}$, where $m = (m_1, \dots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)}) = (m_1h_1, \dots, m_nh_n).$$

Let us denote by H the set of all values of h such that there are $K_0 \in \mathbb{Z}$ and $K = (K_1, \dots, K_n) \in \mathbb{Z}^n$ with the properties: $K_0h_0 = d_0$ and $(K_1h_1, \dots, K_nh_n) = c$. Set

$$R_h^{1+n} = \{ (t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n} \}$$

and $\Omega_h = E_h \cup E_{0,h} \cup \partial_0 E_h$, where

$$E_h = E \cap \mathbb{R}_h^{1+n}, \quad E_{0,h} = E_0 \cap \mathbb{R}_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap \mathbb{R}_h^{1+n}.$$

Let $N \in \mathbb{N}$ be defined by the relations: $Nh_0 \leq a < (N+1)h_0$ and

$$E'_h = \{ (t^{(r)}, x^{(m)}) \in E_h : 0 \leq r \leq N-1 \}.$$

For a function $z : \Omega_h \rightarrow \mathbb{R}$, we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$. Classical difference methods for (1), (2) or (7), (8) consist in replacing partial derivatives with difference operators. Moreover, because equations (1) and (7) contain the functional variable $z_{\alpha(t,x)}$ which is an element of the space $C(D, \mathbb{R})$ and

solutions of difference equations are defined on the mesh, we need an interpolating operator $T_h : \mathbb{F}(\Omega_h, \mathbb{R}) \rightarrow C(\Omega, \mathbb{R})$. We define T_h in the following way. Set

$$\Lambda_+ = \{ \lambda = (\lambda_1, \dots, \lambda_n) : \lambda_i \in \{0, 1\} \text{ for } 1 \leq i \leq n \}.$$

Suppose that $z \in \mathbb{F}(\Omega_h, \mathbb{R})$ and $(t, x) \in \Omega$. Two cases will be distinguished.

I. Suppose that there exists $(r, m) \in \mathbb{Z}^{1+n}$ such that $t^{(r)} \leq t \leq t^{(r+1)}$ and $x^{(m)} \leq x \leq x^{(m+1)}$, where $m+1 = (m_1+1, \dots, m_n+1)$ and $(t^{(r)}, x^{(m)}) \in \Omega_h$, $(t^{(r+1)}, x^{(m+1)}) \in \Omega_h$. Write

$$\begin{aligned} (T_h[z])(t, x) &= \left(1 - \frac{t - t^{(r)}}{h_0}\right) \sum_{\lambda \in \Lambda_+} z^{(r, m+\lambda)} \left(\frac{x - x^{(m)}}{h'}\right)^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} \\ &\quad + \frac{t - t^{(r+1)}}{h_0} \sum_{\lambda \in \Lambda_+} z^{(r+1, m+\lambda)} \left(\frac{x - x^{(m)}}{h'}\right)^\lambda \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda}, \end{aligned}$$

where

$$\begin{aligned} \left(\frac{x - x^{(m)}}{h'}\right)^\lambda &= \prod_{i=1}^n \left(\frac{x_i - x_i^{(m_i)}}{h_i}\right)^{\lambda_i}, \\ \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} &= \prod_{i=1}^n \left(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\right)^{1-\lambda_i} \end{aligned}$$

and take $0^0 = 1$ in the above formulas.

II. Suppose that $(t, x) \in \Omega$ and $Nh_0 < t \leq a$. Thus we put $(T_h[z])(t, x) = (T_h[z])(Nh_0, x)$. Then we have defined $T_h[z] : \Omega \rightarrow \mathbb{R}$ and $T_h[z] \in C(\Omega, \mathbb{R})$.

The above interpolating operator was introduced and widely studied in [9].

We consider the following seminorms in functional spaces. For $z \in C(\Omega, \mathbb{R})$ we put

$$\|z\|_t = \max\{ |z(\tau, y)| : (\tau, y) \in \Omega \cap ([-d_0, t] \times \mathbb{R}^n) \}, \quad 0 \leq t \leq a.$$

For $z \in \mathbb{F}(\Omega_h, \mathbb{R})$ we write

$$\|z\|_{h,r} = \max\{ |z^{(r,m)}| : (t^{(i)}, x^{(m)}) \in \Omega_h \cap ([-d_0, t^{(r)}] \times \mathbb{R}^n) \}, \quad 0 \leq r \leq N.$$

The maximum norm in the space $C(D, \mathbb{R})$ is denoted by $\|\cdot\|_D$. The following properties of the operator T_h are important in the paper.

LEMMA 2.1. *Suppose that the function $v : \Omega \rightarrow \mathbb{R}$ is of class C^1 and $v_h = v|_{\Omega_h}$. Let $\tilde{C} \in \mathbb{R}_+$ be defined by the relations*

$$|\partial_t v(t, x)|, \quad |\partial_{x_i} v(t, x)| \leq \tilde{C} \quad \text{for } i = 1, \dots, n \text{ and } (t, x) \in \Omega.$$

Then $\|T_h[v_h] - v\|_t \leq \tilde{C}\|h\|$ for $0 \leq t \leq Nh_0$, where $\|h\| = h_0 + h_1 + \dots + h_n$. Moreover, for $z \in \mathbb{F}(\Omega_h, \mathbb{R})$, there is $\|T_h[z]\|_{t^{(r)}} = \|z\|_{h,r}$ for $0 \leq r \leq N$.

LEMMA 2.2. Suppose that the function $v : \Omega \rightarrow \mathbb{R}$ is of class C^2 and $v_h = v|_{\Omega_h}$. Let $\bar{C} \in \mathbb{R}_+$ be defined by the relations

$$|\partial_{tt}v(t, x)|, |\partial_{tx_i}v(t, x)|, |\partial_{x_i x_j}v(t, x)| \leq \bar{C} \text{ for } i, j = 1, \dots, n \text{ and } (t, x) \in \Omega.$$

Then $\|T_h[v_h] - v\|_t \leq \bar{C}\|h\|^2$ for $0 \leq t \leq Nh_0$.

The proofs of the above properties of T_h are similar to the proof of Theorem 5.27 in [9].

Write

$$A_h = \{x^{(m)} : -1 \leq m_i \leq 1 \text{ for } 1 \leq i \leq n\}.$$

For a function $\eta : A_h \rightarrow X$, we put $\eta^{(m)} = \eta(x^{(m)})$. If $z : \Omega_h \rightarrow \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E_h$ then the function $z_{\langle r, m \rangle} : A_h \rightarrow X$ is defined by $z_{\langle r, m \rangle}(y) = z(t^{(r)}, x^{(m)} + y)$, $y \in A_h$. Then $z_{\langle r, m \rangle}$ is the restriction of z to the set $(\{t^{(r)}\} \times [x^{(m)} - h', x^{(m)} + h']) \cap R_h^{1+n}$ and this restriction is shifted to the set A_h . The norm in the space $\mathbb{F}(A_h, \mathbb{R})$ is defined by

$$\|\eta\|_{A_h} = \max \{|\eta^{(m)}| : x^{(m)} \in A_h\}.$$

Set $Y_h = E'_h \times C(D, \mathbb{R}) \times \mathbb{F}(A_h, \mathbb{R})$ and suppose that the functions $F_h : Y_h \rightarrow \mathbb{R}$, $\alpha : E'_h \rightarrow \mathbb{R}^{1+n}$, are given. The function α is required to fulfil $\alpha^{(r, m)} \in \bar{E}$ and $\alpha_0^{(r, m)} \leq t^{(r)}$ for $(t^{(r)}, x^{(m)}) \in E'_h$. For $(t^{(r)}, x^{(m)}, z, \eta) \in Y_h$ we write $F_h[w, \eta]^{(r, m)} = F_h(t^{(r)}, x^{(m)}, z, \eta)$. Given $\varphi \in \mathbb{F}(E_{0, h} \cup \partial_0 E_h, \mathbb{R})$, we consider the functional difference equation

$$(9) \quad z^{(r+1, m)} = F_h[(T_h[z])_{\alpha^{(r, m)}}, z_{\langle r, m \rangle}]^{(r, m)}$$

with the initial boundary condition

$$(10) \quad z^{(r, m)} = \varphi_h^{(r, m)} \text{ on } E_{0, h} \cup \partial_0 E_h.$$

It is clear that there exists exactly one solution $z_h : \Omega_h \rightarrow \mathbb{R}$ of (9), (10).

REMARK 2.3. Difference functional equations generated by (1) or (7) have the form

$$(11) \quad z^{(r+1, m)} = \mathbf{F}_h(t^{(r)}, x^{(m)}, z),$$

where $\mathbf{F}_h : E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \rightarrow \mathbb{R}$ is an operator of the Volterra type. We give comments on the functional dependence in (11).

Discretization of partial derivatives $\partial_x z$ and $\partial_{xx} z$ at the point $(t^{(r)}, x^{(m)})$ leads to the following observation: the numbers $z^{(r, m+\kappa)}$, $\kappa = (\kappa_1, \dots, \kappa_n)$, $\kappa_i \in \{-1, 0, 1\}$ for $1 \leq i \leq n$, appear in definitions of difference operators corresponding to these derivatives. Difference operators for the derivatives $\partial_t z$ involve the numbers $z^{(r+1, m)}$ and $z^{(r, m+\kappa)}$, where $-1 \leq \kappa_i \leq 1$ for $i = 1, \dots, n$. Thus we see that the right hand side of (11) depends on the function $z_{\langle r, m \rangle}$.

Since the right hand sides of (1) or (7) depend on the functional variable $z_{\alpha(t,x)}$, then the right hand side of (11) depends on the functional variable $(T_h[z])_{\alpha(r,m)}$.

It is clear that assumptions on $z_{(r,m)}$ and on the functional variable corresponding to $(T_h[z])_{\alpha(r,m)}$ are not the same in convergence theorems. Accordingly, we have decided to consider difference functional equations with two functional variables.

Suppose that the functions $v_h : \Omega_h \rightarrow \mathbb{R}$ and $\tilde{\alpha}, \tilde{\gamma} : H \rightarrow \mathbb{R}_+$ satisfy the conditions

$$|v_h^{(r+1,m)} - F_h[(T_h[v_h])_{\alpha(r,m)}, (v_h)_{(r,m)}]^{(r,m)}| \leq \tilde{\gamma}(h) \text{ on } E'_h,$$

$$|\varphi_h^{(r,m)} - v_h^{(r,m)}| \leq \tilde{\alpha}(h) \text{ on } E_{0,h} \cup \partial_0 E_h \text{ and } \lim_{h \rightarrow 0} \tilde{\alpha}(h) = 0, \quad \lim_{h \rightarrow 0} \tilde{\gamma}(h) = 0.$$

The function v_h satisfying the above relations is considered as an approximate solution of (9), (10). We give a theorem on the estimate of the difference between the exact and approximate solutions of (9), (10).

Write $I = [-d_0, 0]$ and $J = [0, a]$. For a function $\xi : I \cup J \rightarrow \mathbb{R}$ and a point $t \in J$ we define $\xi_t : I \rightarrow \mathbb{R}$ by $\xi_t(\tau) = \xi(t + \tau)$, $\tau \in I$. The maximum norm in the space $C(I, \mathbb{R})$ is denoted by $\|\cdot\|_I$. Put

$$I_h = \{t^{(r)} : -K_0 \leq r \leq 0\}, \quad J_h = \{t^{(r)} : 0 \leq r \leq N\}, \quad J'_h = J_h \setminus \{t^{(N)}\}.$$

For $\zeta : I_h \cup J_h \rightarrow \mathbb{R}$ we write $\zeta^{(r)} = \zeta(t^{(r)})$. We need a discrete version of the operator $t \rightarrow \xi_t$. For $\zeta : I_h \cup J_h \rightarrow \mathbb{R}$ and $t^{(r)} \in J_h$, we define $\zeta_{[r]} : I_h \rightarrow \mathbb{R}$ by $\zeta_{[r]}(\tau) = \zeta(t^{(r)} + \tau)$, $\tau \in I_h$.

Let $T_{h_0} : \mathbb{F}(I_h, \mathbb{R}) \rightarrow C(I, \mathbb{R})$ be an interpolating operator given by

$$(12) \quad T_{h_0}[\vartheta](t) = \frac{t - t^{(r)}}{h_0} \vartheta^{(r+1)} + \left(1 - \frac{t - t^{(r)}}{h_0}\right) \vartheta^{(r)} \text{ for } t^{(r)} \leq t \leq t^{(r+1)},$$

where $\vartheta \in \mathbb{F}(I_h, \mathbb{R})$. It is clear the T_{h_0} is a particular case of T_h . We will need the operator $V : C(D, \mathbb{R}) \rightarrow C(I, \mathbb{R})$, which for $w \in C(D, \mathbb{R})$ is defined as follows:

$$V[w](t) = \max \{ |w(t, x)| : x \in [-d, d] \}, \quad t \in I.$$

We formulate assumptions on comparison operators corresponding to (9), (10).

ASSUMPTION $H[\sigma]$. The function $\sigma : J \times C(I, \mathbb{R}_+) \rightarrow \mathbb{R}_+$ satisfies the conditions:

- 1) σ is continuous and nondecreasing with respect to the both variables,
- 2) $\sigma(t, \theta) = 0$ for $t \in J$, where $\theta \in C(I, \mathbb{R}_+)$ is given by $\theta(\tau) = 0$ for $\tau \in I$,

3) the function $\tilde{\omega}(t) = 0$, for $t \in I \cup J$, is the maximal solution of the Cauchy problem

$$(13) \quad \omega'(t) = \sigma(t, \omega_t), \quad \omega(t) = 0 \text{ for } t \in I.$$

Having done the above preparation, we formulate a theorem on the estimate of the difference between the exact and approximate solutions to problem (9), (10) in the form convenient for our purposes.

THEOREM 2.4. *Suppose that $F_h : Y_h \rightarrow \mathbb{R}$, $\varphi_h : E_{0,h} \rightarrow \mathbb{R}$, $\alpha : E'_h \rightarrow \mathbb{R}^{1+n}$ and*

- 1) $\alpha^{(r,m)} \in \bar{E}$, $\alpha_0^{(r,m)} \leq t^{(r)}$ for $(t^{(r)}, x^{(m)}) \in E'_h$ and $z_h : \Omega_h \rightarrow \mathbb{R}$ is a solution of (9), (10),
- 2) there exists $\sigma : J \times C(I, \mathbb{R}_+) \rightarrow \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and

$$(14) \quad \|F_h[w, \eta]^{(r,m)} - F_h[\bar{w}, \bar{\eta}]^{(r,m)}\| \leq \|\eta - \bar{\eta}\|_{A_h} + h_0 \sigma(t^{(r)}, V[w - \bar{w}]) \text{ on } Y_h,$$

3) $v_h : \Omega_h \rightarrow X$ and there are $\beta_0, \gamma : H \rightarrow \mathbb{R}_+$ such that

$$(15) \quad \left| v_h^{(r+1,m)} - F_h[(T_h[v_h])_{\alpha^{(r,m)}}, (v_h)_{\langle r,m \rangle}]^{(r,m)} \right| \leq h_0 \gamma(h) \text{ on } E'_h \\ \text{and } \lim_{h \rightarrow 0} \gamma(h) = 0,$$

$$(16) \quad |\varphi_h^{(r,m)} - v_h^{(r,m)}| \leq \beta_0(h) \text{ on } E_{0,h} \cup \partial_0 E_h \text{ and } \lim_{h \rightarrow 0} \beta_0(h) = 0.$$

Then there is $\beta : H \rightarrow \mathbb{R}_+$ such that

$$(17) \quad |(z_h - v_h)^{(r,m)}| \leq \beta(h) \text{ on } E_h \text{ and } \lim_{h \rightarrow 0} \beta(h) = 0.$$

PROOF. The proof will be divided into two steps.

I. Let us denote by $\beta_h : I_h \cup J_h \rightarrow \mathbb{R}_+$ the solution of the difference problem

$$(18) \quad \zeta^{(r+1)} = \zeta^{(r)} + h_0 \sigma(t^{(r)}, T_{h_0}[\zeta_{[r]}]) + h_0 \gamma(h), \quad 0 \leq r \leq N-1,$$

$$(19) \quad \zeta^{(r)} = \beta_0(h) \text{ for } -K_0 \leq r \leq 0.$$

We prove that

$$(20) \quad |(z_h - v_h)^{(r,m)}| \leq \beta_h^{(r)} \text{ on } E_h.$$

It follows from (16) that (20) holds for $r = 0$ and $(t^{(0)}, x^{(m)}) \in E_h$. Assuming (20) to hold for $0 \leq i \leq r$, $(t^{(i)}, x^{(m)}) \in E_h$, we will prove it for $r+1$ and $(t^{(r+1)}, x^{(m)}) \in E_h$. It follows easily that

$$|(T_h[z_h - v_h])_{(t,x)}(\tau, y)| \leq T_{h_0}[(\beta_h)_{[r]}](\tau),$$

where $(\tau, y) \in D$ and $(t, x) \in E$, $t \leq t^{(r)}$. We conclude from Assumption $H[\sigma]$, (14), (15) and the above inequality that

$$\begin{aligned} & \left| (z_h - v_h)^{(r+1, m)} \right| \\ & \leq \left| F_h[(T_h[z_h])_{\alpha(r, m)}, (z_h)_{\langle r, m \rangle}]^{(r, m)} - F_h[(T_h[v_h])_{\alpha(r, m)}, (v_h)_{\langle r, m \rangle}]^{(r, m)} \right| \\ & \quad + \left| v_h^{(r+1, m)} - F_h[(T_h[v_h])_{\alpha(r, m)}, (v_h)_{\langle r, m \rangle}]^{(r, m)} \right| \\ & \leq \beta_h^{(r)} + h_0 \sigma(t^{(r)}, T_{h_0}[(\beta_h)_{[r]}]) + h_0 \gamma(h) = \beta_h^{(r+1)}. \end{aligned}$$

Hence the proof of (20) is completed by induction with respect to r , $0 \leq r \leq N$.

II. We prove that there is $\beta : H \rightarrow \mathbb{R}_+$ such that $\beta_h^{(r)} \leq \beta(h)$ for $0 \leq r \leq N$ and $\lim_{h \rightarrow 0} \beta(h) = 0$. Consider the Cauchy problem

$$(21) \quad \omega'(t) = \sigma(t, \omega_t + (\mu(h))_t) + \gamma(h),$$

$$(22) \quad \omega(t) = \beta_0(h) \text{ for } t \in I,$$

where $\mu : H \rightarrow (0, +\infty)$, $\lim_{h \rightarrow 0} \mu(h) = 0$ and $(\mu(h))_t \in C(I, \mathbb{R}_+)$ is a constant function: $(\mu(h))_t(\tau) = \mu(h)$ for $\tau \in I$. It follows from Assumption $H[\sigma]$ that there is $\tilde{\varepsilon} > 0$ such that the maximal solution $\omega(\cdot, h)$ of (21), (22) is defined on $I \cup J$ for $\|h\| < \tilde{\varepsilon}$ and

$$(23) \quad \lim_{h \rightarrow 0} \omega(t, h) = 0 \text{ uniformly on } I \cup J.$$

Suppose that $\tilde{h} \in H$ is fixed and $\|\tilde{h}\| < \tilde{\varepsilon}$. Let us denote by $C[\tilde{h}]$ the set of all $h \in H$ such that $\|h\| < \tilde{\varepsilon}$ and $\mu(h) \leq \mu(\tilde{h})$, $\gamma(h) \leq \gamma(\tilde{h})$. Then the maximal solution $\omega(\cdot, h)$ of (21), (22), where $h \in C[\tilde{h}]$, satisfies the condition

$$(24) \quad \omega(t, h) \leq \omega(t, \tilde{h}) \text{ for } t \in I \cup J.$$

Let $\omega_{h_0}(\cdot, h)$ denote the restriction of $\omega(\cdot, h) : I \cup J \rightarrow \mathbb{R}_+$ to the set $I_h \cup J_h$. It follows from (12) that for $t^{(r)} \in J_h$, $h \in C[\tilde{h}]$ there is

$$(25) \quad (\omega(\cdot, h))_{t^{(r)}}(\tau) - T_{h_0}[(\omega_{h_0}(\cdot, h))_{[r]}](\tau) \geq -h_0 \omega'(a, h) \geq -h_0 \omega'(a, \tilde{h}),$$

where $\tau \in I$. There is $\bar{\varepsilon} > 0$ such that for $h \in C[\tilde{h}]$, $\|h\| < \bar{\varepsilon}$:

$$(26) \quad \mu(\tilde{h}) \geq \mu(h) \geq h_0 \omega'(a, \tilde{h}).$$

We conclude from condition 1) of Assumption $H[\sigma]$ and from (25), (26) that for $h \in C[\tilde{h}]$, $\|h\| < \bar{\varepsilon}$, we have

$$\begin{aligned} & \omega'(t^{(r)}, h) = \sigma(t^{(r)}, (\omega(\cdot, h))_{t^{(r)}} + (\mu(h))_{t^{(r)}}) + \gamma(h) \\ & = \sigma(t^{(r)}, T_{h_0}[(\omega_{h_0}(\cdot, h))_{[r]}] + (\omega(\cdot, h))_{t^{(r)}} - T_{h_0}[(\omega_{h_0}(\cdot, h))_{[r]}] + (\mu(h))_{t^{(r)}}) + \gamma(h) \\ & \geq \sigma(t^{(r)}, T_{h_0}[(\omega_{h_0}(\cdot, h))_{[r]}]) + \gamma(h), \quad 0 \leq r \leq N, \end{aligned}$$

and consequently

$$\omega_{h_0}(t^{(r+1)}, h) \geq \omega_{h_0}(t^{(r)}, h) + h_0 \sigma(t^{(r)}, T_{h_0}[(\omega_{h_0}(\cdot, h))_{[r]}]) + h_0 \gamma(h),$$

$$0 \leq r \leq N - 1.$$

Since β_h satisfies (18), (19), the above relations and (21) show that $\beta_h^{(r)} \leq \omega_{h_0}(t^{(r)}, h) \leq \omega(a, h)$ for $0 \leq r \leq N$. It follows from (23), (24) that condition (17) is satisfied with $\beta(h) = \omega(a, h)$. This proves the theorem. \square

Now we formulate a particular case of Theorem 2.4. We assume that the function $\sigma(t, \cdot)$ is linear and that (13) is a classical Cauchy problem.

LEMMA 2.5. *Suppose that $F_h : Y_h \rightarrow \mathbb{R}$, $\varphi_h : E_{0,h} \cup \partial_0 E_h \rightarrow \mathbb{R}$, $\alpha : E'_h \rightarrow \mathbb{R}^{1+n}$ and*

- 1) *the conditions 1), 2) of Theorem 2.4 are satisfied,*
- 2) *there is $\tilde{L} \in \mathbb{R}_+$ such that the estimate*

$$|F_h[w, \eta]^{(r,m)} - F_h[\bar{w}, \bar{\eta}]^{(r,m)}| \leq \|\eta - \bar{\eta}\|_{A_h} + h_0 \tilde{L} \|w - \bar{w}\|_D \text{ on } E'_h.$$

Then

$$(27) \quad |u_h^{(r,m)} - v_h^{(r,m)}| \leq \tilde{\beta}(h) \text{ on } E_h,$$

where

$$\tilde{\beta}(h) = \beta_0(h) e^{\tilde{L}a} + \gamma(h) \frac{e^{\tilde{L}a} - 1}{\tilde{L}} \text{ if } \tilde{L} > 0,$$

$$\tilde{\beta}(h) = \beta_0(h) + a\gamma(h) \text{ if } \tilde{L} = 0.$$

PROOF. It easily follows that the solution $\beta_h : J_h \rightarrow \mathbb{R}_+$ of the difference problem

$$\zeta^{(r+1)} = (1 + \tilde{L}h_0)\zeta^{(r)} + h_0\gamma(h), \quad 0 \leq r \leq N - 1, \quad \zeta^{(0)} = \beta_0(h),$$

satisfies the condition: $\beta_h^{(r)} \leq \tilde{\beta}(h)$, $0 \leq r \leq N$. Hence (27) follows from Theorem 2.4. \square

The above example is important in simple applications. On the other hand, the connection with functional differential problem (21), (22) is important in our considerations.

EXAMPLE 2.6. If $\nu \geq \mu > 1$, $L_0 \in \mathbb{R}_+$, $\tilde{c} > 0$, then the maximal solution of the Cauchy problem

$$(28) \quad \omega'(t) = \tilde{c} \sqrt[\nu]{\omega(t^\mu)} + L_0 \omega(t), \quad \omega(0) = 0'$$

is $\tilde{\omega}(t) = 0$ for $t \in [0, a]$, where $a \leq 1$.

This property of problem (28) may be proved by using a method of differential inequalities. Note that the maximal solution of (28) for $\nu > 1$, $\mu = 1$ is positive on $(0, a]$.

3. APPLICATIONS

In this part of the paper we give sufficient conditions for the convergence of difference schemes corresponding to nonlinear partial functional differential equations of the evolution type.

3.1. Mixed problems for nonlinear first order partial differential functional equations. Let $D, E, E_0, \partial_0 E, \Omega$ and $E_h, E_{0,h}, \partial_0 E_h, \Omega_h, A_h$ be the sets defined in Sections 1 and 2. We formulate a difference method for initial boundary value problem (1), (2). For $1 \leq i \leq n$, we define $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 standing on the i -th place. Let δ_0 and $(\delta_1, \dots, \delta_n) = \delta$ be the difference operators given by

$$(29) \quad \delta_0 z^{(r,m)} = \frac{1}{h_0} \left[z^{(r+1)} - \frac{1}{2n} \sum_{i=1}^n (z^{(r,m+e_i)} + z^{(r,m-e_i)}) \right],$$

$$(30) \quad \delta_i z^{(r,m)} = \frac{1}{2h_i} [z^{(r,m+e_i)} - z^{(r,m-e_i)}], \quad i = 1, \dots, n,$$

where $z : \Omega_h \rightarrow \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E'_h$. Write $\delta z = (\delta_1 z, \dots, \delta_n z)$. We approximate classical solutions of (1), (2) with solutions of the difference functional equation

$$(31) \quad \delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, (T_h[z])_{\alpha(r,m)}, \delta z^{(r,m)})$$

with the initial boundary condition

$$(32) \quad z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0,h} \cup \partial_0 E_h,$$

where $\varphi_h : E_{0,h} \cup \partial_0 E_h \rightarrow \mathbb{R}$ is a given function. Difference method (31), (32) with δ_0 and δ defined by (29), (30) is called the Lax scheme. We claim that problem (31), (32) is a particular case of (9), (10). Let $F_h : Y_h \rightarrow \mathbb{R}$ be defined by

$$(33) \quad F_h[w, \eta]^{(r,m)} = \Delta \eta^{(\theta)} + h_0 f(t^{(r)}, x^{(m)}, w, \delta \eta^{(\theta)}),$$

where

$$\Delta \eta^{(\theta)} = \frac{1}{2n} \sum_{i=1}^n (\eta^{(e_i)} + \eta^{(-e_i)}), \quad \delta \eta^{(\theta)} = (\delta_1 \eta^{(\theta)}, \dots, \delta_n \eta^{(\theta)}),$$

$$\delta_i \eta^{(\theta)} = \frac{1}{2h_i} [\eta^{(e_i)} - \eta^{(-e_i)}], \quad i = 1, \dots, n.$$

It is easily seen that (31), (32) is equivalent to (9), (10) with F_h defined by (33).

ASSUMPTION $H[f, \alpha]$. The function $f : \Omega \rightarrow \mathbb{R}$ in the variables (t, x, w, q) , $q = (q_1, \dots, q_n)$, is continuous and

1) there exists $\sigma : J \times C(I, \mathbb{R}_+) \rightarrow \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and

$$(34) \quad |f(t, x, w, q) - f(t, x, \bar{w}, q)| \leq \sigma(t, V[w - \bar{w}]) \text{ on } \Sigma,$$

2) the partial derivatives $(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f$ exist on Σ and $\partial_q f \in C(\Sigma, \mathbb{R}^n)$ and $\partial_q f$ is bounded on Σ ,

3) $\alpha \in C(E, \mathbb{R}^{1+n})$ and $\alpha(t, x) \in \bar{E}$, $\alpha_0(t, x) \leq t$ for $(t, x) \in E$.

THEOREM 3.1. *Suppose that Assumption $H[f, \alpha]$ is satisfied and*

1) $h \in H$ and

$$(35) \quad \frac{1}{n} - \frac{h_0}{h_i} \left| \partial_{q_i} f(t, x, w, q) \right| \geq 0, \text{ on } \Sigma, \text{ for } i = 1, \dots, n,$$

2) $z_h : \Omega_h \rightarrow \mathbb{R}$ is a solution of (31), (32) and there is $\beta_0 : H \rightarrow \mathbb{R}_+$ such that

$$(36) \quad |\varphi^{(r,m)} - \varphi_h^{(r,m)}| \leq \beta_0(h) \text{ on } E_{0,h} \cup \partial_0 E_h \text{ and } \lim_{h \rightarrow 0} \beta_0(h) = 0,$$

3) $v : \Omega \rightarrow \mathbb{R}$ is a classical solution of (1), (2) and v is of class C^1 on Ω .

Then there is $\beta : H \rightarrow \mathbb{R}_+$ such that

$$(37) \quad |(v_h - u_h)^{(r,m)}| \leq \beta(h) \text{ on } E'_h \text{ and } \lim_{h \rightarrow 0} \beta(h) = 0,$$

where v_h is the restriction of v to the set Ω_h .

PROOF. We apply Theorem 2.4 to prove (37). Suppose that F_h is given by (33). It follows that z_h satisfies (9), (10) and there are $\gamma, \beta_0 : H \rightarrow \mathbb{R}_+$ such that conditions (15), (16) are satisfied. Now we estimate the difference $F_h[w, \eta] - F_h[\bar{w}, \bar{\eta}]$, where $w, \bar{w} \in C(D, \mathbb{R})$ and $\eta, \bar{\eta} \in \mathbb{F}(A_h, \mathbb{R})$. Write

$$(38) \quad U^{(r,m)} = h_0 [f(t^{(r)}, x^{(m)}, w, \delta\eta^{(\theta)}) - f(t^{(r)}, x^{(m)}, \bar{w}, \delta\eta^{(\theta)})]$$

and

$$(39) \quad Q^{(r,m)}(\tau) = (t^{(r)}, x^{(m)}, \bar{w}, \delta\bar{\eta}^{(\theta)} + \tau\delta(\eta - \bar{\eta})^{(\theta)}).$$

It follows from (33) and from Assumption $H[f, \alpha]$ that

$$(40) \quad \begin{aligned} F_h[w, \eta]^{(r,m)} - F_h[\bar{w}, \bar{\eta}]^{(r,m)} &= U^{(r,m)} \\ &+ \frac{1}{2} \sum_{j=1}^n (\eta - \bar{\eta})^{(e_j)} \left[\frac{1}{n} + \frac{h_0}{h_j} \int_0^1 \partial_{q_j} f(Q^{(r,m)}(\tau)) d\tau \right] \\ &+ \frac{1}{2} \sum_{j=1}^n (\eta - \bar{\eta})^{(-e_j)} \left[\frac{1}{n} - \frac{h_0}{h_j} \int_0^1 \partial_{q_j} f(Q^{(r,m)}(\tau)) d\tau \right]. \end{aligned}$$

It is easily seen that

$$(41) \quad |U^{(r,m)}| \leq h_0 \sigma(t^{(r)}, V[w - \bar{w}]) \text{ on } E'_h.$$

We conclude from (35), (40), (41) that the operator F_h satisfies condition (14). Thus we see that all the assumptions of Theorem 2.4 are satisfied and assertion (37) follows. \square

Now we formulate a result on the error estimate for the difference Lax scheme. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we put $||x|| = (|x_1|, \dots, |x_n|)$ and $\|x\| = |x_1| + \dots + |x_n|$.

LEMMA 3.2. *Suppose that the function $f : \Sigma \rightarrow \mathbb{R}$ is continuous and*

- 1) *the partial derivatives $(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f$ exist on Σ and $\partial_q f \in C(\Sigma, \mathbb{R}^n)$,*
- 2) *$h \in H$ and there are $L, M \in \mathbb{R}_+^n$, $L = (L_1, \dots, L_n)$, $M = (M_1, \dots, M_n)$ such that*

$$(42) \quad [|\partial_q f(t, x, w, q)|] \leq L \quad \text{on } \Sigma \quad \text{and} \quad nh_0 L \leq h' \leq h_0 M$$

- 3) *there is $\tilde{L} \in \mathbb{R}_+$ such that*

$$(43) \quad |f(t, x, w, q) - f(t, x, \bar{w}, q)| \leq \tilde{L} \|w - \bar{w}\|_D \quad \text{on } \Sigma,$$

- 4) *$z_h : \Omega_h \rightarrow \mathbb{R}$ is a solution of (31), (32) and there is $\beta_0 : H \rightarrow \mathbb{R}_+$ such that condition (36) is satisfied,*
- 5) *condition 3) of Assumption $H[f, \alpha]$ is satisfied and $v : \Omega \rightarrow \mathbb{R}_+$ is a classical solution of (1), (2) and v is of class C^2 on Ω ,*
- 6) *the constant $C \in \mathbb{R}_+$ is defined by the relations*

$$|\partial_{tt} v(t, x)|, \quad |\partial_{tx_i} v(t, x)|, \quad |\partial_{x_i x_j} v(t, x)| \leq C, \quad (t, x) \in \Omega, \quad i, j = 1, \dots, n.$$

Then

$$(44) \quad |(z_h - v_h)^{(r,m)}| \leq \tilde{\beta}(h) \quad \text{on } E_h,$$

where $v_h = v|_{\Omega_h}$ and

$$(45) \quad \tilde{\beta}(h) = \alpha_0(h) e^{\tilde{L}a} + \tilde{\gamma}(h) \frac{e^{\tilde{L}a} - 1}{\tilde{L}} \quad \text{if } \tilde{L} > 0,$$

$$(46) \quad \tilde{\beta}(h) = \alpha_0(h) + a\tilde{\gamma}(h) \quad \text{if } \tilde{L} = 0,$$

where $\tilde{\gamma}(h) = Ah_0 + Bh_0^2$ and

$$A = \frac{C}{2} \left[1 + \frac{1}{n} \sum_{i=1}^n M_i^2 + \|L\| \|M\| \right], \quad B = \tilde{L}C(1 + \|M\|)^2.$$

PROOF. It follows from (42), (43) that condition (35) is satisfied and, consequently, difference method (31), (32) is convergent. We deduce from (42), (43) and from Lemma 2.2 that the operator F_h given by (33) satisfies condition (15) with $\gamma(h) = \tilde{\gamma}(h)$. We thus get estimate (44) from Lemma 2.5 and the proof is complete. \square

REMARK 3.3. In the result on the error estimate, we need estimates of the derivatives of the solution v of problem (1), (2). One may obtain them by the method of differential inequalities.

Now we consider functional difference problem (31), (32) with δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ given by

$$(47) \quad \delta_0 z^{(r,m)} = \frac{1}{h_0} [z^{(r+1,m)} - z^{(r,m)}]$$

and

$$(48) \quad \delta_i z^{(r,m)} = \frac{1}{h_i} [z^{(r,m+e_i)} - z^{(r,m)}] \quad \text{for } 1 \leq i \leq \kappa_0,$$

$$(49) \quad \delta_i z^{(r,m)} = \frac{1}{h_i} [z^{(r,m)} - z^{(r,m-e_i)}] \quad \text{for } \kappa_0 + 1 \leq i \leq n,$$

where $0 \leq \kappa_0 \leq n$ is fixed. Difference scheme (31), (32) with δ_0 and δ defined by (47)–(49) is called the Euler method. Let $F_h : Y_h \rightarrow \mathbb{R}$ be defined by

$$(50) \quad F_h[w, \eta]^{(r,m)} = \eta^{(\theta)} + h_0 f(t^{(r)}, x^{(m)}, w, \delta \eta^{(\theta)}),$$

where

$$(51) \quad \delta_i \eta^{(\theta)} = \frac{1}{h_i} [\eta^{(e_i)} - \eta^{(\theta)}] \quad \text{for } 1 \leq i \leq \kappa_0,$$

$$(52) \quad \delta_i \eta^{(\theta)} = \frac{1}{h_i} [\eta^{(\theta)} - \eta^{(-e_i)}] \quad \text{for } \kappa_0 + 1 \leq i \leq n.$$

It is clear that problem (31), (32) with δ_0 and δ given by (47)–(49) is equivalent to (9), (10) with F_h defined by (50)–(52).

THEOREM 3.4. *Suppose that Assumption $H[f, \alpha]$ is satisfied and*

1) $h \in H$ and for $(t, x, w, q) \in \Sigma$ we have

$$(53) \quad \partial_{q_i} f(t, x, w, q) \geq 0 \quad \text{for } 1 \leq i \leq \kappa_0,$$

$$(54) \quad \partial_{q_i} f(t, x, w, q) \leq 0 \quad \text{for } \kappa_0 + 1 \leq i \leq n,$$

and

$$(55) \quad 1 - h_0 \sum_{i=1}^n \frac{1}{h_i} |\partial_{q_i} f(t, x, w, q)| \geq 0,$$

2) $z_h : \Omega_h \rightarrow \mathbb{R}$ is a solution of (31), (32) with δ_0 and δ given by (47)–(49) and there is $\beta_0 : H \rightarrow \mathbb{R}_+$ such that condition (36) is satisfied,

3) $v : \Omega \rightarrow \mathbb{R}$ is a classical solution of (1), (2) and v is of class C^1 on Ω .

Then there is $\beta : H \rightarrow \mathbb{R}_+$ such that condition (37) is satisfied, where $v_h = v|_{\Omega_h}$.

PROOF. We apply Theorem 2.4 to prove (37). It follows that z_h satisfies (9), (10) with F_h defined by (50)–(52) and there is $\gamma : H \rightarrow \mathbb{R}_+$ such that condition (15) is satisfied. Now we estimate the function $F_h[w, \eta] - F_h[\bar{w}, \bar{\eta}]$, where $w, \bar{w} \in C(D, \mathbb{R})$ and $\eta, \bar{\eta} \in \mathbb{F}(A_h, \mathbb{R})$. Let $U^{(r,m)}$ and $Q^{(r,m)}(\tau)$ be defined by (38), (39) with $\delta\eta$ and $\delta\bar{\eta}$ given by (51), (52). It follows from Assumption $H[f, \alpha]$ that

$$(56) \quad \begin{aligned} F_h[w, \eta]^{(r,m)} - F_h[\bar{w}, \bar{\eta}]^{(r,m)} &= U^{(r,m)} \\ &+ \left[1 - h_0 \sum_{i=1}^n \frac{1}{h_i} \left| \int_0^1 \partial_{q_i} f(Q^{(r,m)}(\tau)) d\tau \right| \right] (\eta - \bar{\eta})^{(\theta)} \\ &+ h_0 \sum_{i=1}^{\kappa_0} \frac{1}{h_i} \int_0^1 \partial_{q_i} f(Q^{(r,m)}(\tau)) d\tau (\eta - \bar{\eta})^{(e_i)} \\ &- h_0 \sum_{i=\kappa+1}^n \frac{1}{h_i} \int_0^1 \partial_{q_i} f(Q^{(r,m)}(\tau)) d\tau (\eta - \bar{\eta})^{(-e_i)}. \end{aligned}$$

It follows from (41), (53)–(56) that the operator F_h satisfies (14). Then all the assumptions of Theorem 2.4 are satisfied and the assertion (37) follows. \square

REMARK 3.5. In Theorem 3.4 we have assumed that the function

$$\text{sign } \partial_q f = (\text{sign } \partial_{q_1} f, \dots, \text{sign } \partial_{q_n} f)$$

is constant on Σ . Relations (53), (54) can be considered as a definition of κ_0 .

Now we give a result on the error estimate of the Euler difference method.

LEMMA 3.6. *Suppose that*

- 1) *all the assumptions of Theorem 3.4 are satisfied with $\sigma(t, w) = L_0 \|w\|_D$, where $L_0 \in \mathbb{R}_+$,*
- 2) *$v : \Omega \rightarrow \mathbb{R}$ is a solution of (1), (2) and v is of class C^2 on Ω ,*
- 3) *there are $L, M \in \mathbb{R}_+^n$ such that $h_0 L \leq h' \leq h_0 M$.*

Then there are $A, B \in \mathbb{R}_+$ such that estimate (44) is satisfied with $\tilde{\beta}(h)$ given by (45), (46) and $\tilde{\gamma}(h) = Ah_0 + Bh_0^2$.

The above Lemma is a consequence of Lemmas 2.2 and 2.5.

3.2. Nonlinear parabolic functional differential equations. We formulate a difference method for initial boundary value problem (7), (8). Write

$$\Gamma = \{ (i, j) \in \mathbb{N}^2 : 1 \leq i, j \leq n, i \neq j \}$$

and suppose that we have defined the sets $\Gamma_+, \Gamma_- \subset \Gamma$ such that $\Gamma_+ \cup \Gamma_- = \Gamma$, $\Gamma_+ \cap \Gamma_- = \emptyset$. In particular, it may happen that $\Gamma_+ = \emptyset$ or $\Gamma_- = \emptyset$. Moreover, we

assume that $(i, j) \in \Gamma_+$ when $(j, i) \in \Gamma_+$. Let $z : \Omega_h \rightarrow \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E_h$. We define

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i} [z^{(r,m+e_i)} - z^{(r,m)}], \quad \delta_i^- z^{(r,m)} = \frac{1}{h_i} [z^{(r,m)} - z^{(r,m-e_i)}], \quad 1 \leq i \leq n.$$

We consider the difference operators $(\delta_1, \dots, \delta_n) = \delta$ defined by (30) and (47), respectively. We apply the difference operators $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\dots,n}$ given by

$$(57) \quad \delta_{ii} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)}, \quad i = 1, \dots, n,$$

and

$$(58) \quad \delta_{ij} z^{(r,m)} = \frac{1}{2} [\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)}] \quad \text{for } (i, j) \in \Gamma_-,$$

$$(59) \quad \delta_{ij} z^{(r,m)} = \frac{1}{2} [\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)}] \quad \text{for } (i, j) \in \Gamma_+.$$

In the same way we define the difference expressions $\delta\eta^{(\theta)} = (\delta_1\eta^{(\theta)}, \dots, \delta_n\eta^{(\theta)})$ and $\delta^{(2)}\eta^{(\theta)} = [\delta_{ij}\eta^{(\theta)}]_{i,j=1,\dots,n}$, where $\eta : A_h \rightarrow \mathbb{R}$. Suppose that $\varphi_h : E_{0,h} \cup \partial_0 E_h \rightarrow \mathbb{R}$ is a given function. We approximate classical solutions of (7), (8) with solutions of the difference functional equation

$$(60) \quad \delta_0 z^{(r,m)} = F(t^{(r)}, x^{(m)}, (T_h[z])_{\alpha^{(r,m)}}, \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)})$$

with the initial boundary condition

$$(61) \quad z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0,h} \cup \partial_0 E_h.$$

It is clear that there exists exactly one solution $z_h : \Omega_h \rightarrow \mathbb{R}$ of the above difference scheme. We claim that difference method (60), (61) is a particular case of (9), (10). Let $F_h : Y_h \rightarrow \mathbb{R}$ be defined by

$$(62) \quad F_h[w, \eta]^{(r,m)} = \eta^{(\theta)} + h_0 F(t^{(r)}, x^{(m)}, w, \delta\eta^{(\theta)}, \delta^{(2)}\eta^{(\theta)})$$

with the above defined $\delta\eta^{(\theta)}$ and $\delta^{(2)}\eta^{(\theta)}$. It is clear that problem (60), (61) is equivalent to (9), (10) with F_h defined by (62).

ASSUMPTION $F[F, \alpha]$. Suppose that the function $F : \Xi \rightarrow \mathbb{R}$ in the variables (t, x, w, q, s) , $s = [s_{ij}]_{i,j=1,\dots,n}$, is continuous and

- 1) the partial derivatives

$$\partial_q F = (\partial_{q_1} F, \dots, \partial_{q_n} F), \quad \partial_s F = [\partial_{s_{ij}} F]_{i,j=1,\dots,n}$$

exist on Ξ and $\partial_q F \in C(\Xi, \mathbb{R}^n)$, $\partial_s F \in C(\Xi, M_{n \times n})$ and the functions $\partial_q F$, $\partial_s F$ are bounded on Ξ ,

2) for each $P = (t, x, w, q, s) \in \Xi$ the matrix $\partial_s F(P)$ is symmetric and

$$(63) \quad \partial_{s_{ij}} F(P) \geq 0 \text{ for } (i, j) \in \Gamma_+, \quad \partial_{s_{ij}} F(P) \leq 0 \text{ for } (i, j) \in \Gamma_-,$$

$$(64) \quad 1 - 2h_0 \sum_{i=1}^n \frac{1}{h_i^2} \partial_{s_{ii}} F(P) + h_0 \sum_{(i,j) \in \Gamma} \frac{1}{h_i h_j} |\partial_{s_{ij}} F(P)| \geq 0,$$

$$(65) \quad \frac{1}{h_i} \partial_{s_{ii}} F(P) - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{h_j} |\partial_{s_{ij}} F(P)| - \frac{1}{2} |\partial_{q_i} F(P)| \geq 0, \quad 1 \leq i \leq n,$$

3) there is $\sigma : J \times C(I, \mathbb{R}_+) \rightarrow \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and

$$(66) \quad |F(t, x, w, q, s) - F(t, x, \bar{w}, q, s)| \leq \sigma(t, V[w - \bar{w}]) \text{ on } \Xi,$$

4) $\alpha \in C(E, \mathbb{R}^{1+n})$ and $\alpha(t, x) \in \bar{E}$, $\alpha_0(t, x) \leq t$ for $(t, x) \in E$.

We give comments on assumptions (63)–(66).

REMARK 3.7. Suppose that $h_1 = h_2 = \dots = h_n$ and there is $\varepsilon_0 > 0$ such that

$$\partial_{s_{ii}} F(P) - \sum_{\substack{j=1 \\ j \neq i}}^n |\partial_{s_{ij}} F(P)| \geq \varepsilon,$$

where $P \in \Xi$. Assume also that the function $\partial_q f$ is bounded on Ξ . Then there is $\tilde{\varepsilon} > 0$ such that for $\|h\| < \tilde{\varepsilon}$ condition (64) is satisfied.

REMARK 3.8. We have assumed that the functions $\text{sign } \partial_{s_{ij}} F : \Xi \rightarrow \mathbb{R}$, $(i, j) \in \Gamma$, are constant on Ξ . Relations (63) can be considered as the definition of Γ_+ and Γ_- .

REMARK 3.9. Inequality (64) states that we have assumed relations between h_0 and h' . More precisely, we assume that h_0 is sufficiently small if h' is fixed.

THEOREM 3.10. *Suppose that Assumption $H[F, \alpha]$ is satisfied and*

1) *there is $\tilde{c} > 0$ such that $h_i \leq \tilde{c}h_j$, $i, j = 1, \dots, n$,*

2) *$z_h : \Omega_h \rightarrow \mathbb{R}$ is a solution of (60), (61) and there is $\beta_0 : H \rightarrow \mathbb{R}_+$ such that*

$$(67) \quad |\varphi^{(r,m)} - \varphi_h^{(r,m)}| \leq \beta_0(h) \text{ on } E_{0,h} \cup \partial_0 E_h \text{ and } \lim_{h \rightarrow 0} \beta_0(h) = 0,$$

3) *$v : \Omega \rightarrow \mathbb{R}$ is a classical solution of (7), (8) and v is of class C^2 on Ω .*

Then there is $\beta : H \rightarrow \mathbb{R}$ such that

$$(68) \quad |(v_h - z_h)^{(r,m)}| \leq \beta(h) \text{ on } E_h \text{ and } \lim_{h \rightarrow 0} \beta(h) = 0,$$

where $v_h = v|_{\Omega_h}$.

PROOF. We apply Theorem 2.4 to prove (68). Suppose that F_h is given by (62). It follows that z_h satisfies (9), (10) and there is $\gamma : H \rightarrow \mathbb{R}_+$ such that condition (15) is satisfied. Now we consider the difference $F_h[w, \eta] - F_h[\bar{w}, \bar{\eta}]$ where $w, \bar{w} \in C(D, \mathbb{R})$ and $\eta, \bar{\eta} \in \mathbb{F}(A_h, \mathbb{R})$. Write

$$W^{(r,m)} = h_0 [f(t^{(r)}, x^{(m)}, w, \delta\eta^{(\theta)}, \delta^{(2)}\eta^{(\theta)}) - f(t^{(r)}, x^{(m)}, \bar{w}, \delta\eta^{(\theta)}, \delta^{(2)}\eta^{(\theta)})],$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. It follows from Assumption $H[F, \alpha]$ and from (63)–(65) that there are functions

$$\begin{aligned} \tilde{S}_h : E'_h &\rightarrow \mathbb{R}_+^n, & \tilde{S}_h &= (\tilde{S}_{h.1}, \dots, \tilde{S}_{h.n}), \\ \bar{S}_h : E'_h &\rightarrow \mathbb{R}_+^n, & \bar{S}_h &= (\bar{S}_{h.1}, \dots, \bar{S}_{h.n}), \\ Q_h : E'_h &\rightarrow M_{n \times n}, & Q_h &= [Q_{h.ij}]_{i,j=1,\dots,n} \end{aligned}$$

such that

$$Q_{h.ij}^{(r,m)} \geq 0 \text{ on } E'_h \text{ for } (i,j) \in \Gamma, \quad \sum_{i=1}^n Q_{h.ii}^{(r,m)} \leq 1 \text{ on } E'_h,$$

and

$$(69) \quad \begin{aligned} F_h[w, \eta]^{(r,m)} - F_h[\bar{w}, \bar{\eta}]^{(r,m)} &= W^{(r,m)} + [1 - \sum_{i=1}^n Q_{h.ii}^{(r,m)}] (\eta - \bar{\eta})^{(\theta)} \\ &+ \sum_{i=1}^n \tilde{S}_{h.i}^{(r,m)} (\eta - \bar{\eta})^{(e_i)} + \sum_{i=1}^n \bar{S}_{h.i}^{(r,m)} (\eta - \bar{\eta})^{(-e_i)} \\ &+ \sum_{(i,j) \in \Gamma_+} Q_{h.ij}^{(r,m)} (\eta - \bar{\eta})^{(-e_i + e_j)} + (\eta - \bar{\eta})^{(-e_i - e_j)} \\ &+ \sum_{(i,j) \in \Gamma_-} Q_{h.ij}^{(r,m)} (\eta - \bar{\eta})^{(-e_i + e_j)} + (\eta - \bar{\eta})^{(e_i - e_j)}. \end{aligned}$$

Moreover,

$$(70) \quad \sum_{i=1}^n \tilde{S}_{h.i}^{(r,m)} + \sum_{i=1}^n \bar{S}_{h.i}^{(r,m)} + 2 \sum_{(i,j) \in \Gamma} Q_{h.ij}^{(r,m)} - \sum_{i=1}^n Q_{h.ii}^{(r,m)} = 0 \text{ on } E'_h.$$

We conclude from (66) that

$$|W^{(r,m)}| \leq \sigma(t^{(r)}, V[w - \bar{w}]) \text{ on } E'_h.$$

The above inequality, (69) and (70) imply (14). Thus we see that all the assumptions of Theorem 2.4 are satisfied and the assertion (68) follows. \square

Now we formulate a result on the error estimate for difference method (60), (61). For $X, Y \in M_{n \times n}$, where

$$X = [x_{ij}]_{i,j=1,\dots,n}, \quad Y = [y_{ij}]_{i,j=1,\dots,n},$$

we put

$$[|X|] = [|x_{ij}|]_{i,j=1,\dots,n}, \quad \|X\|_{\star} = \sum_{i,j=1}^n |x_{ij}|$$

and we write $X \leq Y$ if $x_{ij} \leq y_{ij}$ for $i, j = 1, \dots, n$. Let us denote by $M_{n \times n}^+$ the class of all $X \in M_{n \times n}$ such that $x_{ij} \geq 0$ for $i, j = 1, \dots, n$.

LEMMA 3.11. *Suppose that*

- 1) $F : \Xi \rightarrow \mathbb{R}$ satisfies conditions 1), 2), 4) of Assumption $H[F, \alpha]$ and there is $\tilde{L} \in \mathbb{R}_+$ such that

$$|F(t, x, w, q, s) - F(t, x, \bar{w}, q, s)| \leq \tilde{L} \|w - \bar{w}\|_D \quad \text{on } \Xi,$$

- 2) $h \in H$ and there are $L \in \mathbb{R}_+^n$, and $B \in M_{n \times n}^+$, such that

$$[|\partial_q F(P)|] \leq L, \quad [|\partial_s F(P)|] \leq B, \quad P = (t, x, w, q, s) \in \Xi,$$

- 3) $z_h : \Omega_h \rightarrow \mathbb{R}$ is a solution of (60), (61) and there is $\beta_0 : H \rightarrow \mathbb{R}_+$ such that condition (67) is satisfied,
- 4) $v : \Omega \rightarrow \mathbb{R}$ is a classical solution of (7), (8) and v is of class C^4 on Ω and $\tilde{C} \in \mathbb{R}_+$ is defined by the relations

$$|\partial_{x_i x_j} v(t, x)|, \quad |\partial_{x_i x_j x_\mu} v(t, x)|, \quad |\partial_{x_i x_j x_\mu x_\nu} v(t, x)| \leq \tilde{C} \quad \text{on } \Omega,$$

where $i, j, \mu, \nu = 0, 1, \dots, n$ and $x_0 = t$.

Then

$$(71) \quad |(z_h - v_h)^{(r,m)}| \leq \bar{\alpha}(h) \quad \text{on } E_h,$$

where $v_h = v|_{\Omega_h}$ and

$$\bar{\alpha}(h) = \alpha_0(h) e^{\tilde{L}a} + \bar{\gamma}(h) \frac{e^{\tilde{L}a} - 1}{\tilde{L}} \quad \text{if } \tilde{L} > 0,$$

$$\bar{\alpha}(h) = \alpha_0(h) + a\bar{\gamma}(h) \quad \text{if } \tilde{L} = 0$$

and

$$\bar{\gamma}(h) = \frac{1}{2} h_0 + \tilde{L} \tilde{C} \|h\|^2 + \tilde{C} \left(\frac{1}{6} \|L\| + \frac{3}{4} \|M\|_{\star} \right) \max \{h_i : 1 \leq i \leq n\}.$$

PROOF. We deduce from condition 2) and from Lemma 2.2 that the operator $F_h : S_h[\mathbb{R}] \rightarrow \mathbb{R}$ given by (62) satisfies condition (15) with $\gamma(h) = \bar{\gamma}(h)$. We thus get estimate (71) from Lemma 2.5 and the proof is complete. \square

References

1. Brandi P., Marcelli C., *Haar inequality in hereditary setting and applications*, Rend. Sem. Mat. Univ. Padova, **96** (1966), 177–194.
2. Brandi P., Marcelli C., *On mixed problem for first order partial differential functional equations*, Atti. Sem. Mat. Fis. Univ. Modena, **46** (1998), 497–510.
3. Brunner H., *The numerical treatment of ordinary and partial Volterra integro differential equations*, in: D. Bainov, V. Covachev (eds.), Proc. First Internat. Colloquium on Numerical Anal., Plovdiv, 17–23 August 1992, VSP Utrecht, Tokyo, 1993, 13–26.
4. Brzychczy S., *Infinite Systems of Parabolic Differential-Functional Equations*, AGH University of Science and Technology Press, Cracow, 2006.
5. Bychowska A., *Existence of unbounded solutions to parabolic equations with functional dependence*, Math. Nachr., **263–264** (2004), 53–66.
6. Bychowska A., Leszczyński H., *Parabolic equations with functional dependence*, Z. Anal. Anwendungen, **20** (2001), 115–130.
7. Czernous W., *Generalized solutions of mixed problems for first-order partial differential functional equations*, Ukrainian Math. J., **58(6)** (2006), 903–936.
8. Jaruszewska-Walczak D., Kamont Z., *Numerical methods for hyperbolic functional differential problems on the Haar pyramid*, Computing, **65** (2000), 45–72.
9. Kamont Z., *Hyperbolic Functional Differential Inequalities and Applications*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1999.
10. Kamont Z., Leszczyński H., *Stability of difference equations generated by parabolic differential functional equations*, Rend. Mat. VII, **16** (1996), 265–287.
11. Kaughten J. P., *The method of lines for parabolic partial integro-differential equations*, J. Integral Equat. Appl., **4** (1992), 69–81.
12. Malec M., Rosati M., *A convergent scheme for nonlinear systems of differential functional equations of parabolic type*, Rend. Mat. VII, **3** (1983), 211–227.
13. Malec M., Schiaffino A., *Méthode aux différences finies pour une équation non-linéaire différentielle du type parabolique avec une condition initiale de Cauchy*, Boll. Un. mat. Ital. B(7), **1** (1987), No. 1, 99–109.
14. Ntouyas S. K., Tsamatos P. Ch., *Initial an boundary value problems for partial functional differential equations*, J. Appl. Math. Stochastic Anal., **10** (1997), 157–168.
15. Pao C. V., *Numerical methods for systems of nonlinear parabolic equations with time delays*, Journ. Math. Anal. Appl., **240** (1999), 249–279.
16. Pao C. V., *Finite difference solutions of reaction diffusion equations with continuous time delays*, Comput. Math. Anal., **42** (2001), 399–412.
17. Pao C. V., *Finite difference reaction-diffusion systems with coupled boundary conditions and time delays*, J. Math. Anal. Appl., **272** (2002), 407–434.
18. Razgulin A. V., Roganowich I. B., *Convergence of the projection difference scheme for the nonlinear parabolic equation with transformed spatial argument*, Comput. Math. Model., **12** (2001), 262–270.
19. Redheffer R., Walter W., *Comparison theorems for parabolic functional inequalities*, Pacific J. Math., **85** (1979), 447–470.
20. Redlinger R., *Existence theorems for semilinear parabolic systems with functionals*, Non-linear Anal., **8** (1984), 667–682.
21. Samarskii A. A., Matus P. P., Vabishchevich P. N., *Difference Schemes with Operator Factors*, Kluwer Acad. Publ., Dordrecht, 2002.

22. Szarski J., *Uniqueness of the solution to a mixed problem for parabolic functional differential equations in arbitrary domains*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astr. Phys, **24** (1976), 841–849.
23. Voigt W., *On finite difference methods for parabolic functional differential equations on unbounded domains*, Numerical Methods and Applications, Sofia, 1989, Publ. House Bulgar. Acad. Sci., Sofia, 1989.
24. Yuan-Ming Wang, Pao C., *Time delayed finite difference reaction–diffusion systems with nonquasimonotone functional*, Numer. Math., **103** (2006), 485–513.
25. Wu J., *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

Received April 19, 2007

Institute of Mathematics
University of Gdańsk
ul. Wita Stwosza 57
80-952 Gdańsk, Poland